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Some Properties of Ramsey Numbers

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Abstract—In this paper, some properties of Ramsey numbers are studied, and the following results are presented.

(1) For any positive integers $k_1, k_2, \dots, k_m$, $l_1, l_2, \dots, l_m$ $(m > 1)$, we have

$$r\left(\prod_{i=1}^{m} k_i + 1, \prod_{i=1}^{m} l_i + 1\right) \geq \prod_{i=1}^{m} \left[r\left(k_i + 1, l_i + 1\right) - 1\right] + 1.$$

(2) For any positive integers $k_1, k_2, \dots, k_m$, $l_1, l_2, \dots, l_n$, we have

$$r\left(\sum_{i=1}^{m} k_i + 1, \sum_{j=1}^{n} l_j + 1\right) \geq \sum_{i=1}^{m} \sum_{j=1}^{n} r\left(k_i + 1, l_j + 1\right) - mn + 1.$$

Based on the known results of Ramsey numbers, some results of upper bounds and lower bounds of Ramsey numbers can be directly derived by those properties. © 2003 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

DEFINITION 1.1. *For any given positive integers k and l, the Ramsey number $r(k,l)$ is the smallest integer that makes each graph with at least $r(k,l)$ vertices contain a clique with k vertices or an independent set with l vertices.*

It has been generally accepted that it is difficult to determine the Ramsey numbers. So it is meaningful to study the properties of the Ramsey numbers and their upper bounds and lower bounds.

DEFINITION 1.2. *If neither a clique with k vertices nor an independent set with l vertices exists in graph G, then G is called a (k,l)-ingenuity graph.*

According to Definitions 1.1 and 1.2, the following lemmas are obviously true.

LEMMA 1.1. *If graph G is a (k,l)-ingenuity graph of order m, then $r(k,l) > m$.*

LEMMA 1.2. *If $m < r(k,l)$, then there exists a (k,l)-ingenuity graph with m vertices.*

The other terminologies and known results can be found in [1–4].

2. MAIN RESULTS

First, we present a proposition as follows.

PROPOSITION 1. *For any positive integers k_1, k_2, l_1, l_2, we have*

$$r\left(k_1k_2+1, l_1l_2+1\right) \geq \left(r\left(k_1+1, l_1+1\right)-1\right)\left(r\left(k_2+1, l_2+1\right)-1\right)+1.$$

PROOF. Let $G_i(V,E)$, $i=1,2$, be two ingenuity graphs with $r(k_1+1,l_1+1)-1$ and $r(k_2+1, l_2+1)-1$ vertices, respectively, then

$$\omega\left(G_i\right) \leq k_i, \quad \omega\left(\bar{G}_i\right) \leq l_i, \qquad i=1,2.$$

Let $V(G_1)=\{v_1,v_2,\dots,v_m\}$, $V(G_2)=\{u_1,u_2,\dots,u_n\}$, and quote G_2^j, $j=1,2,\dots,m$, denote m isomorphic graphs of G_2 with vertices sets $V(G_2^j)=\{u_1^j,u_2^j,\dots,u_n^j\}$, $j=1,2,\dots,m$, respectively, where u_i^k maps to u_i^p under the isomorphic mappings. We now define a graph $G(V,E)$ as follows:

$$V(G)=\bigcup_{j=1}^{m} V\left(G_2^j\right), \qquad E(G)=E' \cup \left(\bigcup_{j=1}^{m} E\left(G_2^j\right)\right),$$

where $(u_i^p, u_j^k) \in E'$ iff $(v_i, v_j) \in E(G_1)$. Obviously, we have $\omega(G) \leq k_1k_2$, $\omega(\bar{G}) \leq l_1l_2$. Hence,

$$r\left(k_1k_2+1, l_1l_2+1\right)-1 \geq |V(G)| = \left(r\left(k_1+1,l_1+1\right)-1\right)\left(r\left(k_2+1,l_2+1\right)-1\right),$$

which is the conclusion of this proposition.

THEOREM 2.1. *For any positive integers $k_1,k_2,\dots,k_m$, $l_1,l_2,\dots,l_m$ $(m>1)$, we have*

$$r\left(\prod_{i=1}^{m} k_i+1, \prod_{i=1}^{m} l_i+1\right) \geq \prod_{i=1}^{m}\left[r\left(k_i+1,l_i+1\right)-1\right]+1.$$

PROOF. Use induction on m. If $m=1$, then $r(k_1+1,l_1+1) \geq r(k_1+1,l_1+1)-1+1$, and the conclusion is true. If $m=2$, according to Proposition 1, the conclusion is also true.

Suppose the conclusion is true for any integer less than m. Particularly, for $m-1$ $(m \geq 2)$, the conclusion is true, that is,

$$r\left(\prod_{i=1}^{m-1} k_i+1, \prod_{i=1}^{m-1} l_i+1\right) \geq \prod_{i=1}^{m-1}\left[r\left(k_i+1,l_i+1\right)-1\right]+1.$$

We now prove the conclusion is true for m:

$$
\begin{aligned}
r\left(\prod_{i=1}^{m} k_i + 1, \prod_{i=1}^{m} l_i + 1\right) &= r\left(\left(\prod_{i=1}^{m-1} k_i\right) k_m + 1, \left(\prod_{i=1}^{m-1} l_i\right) l_m + 1\right) \\
&\geq \left[r\left(\prod_{i=1}^{m-1} k_i + 1, \prod_{i=1}^{m-1} l_i + 1\right) - 1\right]\left[r\left(k_m + 1, l_m + 1\right) - 1\right] + 1 \\
&\geq \prod_{i=1}^{m-1}\left[r\left(k_i + 1, l_i + 1\right) - 1\right]\left[r\left(k_m + 1, l_m + 1\right) - 1\right] + 1 \\
&= \prod_{i=1}^{m}\left[r\left(k_i + 1, l_i + 1\right) - 1\right] + 1.
\end{aligned}
$$

So the conclusion is true.

If $k_1 = l_1 = k$, $k_2 = l_2 = l$, then we can obtain following corollaries immediately.

COROLLARY 2.2. *(See [1].) For any positive integers k and l, we have*

$$r\left(kl + 1, kl + 1\right) - 1 \geq \left(r\left(k + 1, k + 1\right) - 1\right)\left(r\left(l + 1, l + 1\right) - 1\right).$$

COROLLARY 2.3. *If $r(k + 1, k + 1) = m + 1$, for all $n > 0$, we have*

$$r\left(k^n + 1, k^n + 1\right) \geq m^n + 1.$$

PROOF. When $n = 0$, the conclusion is true obviously.

When $n > 0$, by Theorem 2.1, we have

$$r\left(k^n + 1, k^n + 1\right) \geq \prod_{i=1}^{n}\left(r\left(k + 1, k + 1\right) - 1\right) + 1 = m^n + 1.$$

By Corollary 2.3, we can obtain the following corollary immediately.

COROLLARY 2.4. *(See [1].) For all $n > 0$, we have*

$$r\left(2^n + 1, 2^n + 1\right) \geq 5^n + 1.$$

PROPOSITION 2. *For any positive integers k_1, k_2, l_1, l_2, we have*

$$r\left(k_1 + k_2 + 1, l_1 + l_2 + 1\right) \geq \sum_{i=1}^{2}\sum_{j=1}^{2} r\left(k_i + 1, l_j + 1\right) - 4 + 1.$$

PROOF. Let G_i, $i = 1, 2, \ldots, 4$, be ingenuity graphs with

$$
\begin{aligned}
|V(G_1)| &= r\left(k_1 + 1, l_1 + 1\right) - 1, &\quad |V(G_2)| &= r\left(k_1 + 1, l_2 + 1\right) - 1, \\
|V(G_3)| &= r\left(k_2 + 1, l_1 + 1\right) - 1, &\quad |V(G_4)| &= r\left(k_2 + 1, l_2 + 1\right) - 1.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\omega(G_1) \leq k_1, \quad \omega(\bar{G}_1) \leq l_1; \qquad \omega(G_2) \leq k_1, \quad \omega(\bar{G}_2) \leq l_2; \\
&\omega(G_3) \leq k_2, \quad \omega(\bar{G}_3) \leq l_1; \qquad \omega(G_4) \leq k_2, \quad \omega(\bar{G}_4) \leq l_2.
\end{aligned}
$$

Let $a \in V(G_1)$, $b \in V(G_4)$, $c \in V(G_2)$, and $d \in V(G_3)$. We now construct a graph as follows:

$$V(G) = \bigcup_{i=1}^{4} V(G_i); \qquad E(G) = E' \cup \left(\bigcup_{i=1}^{4} E(G_i) \right),$$

where $E' = \{(u,v) \mid u \in V(G_1) \text{ and } v \in V(G_3), \text{ or } u \in V(G_2) \text{ and } v \in V(G_4)\} \cup \{ab, cd\}$. If a graph G_0 is obtained by joining any vertices between two vertices disjoint graphs G' and G'', it is clear that $\omega(G_0) \le \omega(G') + \omega(G'')$ and $\omega(\bar{G}_0) \le \max\{\omega(\bar{G}'), \omega(\bar{G}'')\}$, hence, we have $\omega(G) \le k_1+k_2$, $\omega(\bar{G}) \le l_1+l_2$, hence, $r(k_1+k_2+1, l_1+l_2+1)-1 \ge |V(G_1)|+|V(G_2)|+|V(G_3)|+|V(G_4)|$, which is the conclusion of this proposition.

When $m = 2$ and $n = 1$ (or $m = 1$ and $n = 2$), let G_1 and G_2 be two ingenuity graphs with $|V(G_1)| = r(k_1+1, l_1+1)-1$ and $|V(G_2)| = r(k_2+1, l_1+1)-1$ (or $|V(G_1)| = r(k_1+1, l_1+1)-1$ and $|V(G_2)| = r(k_1+1, l_2+1)-1$), respectively. Graph G is obtained by joining all of the vertices (or adding just one edge) between $V(G_1)$ and $V(G_2)$. Clearly, the conclusion is true.

When $m = 1$ or $n = 1$, the conclusion is trivial.

THEOREM 2.5. *For any positive integers $k_1, k_2, \dots, k_m$, $l_1, l_2, \dots, l_n$, we have*

$$r\left(\sum_{i=1}^{m} k_i + 1, \sum_{j=1}^{n} l_j + 1\right) \ge \sum_{i=1}^{m}\sum_{j=1}^{n} r(k_i+1, l_j+1) - mn + 1.$$

PROOF. Combining with Proposition 2, we use induction on n.

If $n = 1$, the following conclusion can be proved:

$$r\left(\sum_{i=1}^{m} k_i + 1, l_1 + 1\right) \ge \sum_{i=1}^{m} r(k_i, l_1+1) - m + 1.$$

We now use induction on m.

(1) If $m = 1$, then $r(k_1+1, l_1+1) = r(k_1+1, l_1+1) - 1 + 1$, so the conclusion is obviously true.

(2) Assume the conclusion is true for all integers less than m. Without loss of generality, assume that the conclusion holds for $m-1$ ($m \ge 2$), that is,

$$r\left(\sum_{i=1}^{m-1} k_i + 1, l_1 + 1\right) \ge \sum_{i=1}^{m-1} r(k_i+1, l_1+1) - (m-1) + 1.$$

We now prove the conclusion is true for m:

$$\begin{aligned} r\left(\sum_{i=1}^{m} k_i + 1, l_1 + 1\right) &= r\left(\sum_{i=1}^{m-1} k_i + k_m + 1, l_1 + 1\right) \\ &\ge r\left(\sum_{i=1}^{m-1} k_i + 1, l_1 + 1\right) + r(k_m+1, l_1+1) - 2 + 1 \\ &\ge \sum_{i=1}^{m-1} r(k_i+1, l_1+1) + r(k_m+1, l_1+1) - (m-1) + 1 - 2 + 1 \\ &= \sum_{i=1}^{m} r(k_i+1,\ l_1+1) - m + 1. \end{aligned}$$

By the induction principle, the conclusion is true for any $m \ge 1$ and $n = 1$, that is,

$$r\left(\sum_{i=1}^{m} k_i + 1, l_1 + 1\right) \ge \sum_{i=1}^{m} r(k_i+1, l_1+1) - m + 1.$$

(3) Assume the conclusion is true for all integers less than n, particularly for $n-1$ $(n \geq 2)$, that is,

$$r\left(\sum_{i=1}^{m} k_i + 1, \sum_{j=1}^{n-1} l_j + 1\right) \geq \sum_{i=1}^{m} \sum_{j=1}^{n-1} r\left(k_i + 1, l_j + 1\right) - m\left(n-1\right) + 1.$$

We now prove the conclusion is true for n:

$$\begin{aligned} r\left(\sum_{i=1}^{m} k_i + 1, \sum_{j=1}^{n} l_j + 1\right) &= r\left(\sum_{i=1}^{m} k_i + 1, \sum_{j=1}^{n-1} l_j + l_n + 1\right) \\ &\geq r\left(\sum_{i=1}^{m} k_i + 1, \sum_{j=1}^{n-1} l_j + 1\right) + r\left(\sum_{i=1}^{m} k_i + 1, l_n + 1\right) - 2 + 1 \\ &\geq \sum_{i=1}^{m} \sum_{j=1}^{n-1} r\left(k_i + 1, l_j + 1\right) - m\left(n-1\right) + 1 \\ &\quad + \sum_{i=1}^{m} r\left(k_i + 1, l_n + 1\right) - m + 1 - 1 \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} r\left(k_i + 1, l_j + 1\right) - mn + 1. \end{aligned}$$

Combining the above process, the conclusion holds for any integers $m \geq 1, n \geq 1$, that is,

$$r\left(\sum_{i=1}^{m} k_i + 1, \sum_{j=1}^{n} l_j + 1\right) \geq \sum_{i=1}^{m} \sum_{j=1}^{n} r\left(k_i + 1, l_j + 1\right) - mn + 1.$$

COROLLARY 2.6. *For any positive integers k and l, we have*

$$r\left(k, l\right) \geq \left(k-1\right)\left(l-1\right) + 1.$$

PROOF. Let $k_i = 1$, $i = 1, 2, \ldots, k-1$, $l_j = 1$, $j = 1, 2, \ldots, l-1$, then from Theorem 2.5, we have

$$\begin{aligned} r\left(k, l\right) &= r\left(\sum_{i=1}^{k-1} 1 + 1, \sum_{j=1}^{l-1} 1 + 1\right) \\ &\geq \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} r\left(2, 2\right) - \left(k-1\right)\left(l-1\right) + 1 \\ &= 2\left(k-1\right)\left(l-1\right) - \left(k-1\right)\left(l-1\right) + 1 = \left(k-1\right)\left(l-1\right) + 1. \end{aligned}$$

From Corollary 2.6, it is easy to obtain following corollary.

COROLLARY 2.7. *For any positive integer k, we have $r(k,k) \geq (k-1)^2 + 1$.*

COROLLARY 2.8. *For any positive integers k and l, we have*

$$r\left(2k+1, 2l+1\right) \geq 5kl + 1.$$

PROOF. By the above theorems and corollaries, we have

$$\begin{aligned} r\left(2k+1, 2l+1\right) &\geq \left(r\left(2+1, 2+1\right) - 1\right)\left(r\left(k+1, l+1\right) - 1\right) + 1 \\ &\geq \left(r\left(3, 3\right) - 1\right) kl + 1 = 5kl + 1. \end{aligned}$$

THEOREM 2.9. *Every vertex in a (k,l)-ingenuity graph of order $r(k,l)-1$ is included in some cliques with $k-1$ vertices or in some independent set with $l-1$ vertices.*

PROOF. Let $G=(V,E)$ be a (k,l)-ingenuity graph with $r(k,l)-1$ vertices and let a be any vertex of G.

Suppose each vertex subset with $k-1$ vertices which includes vertex a is not a clique.

We now consider graph $G'=(V',E')$ of order $r(k,l)$, where $V'=V\cup\{b\}$, $E'=E\cup\{by \mid ay\in E\}\cup\{ab\}$, $b\notin V$. We now prove G' is an ingenuity graph.

Let X be any set with k vertices of G. We consider the following three cases.

CASE 1. $a,b\in X$. Then $|X-\{b\}|=k-1$, $X-\{b\}\subseteq V$, and $X-\{b\}$ contains a. From the assumption, $X-\{b\}$ is not a clique in G. So $X-\{b\}$ is not a clique in G', hence, X is not a clique in G'.

CASE 2. $b\notin X$. Then $X\subseteq V$, $|X|=k$. Since G is a (k,l)-ingenuity graph, X is not a clique in G. Hence, X is not a clique in G'.

CASE 3. $b\in X$ and $a\notin X$. Since a and b have the same neighborhood, this is similar to Case 2. X is not a clique in G'.

By the above discussion, X is not a clique in G'. Since X is any set with k vertices, no clique exists with k vertices in G'.

Let Y be any set with l vertices. We consider four cases.

CASE A. $a,b\in Y$. Because of $ab\in E'$, Y is not an independent set in G'.

CASE B. $a\in Y$ and $b\notin Y$. Then $Y\subseteq V$, since G is a (k,l)-ingenuity graph, but $|Y|=l$, Y is not an independent set in G. Hence, Y is not an independent set in G'.

CASE C. $b\in Y$ and $a\notin Y$. Then, since a and b have the same neighborhood, this is the same as Case B, Y is not a clique in G'.

CASE D. $a\notin Y$ and $b\notin Y$. Then $Y\subseteq V$, since G is a (k,l)-ingenuity graph, but $|Y|=l$, Y is not an independent set in G. Hence, Y is not an independent set in G'.

By the above discussion, Y is not an independent set in G'. Since Y is not any set with l vertices, no independent set with l vertices exists in G'.

Hence, G' is a (k,l)-ingenuity graph. But there are $r(k,l)$ vertices in G'. This contradicts Lemma 1.1, hence, we have proved that vertex a is contained in some clique with $k-1$ vertices.

Similarly, we can prove that vertex a is included in some independent set with $l-1$ vertices.

THEOREM 2.10. *Let G be a (k,l)-ingenuity graph with $r(k,l)-1$ vertices, then there exist $k-1$ disjoint independent sets with $l-1$ vertices in G, and $l-1$ disjoint cliques with $k-1$ vertices in G.*

PROOF. Assume $G=G(V,E)$.

(1) First, we prove that there exist $k-1$ disjoint independent sets with $l-1$ vertices.

We now divide the vertex set V into disjoint independent subsets $F_1,F_2,\dots,F_m,S$, where $F_1,F_2,\dots,F_m$ are independent sets with $l-1$ vertices, and S is all of the remaining vertices not in $F_1,F_2,\dots,F_m$. Clearly, the independent set in S has at most $l-1$ vertices. If there is no independent set with $l-1$ vertices in G, let $S=V$, and $m=0$. If there is at least one independent set with $l-1$ vertices in G, the subsets $F_1,F_2,\dots,F_m$ and S are easy to get. We may allow $m=0$ and $S=\emptyset$.

If $m\geq k-1$, then we have the known conclusion.

If $m<k-1$, then we consider the graph $G'=(V',E')$ with $r(k,l)$ vertices, where $b\notin V$, $V'=V\cup\{b\}$, $E'=E\cup\{ab \mid a\notin S\}$.

As in the proof of Theorem 2.9, it follows that G' is a (k,l)-ingenuity graph. But there are $r(k,l)$ vertices in G', which contradicts Lemma 1.1. Hence, $m\geq k-1$, that is, the conclusion is true.

(2) Similar to (1), it follows that there are $l-1$ disjoint cliques with $k-1$ vertices in G.

Finally, what we should state is that for every positive integer $k \geq 2$, according to monotonousness of the function, it is known that:

(1) when $k \leq 15$, $2^{k/2} < (k-1)^2 + 1$;

(2) when $k > 15$, $2^{k/2} > (k-1)^2 + 1$.

Hence, when $2 \leq k \leq 15$, from the formula $r(k,k) \geq (k-1)^2 + 1$, we can get a better lower bound of the Ramsey number than from the formula $r(k,k) \geq 2^{k/2}$; when $k > 15$, the lower bound of Ramsey number by means of $r(k,k) \geq 2^{k/2}$ is better.

Similarly, when $m = \min\{k,l\} \leq 15$, we can get a better lower bound of Ramsey number by means of $r(k,l) \geq (k-1)(l-1)+1$ than $r(k,l) \geq 2^{k/2}$. When $m = \min\{k,l\} > 15$, we can get a better lower bound of Ramsey number by means of $r(k,l) \geq 2^{m/2}$ than $r(k,l) \geq (k-1)(l-1)+1$. Of course, if $\min\{k,l\}$ is not big enough, but $|k-l|$ is bigger, we can get a better lower bound of Ramsey number from $r(k,l) \geq (k-1)(l-1)+1$.

And what is more, in order to get a bigger lower bound of Ramsey number, we can divide k, l into several integers by employing Theorem 2.1.

For Ramsey number $R(R_1, R_2, \ldots, R_m)$, it is not difficult to get the same conclusion as in this paper, and we can deduce other bound formula of Ramsey number from the conclusions in this paper.

REFERENCES

1. J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, New York, (1976).
2. H.L. Abbott, Lower bounds for some Ramsey numbers, *Discrete Math.* **2**, 289–293, (1972).
3. F.R.K. Chung and C.M.A. Crinstead, A survey of bound for classical Ramsey numbers, *J. of Graph Theory* **7**, 25–37, (1983).
4. G. Exoo, A lower bound for $r(5,5)$, *J. of Graph Theory* **13**, 97–98, (1989).