The $n$-motivic $t$-structures for $n = 0, 1$ and $2^{\star}$

Joseph Ayoub $^{a,b,\ast}$

$^{a}$ Institut für Mathematik, Universität Zürich, Winterthurerstr. 190, CH-8057 Zürich, Switzerland

$^{b}$ CNRS, LAGA Université Paris 13, 99 avenue J.B. Clément, 93430 Villetaneuse, France

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Abstract

For a field $k$ and an integer $n \in \{0, 1, 2\}$, we construct a $t$-structure $(^{n}T_{\geq 0}^{M}(k), ^{n}T_{\leq 0}^{M}(k))$ on Voevodsky’s triangulated category of motives $DM_{\text{eff}}(k)$, which we call the $n$-motivic $t$-structure. When $n = 0$, this is simply the usual homotopy $t$-structure, but for $n \in \{1, 2\}$, these are new $t$-structures. We will show that the category of Deligne’s 1-motives can be embedded as a full subcategory in the heart of the 1-motivic $t$-structure. By a rather straightforward analogy, we are led to specify a class of objects in the heart of the 2-motivic $t$-structure which we call mixed 2-motives. We will also check that these objects form an Abelian category.

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* Address for correspondence: Institut für Mathematik, Universität Zürich, Winterthurerstr. 190, CH-8057 Zürich, Switzerland.

E-mail address: joseph.ayoub@math.uzh.ch.

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0. Introduction

A major open problem in the theory of motives is the construction of a motivic t-structure on Voevodsky’s triangulated category $\mathbf{DM}_{\text{eff}}(k)$ whose heart would be the awaited Abelian category of mixed motives. This motivic t-structure should be very different from the existing homotopy t-structure which is an outcome of the construction of $\mathbf{DM}_{\text{eff}}(k)$ and the study of homotopy invariant presheaves with transfers [7]. However, one can speculate about the existence of a sequence of n-motivic t-structures $(n_T M_\geq 0(k), n_T M_\leq 0(k))$ on $\mathbf{DM}_{\text{eff}}(k)$, which interpolate between the homotopy t-structure and the motivic t-structure. More precisely, we expect the following to hold:

1. $(0_T M_\geq 0(k), 0_T M_\leq 0(k))$ is the homotopy t-structure.
2. $n_T M_\geq 0(k) \subset n+i_T M_\geq 0(k)$ and $n_T M_\leq 0(k) \supset n+i_T M_\leq 0(k)$ for $i \geq 0$.
3. Denote $\infty_T M_\leq 0(k) = \bigcap_{n \in \mathbb{N}} n_T M_\leq 0(k)$ and $\infty_T M_\geq 0(k) \subset \mathbf{DM}_{\text{eff}}(k)$ the full subcategory of $P \in \mathbf{DM}_{\text{eff}}(k)$ such that

$$\text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(P, N[-1]) = 0$$

for all $N \in \infty_T M_\leq 0(k)$. Then $(\infty_T M_\geq 0(k), \infty_T M_\leq 0(k))$ is the motivic t-structure.
4. $n_T M_\geq 0(k) \cap n+i_T M_\leq 0(k)$ is independent of $i \geq 1$ and is the Abelian category of mixed n-motives.

The last property above, justifies our terminology. For us, a mixed n-motive is an object of the heart of the motivic t-structure which is also in the smallest triangulated subcategory of $\mathbf{DM}_{\text{eff}}(k)$ stable under small sums and containing the motives of smooth varieties of dimension at most $n$. In particular, contrary to the usual, we allow non-geometric (or non-constructible) objects.
In this paper, we propose a definition of the $n$-motivic $t$-structure for $n \in \{0, 1, 2\}$. Our construction relies on [2]. We will see that objects in $\mathcal{DM}^0_{\leq 0}(k) \cap \mathcal{DM}^1_{\leq 1}(k)$ are (possibly non-compact) 0-motives. We also see that objects in $\mathcal{DM}^0_{\leq 1}(k) \cap \mathcal{DM}^1_{\leq 2}(k)$ are (possibly non-compact) Deligne’s 1-motives. Finally, we specify a class of objects in the heart of the 2-motivic $t$-structure which we call mixed 2-motives. We check that the category of mixed 2-motives is Abelian.

1. Preliminaries

1.1. Notation and general facts

If not otherwise stated, we work with rational coefficients. In particular, our sheaves take values in the category of $\mathbb{Q}$-vectorspaces and we think of an isogeny of semi-Abelian groupschemes as an invertible morphism (this will be made more precise later).

Fix a ground field $k$ and denote by $Sm/k$ the category of smooth $k$-schemes. Given two smooth $k$-schemes $X$ and $Y$, we denote by $\text{Cor}(X, Y)$ the group of finite correspondences, i.e., the $\mathbb{Z}$-module freely generated by closed and integral subschemes $Z \subset X \times_k Y$ which are finite and surjective on $X$. There is an additive category $\text{Cor}(k)$ whose objects are smooth $k$-schemes and whose morphisms are finite correspondences (see [7, Lect. 1]). The graph of a morphism yields an inclusion $Sm/k \hookrightarrow \text{Cor}(k)$.

A presheaf with transfers (on $Sm/k$) is an additive contravariant functor $F$ from $\text{Cor}(k)$ to the category of Abelian groups. $F$ is called a Nisnevich (resp. an étale) sheaf with transfers if its restriction to $Sm/k$ is a sheaf for the Nisnevich (resp. the étale) topology. If not otherwise stated, a presheaf with transfers $F$ is assumed to be uniquely divisible, i.e., takes values in the category of $\mathbb{Q}$-vectorspaces. Under this assumption, the restriction of $F$ to $Sm/k$ is a Nisnevich sheaf if and only if it is an étale sheaf. Thus, there will be no ambiguity in saying: $F$ is a sheaf with transfers. We denote $\text{Shv}_{tr}(k)$ the Abelian category of sheaves with transfers on $Sm/k$. There is an embedding $\mathbb{Q}_{tr}(-) : \text{Cor}(k) \hookrightarrow \text{Shv}_{tr}(k)$ which takes a smooth $k$-scheme $X$ to the sheaf with transfers $\mathbb{Q}_{tr}(X) = \text{Cor}(-, X) \otimes \mathbb{Q}$ represented by $X$.

We denote by $K(\text{Shv}_{tr}(k))$ the category of complexes of sheaves with transfers endowed with its injective model structure (i.e., $W = \{\text{quasi-isomorphisms}\}$ and $\text{Cof} = \{\text{monomorphisms}\}$). The homotopy category of this model structure is the derived category $D(\text{Shv}_{tr}(k))$. Following Voevodsky [7], we define $\mathcal{DM}_{\text{eff}}(k)$ to be the homotopy category of the Bousfield localization of $K(\text{Shv}_{tr}(k))$ with respect to the class of arrows $\mathbb{Q}_{tr}(\mathbb{A}^1_X)[n] \to \mathbb{Q}_{tr}(X)[n]$ for $X \in Sm/k$ and $n \in \mathbb{Z}$. Given a smooth $k$-scheme $X$, we denote by $M(X)$ the complex $\mathbb{Q}_{tr}(X)[0]$ considered as an object of $\mathcal{DM}_{\text{eff}}(k)$. This is the motive of $X$. From the general theory of Bousfield localizations (see [5]), we may identify (up to an equivalence) $\mathcal{DM}_{\text{eff}}(k)$ with the triangulated subcategory of $D(\text{Shv}_{tr}(k))$ whose objects are the $\mathbb{A}^1$-local complexes. Recall that $K \in K(\text{Shv}_{tr}(k))$ is $\mathbb{A}^1$-local if the natural homomorphism

$$H^n(X, K) \to H^n(\mathbb{A}^1_X, K)$$

is invertible for all $n \in \mathbb{Z}$ and $X \in Sm/k$. (Here, $H^n(\cdot, K)$ stands for the Nisnevich (or equivalently the étale) hypercohomology with values in $K$.) A central result of Voevodsky [7, Th. 24.1] asserts that this condition holds if and only if the homology sheaves $H_i(K)$ are homotopy invariant for all $i \in \mathbb{Z}$. In particular, this implies that the canonical $t$-structure on $D(\text{Shv}_{tr}(k))$ restricts to a $t$-structure on $\mathcal{DM}_{\text{eff}}(k)$. This is the so-called homotopy $t$-structure whose heart is identified with the category $HI(k)$ of homotopy invariant sheaves with transfers.
N.B. — In this paper, we will use the expression $\mathcal{H}$-sheaf as a shorthand for: “homotopy invariant sheaf with transfers”.

1.2. Some recollection from [2]

For $n \in \mathbb{N}$, we denote $Sm/k \leq n$ the full subcategory of $Sm/k$ whose objects are the smooth $k$-schemes of dimension less or equal to $n$. Similarly, we denote $\text{Cor}(k \leq n)$ the full-subcategory of $\text{Cor}(k)$ having the same objects as $Sm/k \leq n$. A presheaf with transfers on $Sm/k \leq n$ is an additive contravariant functor $F$ from $\text{Cor}(k \leq n)$ to the category of $\mathbb{Q}$-vectorspaces. $F$ is a sheaf with transfers if its restriction to $Sm/k \leq n$ is a sheaf for the Nisnevich topology (or equivalently, for the étale topology). We denote by $\text{Shv}_{tr}(k \leq n)$ the category of étale sheaves with transfers on $Sm/k \leq n$. There is an adjunction [2, Lem. 1.1.12]:

$$\text{Shv}_{tr}(k \leq n) \xrightarrow{\sigma_n^*} \text{Shv}_{tr}(k). \quad (1)$$

**Definition 1.1.** An $\mathcal{H}$-sheaf $F \in \text{HI}(k)$ is $n$-presented if the obvious morphism

$$h_0 \sigma_n^* \sigma_{n*} F \to F$$

is an isomorphism. (Here, $h_0$ is the left adjoint to the inclusion $\text{HI}(k) \hookrightarrow \text{Shv}_{tr}(k)$.)

**Remark 1.2.** In [2] (see Def. 1.1.20 of [2]) $n$-presented $\mathcal{H}$-sheaves where called “$n$-motivic sheaves”. In this paper, we use a different terminology because of an eventual conflict with the notion of $(n, \mathcal{H})$-sheaf which will be introduced later (for $n \in \{0, 1, 2\}$).

Let $\text{HI}_{\leq n}(k)$ denotes the full-subcategory of $\text{HI}(k)$ whose objects are the $n$-presented $\mathcal{H}$-sheaves. This is an Abelian category, and the inclusion $\text{HI}_{\leq n}(k) \hookrightarrow \text{HI}(k)$ is right exact. It is conjectured that this inclusion is also left exact (see [2, Cor. 1.4.5] for a conjectural proof). This conjecture is known to hold for $n = 0, 1$ due to the following result (see [2, Cor. 1.2.5 and Prop. 1.3.11]).

**Proposition 1.3.** For $n = 0, 1$, the inclusions $\text{H}_{\leq n}(k) \hookrightarrow \text{Shv}_{tr}(k)$ admit left adjoints denoted respectively by

$$\pi_0 : \text{Shv}_{tr}(k) \longrightarrow \text{HI}_{\leq 0}(k) \quad \text{and} \quad \text{Alb} : \text{Shv}_{tr}(k) \longrightarrow \text{HI}_{\leq 1}(k). \quad (2)$$

**Definition 1.4.** An $\mathcal{H}$-sheaf $F \in \text{HI}(k)$ is called 0-connected if $\pi_0(F) = 0$. It is called 1-connected if $\text{Alb}(F) = 0$.

By [2, Prop. 2.3.2 and Th. 2.4.1], the functors $\pi_0$ and $\text{Alb}$ from (2) can be left derived, yielding functors

$$L\pi_0 : \text{D}(\text{Shv}_{tr}(k)) \longrightarrow \text{D}(\text{HI}_{\leq 0}(k)) \quad \text{and} \quad L\text{Alb} : \text{D}(\text{Shv}_{tr}(k)) \longrightarrow \text{D}(\text{HI}_{\leq 1}(k)).$$

The above functors pass to the Bousfield localization, yielding functors
\[ L\pi_0 : \text{DM}_{\text{eff}}(k) \rightarrow D(\text{HI}_{\leq 0}(k)) \quad \text{and} \quad \text{LAlb} : \text{DM}_{\text{eff}}(k) \rightarrow D(\text{HI}_{\leq 1}(k)) \]

which are left adjoints to the obvious functors

\[ D(\text{HI}_{\leq 0}(k)) \rightarrow \text{DM}_{\text{eff}}(k) \quad \text{and} \quad D(\text{HI}_{\leq 1}(k)) \rightarrow \text{DM}_{\text{eff}}(k). \]

These are fully faithful embedding with essential images \( \text{DM}_{\leq 0}(k) \) and \( \text{DM}_{\leq 1}(k) \) respectively. Recall that \( \text{DM}_{\leq n}(k) \) (with \( n \in \mathbb{N} \)) is the smallest triangulated subcategory of \( \text{DM}_{\text{eff}}(k) \) stable under small sums and containing \( M(X) \) with \( X \in \text{Sm}/k \) of dimension at most \( n \). It follows that the obvious inclusions \( \text{DM}_{\leq 0}(k) \hookrightarrow \text{DM}_{\text{eff}}(k) \) and \( \text{DM}_{\leq 1}(k) \hookrightarrow \text{DM}_{\text{eff}}(k) \) have left adjoints, which we also denote as follows:

\[ L\pi_0 : \text{DM}_{\text{eff}}(k) \rightarrow \text{DM}_{\leq 0}(k) \quad \text{and} \quad \text{LAlb} : \text{DM}_{\text{eff}}(k) \rightarrow \text{DM}_{\leq 1}(k). \]

### 1.3. Generating \( t \)-structures

Let \( T \) be a triangulated category. Recall from [3] that a \( t \)-structure on \( T \) is a couple of full subcategories \( (T_{\geq 0}, T_{\leq 0}) \) satisfying three simple axioms. Contrary to [3], we will use the homological convention for \( t \)-structures. One passes back and forth between the homological and cohomological conventions via the usual rule: \( T_{\geq n} = T_{\leq -n} \) and \( T_{\leq n} = T_{\geq -n} \).

In this paragraph we recall the technique of generating \( t \)-structures which is described in [1, §2.1.3]. Let \( G \) a class of objects in \( T \).

**Definition 1.5.** (Compare with [1, Déf. 2.1.68].)

(a) An object \( N \in T \) is \( G \)-negative if for every \( A \in G \) and \( n \in \mathbb{N} \), we have

\[ \text{hom}_T(A[n + 1], N) = 0. \]

We denote \( T^G_{\leq 0} \) the full subcategory of \( G \)-negative objects and set \( T^G_{\leq d} = T^G_{\leq 0}[d] \) for \( d \in \mathbb{Z} \).

(b) An object \( P \in T \) is \( G \)-positive if for every \( N \in T^G_{\leq -1} \), we have

\[ \text{hom}_T(P, N) = 0. \]

We denote \( T^G_{\geq 0} \) the full subcategory of \( G \)-positive objects and set \( T^G_{\geq d} = T^G_{\geq 0}[d] \) for \( d \in \mathbb{Z} \).

Recall that an object \( E \) of \( T \) is said to be an extension of \( E' \) and \( E'' \) if there exists a distinguished triangle in \( T \):

\[ E' \rightarrow E \rightarrow E'' \rightarrow E'[1]. \]

We record the following fact (see [1, Prop. 2.1.70]).
Proposition 1.6. Assume that $T$ has small sums and that $G$ is essentially small (i.e., the isomorphism classes of objects in $G$ form a set) and consists of compact objects. Then $(T^G_{\geq 0}, T^G_{\leq 0})$ is a $t$-structure on $T$. Moreover, $T^G_{\geq 0}$ is the smallest full subcategory of $T$ containing $G$, and stable under small sums, suspensions and extensions.

The $t$-structure $(T^G_{\geq 0}, T^G_{\leq 0})$ is said to be generated by $G$. Clearly $G \subset T^G_{\geq 0}$, and $(T^G_{\geq 0}, T^G_{\leq 0})$ is the universal $t$-structure with this property in the following sense (see [1, Lem. 2.1.78]).

Lemma 1.7. Keep the hypothesis in Proposition 1.6. Let $S$ be a triangulated category endowed with a $t$-structure $(S_{\geq 0}, S_{\leq 0})$. Let $F : T \to S$ be a triangulated functor. We assume that $F$ commutes with small sums and that $F(G) \subset S_{\geq 0}$. Then $F$ is $t$-positive, i.e., takes an object in $T^G_{\geq 0}$ to an object in $S_{\geq 0}$.

2. Perverting $t$-structures

2.1. The abstract construction

In this paragraph we present a simple way to construct new $t$-structures out of olds. This will be applied in the next section. We begin by describing the abstract setting.

Let $T$ be a triangulated category endowed with a $t$-structure $(T_{\geq 0}, T_{\leq 0})$. For $n \in \mathbb{Z}$, we denote by $\tau_{\geq n}$ and $\tau_{\leq n}$ the truncation functors with respect to this $t$-structures. Thus, we have a canonical distinguished triangle

$$\tau_{\geq n}(A) \to A \to \tau_{\leq n-1}(A) \to \tau_{\geq n}(A)[1]$$

for every $A \in T$. We also set $H_n(A) = \tau_{\geq n} \circ \tau_{\leq n}(A)[-n]$. This is an object of the heart $H_T = T_{\geq 0} \cap T_{\leq 0}$.

Let $A \subset H_T$ be a full subcategory. We assume the following.

Hypothesis 2.1.

(i) $A$ is a thick Abelian subcategory of $H_T$, i.e., stable under extensions, subobjects and quotients.

(ii) The inclusion $A \hookrightarrow H_T$ admits a left adjoint $F : H_T \to A$.

(iii) Let

$$0 \to A' \to A \to A'' \to 0,$$

be a short exact sequence in $H_T$. If $A'' \in A$, then $F(A') \to F(A)$ is a monomorphism.

Remark 2.2.

(a) It follows from (i) that the inclusion $A \hookrightarrow H_T$ is an exact functor. As it is also a full embedding, the unit of the adjunction $\phi_A : A \to F(A)$ is the universal morphism from $A \in H_T$ to an object in $A$. In particular, when $A \in A$, $\phi_A$ is invertible. For general $A \in H_T$, we claim
that \( \phi_A \) is surjective. Indeed, \( \text{im}(\phi_A) \) is a sub-object of \( \mathcal{F}(A) \). Hence by (i), it is in \( \mathcal{A} \). Applying the universal property to \( A \rightarrow \text{im}(\phi_A) \), we get a retraction \( \mathcal{F}(A) \rightarrow \text{im}(\phi_A) \). As the composition \( \mathcal{F}(A) \rightarrow \text{im}(\phi_A) \rightarrow \mathcal{F}(A) \) is the identity, \( \text{im}(\phi_A) \rightarrow \mathcal{F}(A) \) is an isomorphism.

(b) Being a left adjoint, the functor \( \mathcal{F} \) is right exact. Under the conditions of (iii), we thus have a short exact sequence in \( \mathcal{A} \):

\[
0 \rightarrow \mathcal{F}(A') \rightarrow \mathcal{F}(A) \rightarrow A'' \rightarrow 0. \tag{3}
\]

(Here we use that \( A'' \cong \mathcal{F}(A'') \).) For \( A \in \mathcal{H}_T \), we set

\[
\mathcal{G}(A) = \text{ker}\{A \rightarrow \mathcal{F}(A)\}. \tag{4}
\]

Thus, we have a canonical exact sequence in \( \mathcal{H}_T \):

\[
0 \rightarrow \mathcal{G}(A) \rightarrow A \rightarrow \mathcal{F}(A) \rightarrow 0. \tag{5}
\]

It follows from the exact sequence (3) that \( \mathcal{F}(\mathcal{G}(A)) = 0 \).

**Definition 2.3.** An object \( A \in \mathcal{H}_T \) is said to be \( \mathcal{F} \)-connected if \( \mathcal{F}(A) = 0 \). Equivalently, any morphism from \( A \) to an object of \( \mathcal{A} \) is zero.

We have seen that for any \( A \in \mathcal{H}_T \), \( \mathcal{G}(A) \) is \( \mathcal{F} \)-connected. Moreover, this is the largest \( \mathcal{F} \)-connected subobject of \( A \). Indeed, let \( a : B \rightarrow A \) be a morphism in \( \mathcal{H}_T \) from an \( \mathcal{F} \)-connected object \( B \). Then the composition \( \phi_A \circ a : B \rightarrow \mathcal{F}(A) \) is zero and hence, \( a \) factors through \( \mathcal{G}(A) \).

This also proves that \( \mathcal{G} \) is the right adjoint to the inclusion of the full subcategory of \( \mathcal{F} \)-connected objects in \( \mathcal{H}_T \).

We now come to the main construction of this paragraph.

**Proposition 2.4.** We define a \( t \)-structure \( (\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0}) \) on \( \mathcal{T} \) as follows:

- \( \mathcal{T}_{\geq 0} \) is the full subcategory of \( P \in \mathcal{T} \) such that \( H_i(P) = 0 \) for \( i < -1 \) and \( H_{-1}(P) \) is \( \mathcal{F} \)-connected.
- \( \mathcal{T}_{\leq 0} \) is the full subcategory of \( N \in \mathcal{T} \) such that \( H_i(N) = 0 \) for \( i > 0 \) and \( H_0(N) \in \mathcal{A} \).

**Proof.** As usual, we set \( \mathcal{T}_{\geq n} = \mathcal{T}_{\geq 0}[n] \) and \( \mathcal{T}_{\leq n} = \mathcal{T}_{\leq 0}[n] \) for \( n \in \mathbb{Z} \). We clearly have \( \mathcal{T}_{\geq 1} \subset \mathcal{T}_{\geq 0} \) and \( \mathcal{T}_{\leq 1} \supset \mathcal{T}_{\leq 0} \).

Let \( P \in \mathcal{T}_{\geq 0} \) and \( N \in \mathcal{T}_{\leq 1} \). Then \( P \in \mathcal{T}_{\geq -1} \) and \( N \in \mathcal{T}_{\leq -1} \). Thus, we have

\[
\text{hom}_\mathcal{T}(P, N) \cong \text{hom}_\mathcal{T}(\mathcal{\tau}_{\leq -1}(P), N) \cong \text{hom}_\mathcal{T}(\mathcal{\tau}_{\leq -1}(P), \mathcal{\tau}_{\geq -1}(N)) \cong \text{hom}_{\mathcal{H}_T}(H_{-1}(P), H_{-1}(N)).
\]

As \( H_{-1}(P) \) is \( \mathcal{F} \)-connected and \( H_{-1}(N) \in \mathcal{A} \), every morphism from \( H_{-1}(P) \) to \( H_{-1}(N) \) is zero. This shows that \( \text{hom}_\mathcal{T}(P, N) = 0 \).

To end the proof, we still need to check axiom (iii) of [3, Déf. 1.3.1]. Let \( A \in \mathcal{T} \). There is a distinguished triangle
where \( P_0 \in \mathcal{T}_{\geq -1} \) and \( N_0 \in \mathcal{T}_{\leq -2} \). Consider the composition

\[
t : P_0 \to H_{-1}(P_0)[-1] \to F(H_{-1}(P_0))[-1],
\]

and form a distinguished triangle

\[
P \to P_0 \to F(H_{-1}(P_0))[-1] \to P[1].
\]

Let \( u = u_0 \circ s : P \to A \). Clearly \( H_i(P) = 0 \) for \( i < -1 \), and we have an isomorphism \( H_i(u) : H_i(P) \sim H_i(A) \) for \( i > -1 \). Moreover, there is a short exact sequence

\[
0 \to H_{-1}(P) \xrightarrow{H_{-1}(u)} H_{-1}(A) \xrightarrow{F} F(H_{-1}(A)) \to 0.
\]

In particular \( H_{-1}(P) \) is F-connected. It follows that \( P \in \mathcal{T}_{\geq 0} \).

Now, form a distinguished triangle

\[
P \xrightarrow{u} A \xrightarrow{v} N \to P[+1].
\]

Then \( H_i(N) = 0 \) for \( i \leq 0 \) and

\[
H_{-1}(N) \simeq coker \{ H_{-1}(P) \to H_{-1}(A) \} \simeq F(H_{-1}(A)).
\]

This shows that \( N \in \mathcal{T}_{\leq -1} \). The proposition is proven. \( \square \)

**Definition 2.5.** Keep the above notation and assumption. The \( t \)-structure \( (\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0}) \) is called the \( \mathcal{A} \)-perverted \( t \)-structure.

**Remark 2.6.** We denote \( \mathcal{H}_T = \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0} \) the heart of the \( \mathcal{A} \)-perverted \( t \)-structure. Clearly, and object \( A \in \mathcal{T} \) is in \( \mathcal{H}_T \) if and only if it satisfies the following properties:

1. \( H_i(A) = 0 \) for \( i \notin \{0, -1\} \);
2. \( H_0(A) \in \mathcal{A} \);
3. \( H_{-1}(A) \) is F-connected.

From this, it follows immediately that \( \mathcal{A} = \mathcal{H}_T \cap \mathcal{H}_T \).

### 2.2. The case of a generated \( t \)-structure

We keep the notation and assumption of Section 2.1. Suppose that the \( t \)-structure \( (\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0}) \) is generated by an essentially small class \( \mathcal{G} \) of compact objects in \( \mathcal{T} \). Assume that for every \( A \in \mathcal{G} \), we can find a distinguished triangle
such that $A^F$ is compact and $t$-positive, $H_0(A^F) \in \mathcal{A}$, and the obvious morphism $F(H_0(A)) \to H_0(A^F)$ is invertible. Let $\mathcal{G}^\perp[-1] = \{ A^\perp[-1] \mid A \in \mathcal{G} \}$. We choose the above triangles so that $\mathcal{G}^\perp[-1]$ is again essentially small (this is clearly possible). Remark also that $\mathcal{G}^\perp[-1]$ consists of compact objects.

**Proposition 2.7.** The $t$-structure $(\mathcal{T}' \geq 0, \mathcal{T}' \leq 0)$ is generated by the essentially small class of compact objects $\mathcal{G}' = \mathcal{G} \cup \mathcal{G}^\perp[-1]$.

**Proof.** Denote by $(\mathcal{T}' \geq 0, \mathcal{T}' \leq 0)$ the $t$-structure on $\mathcal{T}$ generated by $\mathcal{G}$. It suffices to check that $\mathcal{T}' \geq 0 \subset \mathcal{T}' \geq 0$ and $\mathcal{T}' \leq 0 \subset \mathcal{T}' \leq 0$. It is easy to see that $\mathcal{G} \subset \mathcal{T}' \geq 0$. We thus have $\mathcal{T}' \geq 0 \subset \mathcal{T}' \geq 0$ by Lemma 1.7. To check the second inclusion, we fix $N \in \mathcal{T}' \leq 0$. As $\mathcal{G} \subset \mathcal{G}'$, we have $\mathcal{T}' \leq 0 \subset \mathcal{T}' \leq 0$ and thus $H_i(N) = 0$ for $i > 0$. It remains to show that $H_0(N) \in \mathcal{A}$.

Let $A \in \mathcal{G}$. Clearly, $A^\perp$ is $t$-positive. It follows that $\text{hom}_T (A^\perp, N) \cong \text{hom}_{\mathcal{H}_T} (H_0(A^\perp), H_0(N))$. On the other hand, $\text{hom}_T (A^\perp, N) = 0$ by the definition of the class of $\mathcal{G}$-negative objects. Thus we get $\text{hom}_{\mathcal{H}_T} (H_0(A^\perp), H_0(N)) = 0$.

From the statement of the proposition, we have an exact sequence

$$H_0(A^\perp) \longrightarrow H_0(A) \longrightarrow F(H_0(A)) \longrightarrow 0,$$

and hence a surjective morphism $H_0(A^\perp) \to G(H_0(A))$. We deduce from this an inclusion

$$\text{hom}_{\mathcal{H}_T} (G(H_0(A)), H_0(N)) \hookrightarrow \text{hom}_{\mathcal{H}_T} (H_0(A^\perp), H_0(N)).$$

This shows that

$$\text{hom}_{\mathcal{H}_T} (G(H_0(A)), H_0(N)) = 0. \tag{6}$$

Now, the Abelian category $\mathcal{H}_T$ is generated by $H_0(A)$ for $A \in \mathcal{G}$. Thus, we may find a family $(A_i)_{i \in I} \in \mathcal{G}_I$ and a surjective morphism $\bigoplus_{i \in I} H_0(A_i) \to H_0(N)$. Consider the induced morphism of short exact sequences

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \bigoplus_{i \in I} G(H_0(A_i)) & \longrightarrow & \bigoplus_{i \in I} H_0(A_i) & \longrightarrow & \bigoplus_{i \in I} F(H_0(A_i)) & \longrightarrow & 0 \\
& & \downarrow \alpha & & \downarrow & & \downarrow \beta & & \\
0 & \longrightarrow & G(H_0(N)) & \longrightarrow & H_0(N) & \longrightarrow & F(H_0(N)) & \longrightarrow & 0.
\end{array}
$$

As $G(H_0(N)) \to H_0(N)$ is injective, we deduce from (6) that $\alpha = 0$. By the Snake Lemma, we have a surjective morphism $\ker(\beta) \to G(H_0(N))$. As $\ker(\beta) \in \mathcal{A}$, we get from Hypothesis 2.1(i) that $G(H_0(N)) \in \mathcal{A}$. This implies that $G(H_0(N)) = 0$. Indeed, the identity morphism of an $F$-connected object which is in $\mathcal{A}$ is necessarily zero. The proposition is proven. \(\Box\)
2.3. Perverting subcategories of the heart

We keep the notation and assumption of Section 2.1. Let $\mathcal{B} \subset \mathcal{H}_T$ be a full subcategory satisfying the following conditions.

**Hypothesis 2.8.**

(i) $\mathcal{B}$ contains $\mathcal{A}$. Moreover, if $B \in \mathcal{B}$, then $G(B)$ is also in $\mathcal{B}$.

(ii) The category $\mathcal{B}$ is Abelian and the inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_T$ admits a right adjoint $Q : \mathcal{H}_T \to \mathcal{B}$.

It follows from (ii) that the inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_T$ is right exact. Given a morphism $b : B \to C$ in $\mathcal{B}$, its cokernel taken in $\mathcal{B}$ coincides with its cokernel taken in $\mathcal{H}_T$. It will be denoted by $\text{coker}(b)$. This is a priori not the case for kernels. We will reserve the notation $\text{ker}(b)$ for the kernel taken in $\mathcal{H}_T$ and denote $\text{ker}_{\mathcal{B}}(b)$ the kernel taken in $\mathcal{B}$. We have a canonical isomorphism $\text{ker}_{\mathcal{B}}(b) \simeq Q(\text{ker}(b))$.

**Definition 2.9.** Let $\mathcal{B} \subset \mathcal{T}$ be the full subcategory whose objects are the $A \in \mathcal{T}$ such that:

1. $H_i(A) = 0$ for $i \neq \{0, -1\}$;
2. $H_0(A) \in \mathcal{A}$;
3. $H_{-1}(A) \in \mathcal{B}$ and is $F$-connected.

We call $\mathcal{B}$ the $A$-perverted subcategory associated to $\mathcal{B}$. Clearly, $\mathcal{B}$ is contained in $\mathcal{H}_T$.

**Lemma 2.10.** The inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_T$ has a right adjoint $\mathcal{Q} : \mathcal{H}_T \to \mathcal{B}$. Moreover, for $A \in \mathcal{H}_T$, the counit of the adjunction $\mathcal{Q}(A) \to A$ induces isomorphisms $H_0(\mathcal{Q}(A)) \simeq H_0(A)$ and $H_{-1}(\mathcal{Q}(A)) \simeq G \circ Q(H_{-1}(A))$.

**Proof.** It suffices to construct for every $A \in \mathcal{H}_T$ a universal morphism $\mathcal{Q}(A) \to A$ from an object $\mathcal{Q}(A) \in \mathcal{B}$. We have a functorial distinguished triangle

$$H_0(A) \to A \to H_{-1}(A)[-1] \to H_0(A)[1].$$

As both $H_0(A)$ and $H_{-1}(A)[-1]$ are in $\mathcal{H}_T$, this determines a functorial short exact sequence

$$0 \to H_0(A) \to A \to H_{-1}(A)[-1] \to 0.$$

Consider $G \circ Q(H_{-1}(A))[-1]$. This is an object of $\mathcal{H}_T$. We define

$$\mathcal{Q}(A) = A \times_{H_{-1}(A)[-1]} (G \circ Q(H_{-1}(A)))[-1],$$

the fiber product being taken in the Abelian category $\mathcal{H}_T$. We thus have a Cartesian square in $\mathcal{H}_T$:
\[
\begin{array}{cccc}
\varepsilon(A) & \to & G \circ Q(H_{-1}(A))[1] & \\
\downarrow & & \downarrow & \\
A & \to & H_{-1}(A)[1]. & 
\end{array}
\]

From the construction, \(H_0(\varepsilon(A)) \simeq H_0(A)\) and \(H_{-1}(\varepsilon(A)) \simeq G \circ Q(H_{-1}(A))\). In particular, \(\varepsilon(A) \in \mathcal{B}\) as it follows from Hypothesis 2.8(i).

We claim that \(\varepsilon(A) \to A\) is the universal morphism from an object of \(\mathcal{B}\). Indeed, let \(B\) be an object of \(\mathcal{B}\). With \(A = \varepsilon(A)\), we have a commutative diagram of Abelian groups

\[
\begin{array}{cccc}
0 & \to & \text{hom}_T(B, H_0(\varepsilon(A))) & \to & \text{hom}_T(B, A) & \to & \text{hom}_T(B, H_{-1}(\varepsilon(A))[1]) & \\
\downarrow & & \sim & & \downarrow & & \sim & \\
0 & \to & \text{hom}_T(B, H_0(A)) & \to & \text{hom}_T(B, H_{-1}(A)[1]) & \to & \text{hom}_T(B, H_{-1}(A)) & \\
\end{array}
\]

with exact rows. By the Five Lemma, we are reduced to show that third vertical homomorphism is bijective. The latter can be identified with

\[
\text{hom}_{\mathcal{H}_T}(H_{-1}(B), G \circ Q(H_{-1}(A))) \to \text{hom}_{\mathcal{H}_T}(H_{-1}(B), H_{-1}(A)).
\]

This is a bijection as \(H_{-1}(B)\) is F-connected and in \(\mathcal{B}\). \(\square\)

**Proposition 2.11.** Keep the above notation and assumptions. The category \(\mathcal{B}\) is Abelian.

**Proof.** We split the proof in two parts.

**Part A.** Let \(a : A \to B\) be a morphism in \(\mathcal{B}\). Here we prove that \(\text{coker}(a)\), taken in \(\mathcal{H}_T\) is an object of \(\mathcal{B}\). Denote \(N = \ker(a)\), \(C = \text{im}(a)\) and \(D = \text{coker}(a)\), all taken in \(\mathcal{H}_T\). From the two short exact sequences:

\[
\begin{array}{cccc}
0 & \to & N & \to & A & \to & C & \to & 0 \\
0 & \to & C & \to & B & \to & D & \to & 0 \\
\end{array}
\]

we deduce two exact sequences in \(\mathcal{H}_T\):

\[
\begin{array}{cccc}
H_{-1}(N) & \to & H_{-1}(A) & \to & H_{-1}(C) & \to & 0 \\
H_{-1}(C) & \to & H_{-1}(B) & \to & H_{-1}(D) & \to & 0 \\
\end{array}
\]

which can be put together to get another exact sequence:

\[
H_{-1}(A) \to H_{-1}(B) \to H_{-1}(D) \to 0.
\]

Now, as \(H_{-1}(A)\) and \(H_{-1}(B)\) are in \(\mathcal{B}\), and the inclusion \(\mathcal{B} \hookrightarrow \mathcal{H}_T\) is right exact, we deduce that \(H_{-1}(D) \in \mathcal{B}\). This shows that \(\text{coker}(a) \in \mathcal{B}\).
Part B. The previous part shows that cokernels exist in \( \mathcal{B} \) and can be computed in \( \mathcal{H}_T \). On the other hand, by Lemma 2.10, the kernels also exist in \( \mathcal{B} \). Indeed, if \( a : A \to B \) is a morphism in \( \mathcal{B} \), then its kernel in \( \mathcal{B} \) is \( \ker_{\mathcal{B}}(a) = \mathcal{Q}(\ker(a)) \). It remains to show that images and coimages coincide in \( \mathcal{B} \).

Fix a morphism \( a : A \to B \) in \( \mathcal{B} \). \( \mathcal{H}_T \) being an Abelian category, the canonical morphism

\[
\text{coker}\{\ker(a) \to A\} \to \ker\{B \to \text{coker}(a)\}
\]

is invertible. Applying \( \mathcal{Q} \), we get an isomorphism

\[
\mathcal{Q}\left(\text{coker}\{\ker(a) \to A\}\right) \sim \mathcal{Q}\left(\ker\{B \to \text{coker}(a)\}\right).
\]

\( \mathcal{Q} \) being a left adjoint, it commutes with cokernels. It follows that:

\[
\mathcal{Q}
\left(\text{coker}\{\ker_{\mathcal{B}}(a) \to A\}\right) \simeq \text{coker}\left(\mathcal{Q}(\ker(a)) \to \mathcal{Q}(A)\right) \simeq \text{coker}\{\ker_{\mathcal{B}}(a) \to A\}.
\]

Thus the obvious morphism

\[
\text{coker}\{\ker_{\mathcal{B}}(a) \to A\} \to \ker_{\mathcal{B}}\{B \to \text{coker}(a)\}
\]

is invertible. This finishes the proof of the proposition. \( \square \)

3. The \( n \)-motivic \( t \)-structure for \( n = 0 \) and 1

3.1. The 0-motivic \( t \)-structure

We bring in the notation from Section 1.1. From the introduction, we are led to make the following definition.

**Definition 3.1.** The 0-motivic \( t \)-structure \( (0^\tau_{\geq 0} M(k), 0^\tau_{\leq 0} M(k)) \) on \( \text{DM}_{\text{eff}}(k) \) is the usual homotopy \( t \)-structure. An object in \( 0^\tau_{\geq 0} M(k) \) will be called \( 0^\tau_{\geq 0} M \)-positive. An object in \( 0^\tau_{\leq 0} M(k) \) will be called \( 0^\tau_{\leq 0} M \)-negative. For \( n \in \mathbb{Z} \), we denote \( 0^\tau_{\geq n}, 0^\tau_{\leq n} \) the truncation functors, and \( 0^H_n M(-) = 0^\tau_{\geq n} \circ 0^\tau_{\leq n}(-)[-n] \). We also denote \( 0^H M(k) = 0^T_{\geq 0} M(k) \cap 0^T_{\leq 0} M(k) \), the heart of 0-motivic \( t \)-structure. An object of \( 0^H M(k) \) is a called a \((0, H)\)-sheaf.

Strictly speaking, the category \( 0^H M(k) \) is equivalent (and not isomorphic) to the category \( \mathcal{H}I(k) \) of \( \mathcal{H} \)-sheaves (i.e., homotopy invariant sheaves with transfers). This equivalence takes a \((0, \mathcal{H})\)-sheaf to its zero homology \( \mathcal{H} \)-sheaf. However, it is safe enough to identify both categories, and we will often do this.

**Remark 3.2.** In the sequel, we will keep using the notation \( \tau_{\leq n}, \tau_{\geq n}, H_n \) and \( \mathcal{H}I(k) \) relative to the homotopy \( t \)-structure. In fact, the only reason we introduced the new terminology in Definition 3.1, is to stress the analogy between the 0-motivic, 1-motivic and 2-motivic \( t \)-structures.
**Proposition 3.3.** The $0$-motivic t-structure on $\text{DM}_{\text{eff}}(k)$ is generated by the essentially small class \{$M(X) \mid X \in \text{Sm}/k$\}.

**Proof.** This is well-known (see [8]). For the sake of completeness, we provide an argument. For $K \in \mathcal{K}(\text{Shv}_{tr}(k))$, the following two conditions are equivalent:

(i) The homology sheaves $H_i(K)$ are zero for $i < 0$;
(ii) The (Nisnevich) hyper-cohomology groups $H^{-i}(X, K)$ are zero for $i > 0$ and $X \in \text{Sm}/k$.

If moreover we assume that $K$ is $\mathbb{A}^1$-local, the second condition can be rewritten as follows:

(ii') The groups $\text{hom}_{\text{DM}_{\text{eff}}(k)}(M(X)[i], K)$ are zero for $i > 0$ and $X \in \text{Sm}/k$.

In other words, $K$ is $\{M(X) \mid X \in \text{Sm}/k\}$-negative. This proves the proposition. \qed

**Definition 3.4.** We denote by $M_0(k) \subset H^M(k)$ the full subcategory whose objects are the $0$-presented $\mathcal{H}$-sheaves which we will also call *mixed* $0$-motives (or simply $0$-motive).

In [2], the category $M_0(k)$ is denoted by $H^{\leq 0}(k)$ and their objects were called $0$-motivic sheaves. It is the heart of the restriction of the homotopy $t$-structure on $\text{DM}^{\leq 0}(k)$. By [2, Lem. 1.2.2], $M_0(k)$ is canonically equivalent to the category $\text{Shv}_{tr}(k^{\leq 0})$. The latter is equivalent to the category of $\mathbb{Q}$-linear representations $V$ of the absolute Galois group $\text{Gal}(k^s/k)$ of $k$ such that the stabilizer of each element of $V$ is open (i.e., of finite index). This justifies our terminology.

### 3.2. The $1$-motivic $t$-structure

The subcategory $M_0(k) \subset H^M(k)$ satisfies Hypothesis 2.1. Indeed, (i) and (ii) are contained in [2, Cor. 1.2.5 and Prop. 1.2.7]. To check (iii), we use [2, Cor. 2.3.3]. It asserts that the left adjoint $\pi_0 : H^M \rightarrow M_1(k)$ is induced on the hearts by a $t$-positive triangulated functor $L\pi_0 : \text{DM}_{\text{eff}}(k) \rightarrow \text{DM}_{\leq 0}(k)$. Given an exact sequence of $\mathcal{H}$-sheaves

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we deduce a distinguished triangle

$$L\pi_0(M') \longrightarrow L\pi_0(M) \longrightarrow L\pi_0(M'') \longrightarrow L\pi_0(M')[1],$$

and thus an exact sequence of $0$-motives

$$H_1(L\pi_0(M'')) \longrightarrow \pi_0(M') \longrightarrow \pi_0(M).$$

Now, assume that $M''$ is a $0$-motive. Then $M'' \simeq L\pi_0(M'')$ and $H_1(L\pi_0(M'')) = 0$. This proves that $\pi_0(M') \rightarrow \pi_0(M)$ is injective.

We are now in position to apply the construction from Section 2.1.
Definition 3.5. The 1-motivic t-structure \((1^T_{\geq 0}M(k), 1^T_{\leq 0}M(k))\) on \(DM_{\text{eff}}(k)\) is the \(M_0(k)\)-perverted t-structure associated to the 0-motivic t-structure. An object in \(1^T_{\geq 0}M(k)\) will be called \(1^tM\)-positive. An object in \(1^T_{\leq 0}M(k)\) will be called \(1^tM\)-negative. For \(n \in \mathbb{Z}\), we denote \(1^t_{\geq n}M\) and \(1^t_{\leq n}M\) the truncation functors and \(1^T_{\geq n}M(\cdot) = 1^t_{\geq n}M \circ 1^t_{\leq n}(\cdot)\). We also denote \(1^H_{\geq 0}M = 1^T_{\geq 0}M \cap 1^T_{\leq 0}M\) the heart of 1-motivic t-structure. An object of \(1^H_{\geq 0}M\) is called a \((1, H)\)-sheaf.

Remark 3.6. From the construction, we have the following description of the 1-motivic t-structure.

1. An object \(P \in DM_{\text{eff}}(k)\) is \(1^tM\)-positive if and only if it satisfies:
   a. \(H_n(P) = 0\) for \(n < -1\);
   b. \(H_{-1}(P)\) is a 0-connected \(H\)-sheaf.
2. An object \(N \in DM_{\text{eff}}(k)\) is \(1^tM\)-negative if and only if it satisfies:
   a. \(H_n(N) = 0\) for \(n > 0\);
   b. \(H_0(N)\) is a 0-motive.
3. An object \(M \in DM_{\text{eff}}(k)\) is a \((1, H)\)-sheaf if and only if it satisfies:
   a. \(H_i(M) = 0\) for \(i \notin \{0, -1\}\);
   b. \(H_0(M)\) is a 0-motive;
   c. \(H_{-1}(M)\) is a 0-connected \(H\)-sheaf.

For \(X \in \text{Sm}/k\) we choose a distinguished triangle

\[M_{\geq 1}(X) \longrightarrow M(X) \longrightarrow L\pi_0(M(X)) \longrightarrow M_{\geq 1}(X)[1].\]

From the construction in [2, §2.3], we have \(L\pi_0(M(X)) \simeq M(\pi_0(X))\) where \(\pi_0(X)\) is the étale \(k\)-scheme of connected components of \(X\). It follows that \(M_{\geq 1}(X)\) is isomorphic in \(DM_{\text{eff}}(k)\) to \(\ker[\mathbb{Q}_{tr}(X) \rightarrow \mathbb{Q}_{tr}(\pi_0(X))]\). The following result is a direct consequence of Proposition 2.7.

Proposition 3.7. The 1-motivic t-structure on \(DM_{\text{eff}}(k)\) is generated by the essentially small class \(\{M(X), M_{\geq 1}(X)[-1] \mid X \in \text{Sm}/k\}\).

3.3. Mixed 1-motives

Definition 3.8. An object \(M \in DM_{\text{eff}}(k)\) is called a mixed 1-motive if it satisfies the following conditions:

1. \(H_i(M) = 0\) for \(i \notin \{0, -1\}\);
2. \(H_0(M)\) is a 0-motivic sheaf;
3. \(H_{-1}(M)\) is a 0-connected 1-presented \(H\)-sheaf.

The full subcategory of mixed 1-motives will be denoted by \(M_1(k)\).

Clearly, \(M_1(k)\) is the \(M_0(k)\)-perverted subcategory associated to \(H_{\leq 1}(k) \subset H(k) \simeq 0^H_{\geq 0}M(k)\). In particular, it is Abelian. In fact, we have more as the following result shows.
Proposition 3.9. $M_1(k)$ is a thick Abelian subcategory of $^1\mathcal{H}^M(k)$, i.e., stable under extensions, subobjects and quotients.

Proof. Indeed, consider a short exact sequence of $(1, \mathcal{H})$-sheaves:

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.$$  \hspace{1cm} (8)

It suffices to show that $M$ is a mixed 1-motive if and only if $M'$ and $M''$ are mixed 1-motives. In other words, we need to show that the $\mathcal{H}$-sheaf $H_{-1}(M)$ is 1-presented if and only if the $\mathcal{H}$-sheaves $H_{-1}(M')$ and $H_{-1}(M'')$ are 1-presented.

From (8), we get an exact sequence of $\mathcal{H}$-sheaves:

$$0 \rightarrow \text{im}\{H_0(M) \rightarrow H_0(M'')\} \rightarrow H_{-1}(M') \rightarrow H_{-1}(M) \rightarrow H_{-1}(M'') \rightarrow 0.$$

The $\mathcal{H}$-sheaf $\text{im}\{H_0(M) \rightarrow H_0(M'')\}$ is 0-presented and hence 1-presented. The lemma follows now as $HI_{\leq 1}(k)$ is a thick Abelian subcategory of $HI(k)$ (see [2, Cor. 1.3.5]). □

Lemma 3.10. Let $M$ be a mixed 1-motive. Then, $M$ decomposes into a direct sum

$$M \simeq H_0(M)[0] \oplus H_{-1}(M)[-1].$$

Proof. We have a distinguished triangle in $DM_{\text{eff}}(k)$:

$$H_0(M) \rightarrow M \rightarrow H_{-1}(M)[-1] \rightarrow H_0(M)[1].$$  \hspace{1cm} (9)

We need to show that $\epsilon$ is zero. By [2, Th. 2.4.1(i)], $HI_{\leq 1}(k)$ is contained in $DM_{\leq 1}(k)$, and hence $M \in DM_{\leq 1}(k)$. Also, by [2, Th. 2.4.1(i)], we have an equivalence of categories $D(HI_{\leq 1}(k)) \simeq DM_{\leq 1}(k)$. Thus, we may view (9) as a triangle in the derived category $D(HI_{\leq 1}(k))$ and $\epsilon$ as an element of:

$$\text{hom}_{D(HI_{\leq 1}(k))}(H_{-1}(M)[-1], H_0(M)[1]) \simeq \text{ext}^2_{HI_{\leq 1}(k)}(H_{-1}(M), H_0(M)).$$

On the other hand, the cohomological dimension of $HI_{\leq 1}(k)$ is 1 by [2, Prop. 2.4.10]. This shows that $\text{ext}^2_{HI_{\leq 1}(k)}(H_{-1}(M), H_0(M)) = 0$, and hence $\epsilon = 0$. □

In the reminder of this paragraph, we describe the link between our notion of mixed 1-motives and the classical notion of Deligne’s 1-motives. We do this in order to justify our terminology. However, this material will not be used elsewhere in the paper and can safely be skipped by the reader.

Recall (cf. [4]) that a Deligne 1-motive is a morphism of group-schemes $[L \rightarrow G]$ with $L$ a lattice (i.e., locally for the étale topology isomorphic to $\mathbb{Z}^r$) and $G$ a semi-Abelian variety. We denote by $M_1(k)$ the category of 1-motives. Given two 1-motives $M_1 = [L_1 \rightarrow G_1]$ and $M_2 = [L_2 \rightarrow G_2]$, we have

$$\text{hom}_{M_1}(M_1, M_2) = \{ (a : L_1 \rightarrow L_2, b : G_1 \rightarrow G_2) \mid b \circ u_1 = u_2 \circ a \} \otimes_{\mathbb{Z}} \mathbb{Q}.$$
(Where a and b above are morphisms of group-schemes.) There is a functor
\[ T : \mathcal{M}_1(k) \to \mathbf{D}_\text{eff}(k) \]
which takes a 1-motive \([L \to G]\) to the complex
\[ \cdots \to 0 \to L \otimes \mathbb{Q} \to G \otimes \mathbb{Q} \to 0 \to \cdots \]
where \(L\) and \(G\) are identified with the sheaves they represent and \(L \otimes \mathbb{Q}\) is placed in degree 0.

**Proposition 3.11.** The functor \(T\) induces an exact full embedding of \(\mathcal{M}_1(k)\) into \(\mathbf{M}_1(k)\).

**Proof.** This is a special case of the main result in [9]. For the sake of completeness, we give a proof. Clearly, the image of \(T\) is contained in \(\mathbf{M}_1(k)\). Let \(M_i = [L_i \to G_i]\) (for \(i \in \{1, 2\}\)) be two Deligne 1-motives. We need to show that
\[ \hom_{\mathcal{M}_1}(M_1, M_2) \to \hom_{\mathbf{M}_1(k)}(T(M_1), T(M_2)) \]
(10)
is a bijection. We can decompose \(M_i\) as follows
\[ M_i \simeq L'_i \oplus [L''_i \to G_i] \]
where \(L'_i\) and \(L''_i\) are sub-lattices of \(L_i\) such that:
- \(L'_i \cap L''_i = 0\) and \(L'_i + L''_i\) is of finite index in \(L_i\);
- \(L''_i \to G_i\) is injective.

We are then reduced to check that (10) is bijective in the following cases:

(a) \(G_i\) is zero for \(i \in \{1, 2\}\);
(b) \(G_1\) is zero and \(L_2 \hookrightarrow G_2\) is injective;
(c) \(L_1 \hookrightarrow G_1\) is injective and \(G_2 = 0\);
(d) \(L_i \hookrightarrow G_i\) is injective for \(i \in \{1, 2\}\).

Case (a) is easy. In case (b), both sides of (10) are zero. In case (c), both sides of (10) are canonically isomorphic to \(\hom(L_1, L_2) \otimes \mathbb{Q}\). Finally, in case (d), both sides of (10) are given by the sub-vectorspace of \(e \in \hom(G_1, G_2) \otimes \mathbb{Q}\) such that \(e(L_1 \otimes \mathbb{Q}) \subset e(L_2 \otimes \mathbb{Q})\).

**Remark 3.12.** Using [2, Th. 1.3.10] and Lemma 3.10, it is possible to show that \(\mathbf{M}_1(k)\) is equivalent to the category of ind-objects in \(\mathcal{M}_1(k)\). We leave the details to the reader.

### 3.4. \(n\)-Presented \((1, \mathcal{H})\)-sheaves

**Definition 3.13.** Let \(n \geq 0\) be an integer. A \((1, \mathcal{H})\)-sheaf \(M\) is \(n\)-presented if the \(\mathcal{H}\)-sheaf \(H_{-1}(M)\) is \(n\)-presented. We denote by \(\mathcal{M}_{\leq n}^{\mathcal{H}}(k) \subset \mathcal{H}^{\mathcal{M}}(k)\) the full subcategory of \(n\)-presented \((1, \mathcal{H})\)-sheaves.
Remark 3.14. A \((1, \mathcal{H})\)-sheaf \(M\) is 0-presented if and only if \(H_{-1}(M) = 0\). It follows that \(\mathcal{H}^M_{\leq 0}(k) = M_0(k)\). Also, 1-presented \((1, \mathcal{H})\)-sheaves are exactly the mixed 1-motives, i.e., \(\mathcal{H}^M_{\leq 1}(k) = M_1(k)\).

Clearly, \(\mathcal{H}^M_{\leq n}(k)\) is the \(M_0(k)\)-perverted subcategory of \(\mathcal{H}^M_{\leq n}(k)\) associated to the full subcategory \(\mathcal{H}^M_{\leq n}(k) = HI_{\leq n}(k)\) of \(\mathcal{H}^M_{\leq n}(k)\). It is easy to check Hypothesis 2.8 for the 0-motivic \(t\)-structure with \(A = M_0(k)\) and \(B = HI_{\leq n}(k)\). Indeed, the inclusion \(HI_{\leq n}(k) \hookrightarrow HI(k)\) admits a right adjoint given by \(Q_n = h_0\sigma^*_n\sigma_n\). Moreover, a short exact sequence of \(\mathcal{H}\)-sheaves

\[
0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0
\]

such that \(F''\) is 0-presented has a splitting. This clearly implies that \(Q_n(F') \to Q_n(F)\) is injective. From Lemma 2.10 and Proposition 2.11, we deduce the following result.

Corollary 3.15. \(\mathcal{H}^M_{\leq n}(k)\) is an Abelian category and there is a functor

\[
Q_n : \mathcal{H}^M(k) \longrightarrow \mathcal{H}^M_{\leq n}(k),
\]

which is a right adjoint to the obvious inclusion.

4. The 2-motivic \(t\)-structure

4.1. The construction

By Proposition 3.9, \(M_1(k) \subset \mathcal{H}^M_{\leq n}(k)\) is a thick Abelian subcategory. Thus the first condition in Hypothesis 2.1 is satisfied. We will see in a moment that the two other conditions are satisfied as well. First, we note the following result which is of independent interest.

Proposition 4.1. The 1-motivic \(t\)-structure restricts to a \(t\)-structure on \(DM_{\leq 1}(k)\) whose heart is \(M_1(k)\).

Proof. We know that the homotopy \(t\)-structure restricts to a \(t\)-structure on \(DM_{\leq 1}(k)\) whose heart is \(HI_{\leq 1}(k)\). The subcategory \(M_0(k) \subset HI_{\leq 1}(k)\) satisfies the conditions in Hypothesis 2.1. Thus, we may consider the \(M_0(k)\)-perverted \(t\)-structure on \(DM_{\leq 1}(k)\) associated to the homotopy \(t\)-structure. By a straightforward inspection, we see that the inclusion \(DM_{\leq 1}(k) \hookrightarrow DM_{\text{eff}}(k)\) is exact with respect to the \(M_0(k)\)-perverted \(t\)-structures. This proves the proposition. \(\square\)

Definition 4.2. The restriction of the 1-motivic \(t\)-structure to \(DM_{\leq 1}(k)\) is also called the 1-motivic \(t\)-structure.

Lemma 4.3. The functor \(L\text{Alb} : DM_{\text{eff}}(k) \to DM_{\leq 1}(k)\) is \(1^t^M\)-positive (i.e., \(t\)-positive with respect to the 1-motivic \(t\)-structures).

Proof. This is clear as \(L\text{Alb}\) is the left adjoint to the inclusion \(DM_{\leq 1}(k) \hookrightarrow DM_{\text{eff}}(k)\) which is \(1^t^M\)-exact. \(\square\)
We are now in position to check Hypothesis 2.1(ii) and (iii) for $M_1(k) \subset ^1H^M(k)$.

**Lemma 4.4.** The inclusion $M_1(k) \hookrightarrow ^1H^M(k)$ has a left adjoint

$$\text{Alb}^M : ^1H^M(k) \longrightarrow M_1(k).$$

Moreover, given an exact sequence of $(1, \mathcal{H})$-sheaves

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

with $M''$ a mixed 1-motive, the morphism $\text{Alb}^M(M') \rightarrow \text{Alb}^M(M)$ is injective.

**Proof.** Given a $(1, \mathcal{H})$-sheaf $M$, we set

$$\text{Alb}^M(M) = ^1H^M_0(L\text{Alb}(M)).$$

That this is a left adjoint to the obvious inclusion, follows immediately from Lemma 4.3. To prove the second part, we use the distinguished triangle

$$L\text{Alb}(M') \longrightarrow L\text{Alb}(M) \longrightarrow L\text{Alb}(M'') \longrightarrow L\text{Alb}(M')[1].$$

We deduce an exact sequence of mixed 1-motives

$$^1H^M_1(L\text{Alb}(M'')) \longrightarrow ^1H^M_0(L\text{Alb}(M')) \longrightarrow ^1H^M_0(L\text{Alb}(M)).$$

Now, if $M''$ is a mixed 1-motive, $M'' \simeq L\text{Alb}(M'')$ and thus $^1H^M_1(L\text{Alb}(M'')) = 0$. This finishes the proof of the lemma. \(\square\)

**Definition 4.5.** The 2-motivic $t$-structure $(^2T_{\geq 0}^M(k), ^2T_{\leq 0}^M(k))$ on $\text{DM}_{\text{eff}}(k)$ is the $M_1(k)$-perverted $t$-structure associated to the 1-motivic $t$-structure. An object in $^2T_{\geq 0}^M(k)$ will be called $^2t^M$-positive. An object in $^2T_{\leq 0}^M(k)$ will be called $^2t^M$-negative. For $n \in \mathbb{Z}$, we denote $^2t_n^M$ and $^2t_n^M$ the truncation functors and $^2H_n^M(\cdot) = ^2t_n^M \circ ^2t_n^M(\cdot)[−n]$. We also denote $^2H_n^M(k) = ^2T_{\geq 0}^M(k) \cap ^2T_{\leq 0}^M(k)$ the heart of 2-motivic $t$-structure. An object of $^2H_n^M(k)$ is a called a $(2, \mathcal{H})$-sheaf.

**Remark 4.6.** We will say that a $(1, \mathcal{H})$-sheaf $M$ is 1-connected if $\text{Alb}^M(M) = 0$. From the construction, we have the following.

1. An object $P \in \text{DM}_{\text{eff}}(k)$ is $^2t^M$-positive if and only if it satisfies:
   (a) $^1H^n_n^M(P) = 0$ for $n < −1$;
   (b) $^1H^M_{−1}(P)$ is a 1-connected $(1, \mathcal{H})$-sheaf.
2. An object $N \in \text{DM}_{\text{eff}}(k)$ is $^2t^M$-negative if and only if it satisfies:
   (a) $^1H^n_n^M(N) = 0$ for $n > 0$;
   (b) $^1H^M_0(N)$ is a mixed 1-motive.
(3) An object \( M \in \text{DM}_{\text{eff}}(k) \) is a \((2, \mathcal{H})\)-sheaf if and only if it satisfies:

(a) \( 1^H_M(M) = 0 \) for \( i \notin \{0, -1\} \);
(b) \( 1^H_0(M) \) is a 0-motive;
(c) \( 1^H_{-1}(M) \) is a 0-connected \( \mathcal{H} \)-sheaf.

In the next paragraph, we will give equivalent formulations of the above conditions in terms of the homotopy \( t \)-structure.

For \( X \in \text{Sm}/k \) we choose a distinguished triangle

\[
\begin{array}{ccc}
M \geq 2(X) & \rightarrow & M(X) \\
\downarrow & & \downarrow \\
\text{LAlb}(M(X)) & \rightarrow & M \geq 2(X)[1].
\end{array}
\]

We have the following result.

**Proposition 4.7.** The 2-motivic \( t \)-structure on \( \text{DM}_{\text{eff}}(k) \) is generated by the essentially small class \( \{M(X), M \geq 1(X)[-1], M \geq 2(X)[-2] \mid X \in \text{Sm}/k\} \).

**Proof.** For \( X \in \text{Sm}/k \), we choose a distinguished triangle

\[
\begin{array}{ccc}
M' \geq 2(X) & \rightarrow & M \geq 1(X) \\
\downarrow & & \downarrow \\
\text{LAlb}(M \geq 1(X)) & \rightarrow & M' \geq 2(X)[1].
\end{array}
\]

A direct application of Propositions 2.7 and 3.7 yields the following generating class for the 2-motivic \( t \)-structure:

\[
\{M(X), M \geq 1(X)[-1], M \geq 2(X)[-1], M' \geq 2(X)[-2] \mid X \in \text{Sm}/k\}.
\]

The proposition would follow if we prove that \( M' \geq 2(X) \simeq M \geq 2(X) \). There is a morphism of distinguished triangles

\[
\begin{array}{ccc}
M' \geq 2(X) & \rightarrow & M \geq 1(X) \\
\downarrow & & \downarrow \\
M \geq 2(X) & \rightarrow & \text{LAlb}(M(X)) \\
\downarrow & & \downarrow \\
M(X) & \rightarrow & \text{LAlb}(M(X)) \\
\downarrow & & \downarrow \\
\text{LAlb}(M(X)) & \rightarrow & M \geq 2(X)[1].
\end{array}
\]

By the Verdier’s octahedral axiom (and the fact that \( \text{LAlb} \) is a triangulated functor), it suffices to show that

\[
\text{Cone}\{M \geq 1(X) \rightarrow M(X)\} \simeq \text{LAlb}(\text{Cone}\{M \geq 1(X) \rightarrow M(X)\}).
\]

This is indeed the case as \( \text{Cone}\{M \geq 1(X) \rightarrow M(X)\} \simeq \text{L}\pi_0(\text{M}(X)) \in \text{DM}_{\leq 0}(k) \subset \text{DM}_{\leq 1}(k) \).

The proposition is proven. \( \square \)
4.2. A more explicit description

The description of the 2-motivic $t$-structure in term of the 1-motivic $t$-structure given in Remark 4.6 is rather abstract and not intuitive. Here we give a more down-to-earth description which only uses the homotopy $t$-structure.

**Proposition 4.8.**

1. An object $P \in \text{DM}_{\text{eff}}(k)$ is $2t$-positive if and only if it satisfies:
   - (a') $H_i(P) = 0$ for $i < -2$;
   - (b') $H_{-2}(P)$ is a 1-connected $\mathcal{H}$-sheaf;
   - (c') For every 0-motive $L$, $\text{hom}_{\text{DM}_{\text{eff}}(k)}(P, L[-1]) = 0$.

2. An object $N$ in $\text{DM}_{\text{eff}}(k)$ is $2t$-negative if and only if it satisfies:
   - (a') $H_i(N) = 0$ for $i > 0$;
   - (b') $H_0(N)$ is a 0-presented $\mathcal{H}$-sheaf;
   - (c') $H_{-1}(N)$ is a 1-presented $\mathcal{H}$-sheaf.

**Proof.** We will compare the conditions of the statement with those in Remark 4.6. We split the proof into four parts.

**Part A.** Let $P \in \text{DM}_{\text{eff}}(k)$ be a $2t$-positive object. We will show that $P$ satisfies conditions (1a'), (1b') and (1c').

We have a chain of inclusions $2T^M_{\geq 0}(k) \subset 1T^M_{\geq -1}(k) \subset 0T^M_{\geq -2}(k)$. It follows that $H_i(P) = 0$ for $i < -2$. This is condition (1a') of the statement.

By Remark 4.6(1b), the $(1, \mathcal{H})$-sheaf $1H^M_{-1}(P)$ is 1-connected. As $1H^M_{-1}(\text{LAlb}(P)) \simeq \text{Alb}(1H^M_{-1}(P)) = 0$, we deduce that $\text{LAlb}(P)$ is $1t$-positive. This implies that $\text{LAlb}(P)[1]$ is $t$-positive. It follows that $\text{Alb}(H_{-2}(P)) \simeq H_{-2}(\text{LAlb}(P)) = 0$.

Hence $H_{-2}(P)$ is a 1-connected $\mathcal{H}$-sheaf. This is condition (1b') of the statement.

Condition (1c') of the statement is clear as $P$ is $2t$-positive and $L[-1]$ is strictly $2t$-negative.

**Part B.** Let $P \in \text{DM}_{\text{eff}}(k)$ satisfying conditions (1a'), (1b') and (1c'). We will show that $P$ is $2t$-positive.

The $\mathcal{H}$-sheaf $H_{-2}(P)$ being 1-connected, is also 0-connected. It follows that $P \in 1T^M_{\geq -1}(k)$, i.e., $1H^M_i(P) = 0$ for $i < -1$. It remains to show that the $(1, \mathcal{H})$-sheaf $1H^M_{-1}(P)$ is 1-connected.

Consider $\text{LAlb}(P)$. We have $H_{-2}(\text{LAlb}(P)) \simeq \text{Alb}(H_{-2}(P)) = 0$ by condition (1b'). It follows that $\text{LAlb}(P)[1]$ is $t$-positive. On the other hand, we claim that $H_{-1}(\text{LAlb}(P))$ is a 0-connected $\mathcal{H}$-sheaf. Indeed, given any 0-motive $L$, we have

$$\text{hom}_{\mathcal{H}(k)}(\pi_0(\text{LAlb}(P)), L) \simeq \text{hom}_{\mathcal{H}(k)}(H_{-1}(\text{LAlb}(P)), L) \simeq \text{hom}_{\text{DM}_{\text{eff}}(k)}(\text{LAlb}(P), L[-1]) \simeq \text{hom}_{\text{DM}_{\text{eff}}(k)}(P, L[-1]) = 0.$$
By Yoneda’s Lemma, this shows that $\pi_0(H_{-1}(\text{Alb}(P))) = 0$. Thus, we have proven that $\text{Alb}(P) \in T^M_{\geq 0}(k)$. But, we have

$$\text{Alb}^M(1H_{-1}^M(P)) \simeq 1H_{-1}^M(\text{Alb}(P)) = 0.$$ 

This proves that $P$ satisfies condition (1b) of Remark 4.6.

**Part C.** Let $N \in \mathbf{DM}_{\text{eff}}(k)$ be a $2^t\mathcal{M}$-negative object. We will show that $N$ satisfies conditions (2a′), (2b′) and (2c′).

We have a chain of inclusions $2^T_{<0}(k) \subset 1^T_{<0}(k) \subset 0^T_{<0}(k)$. It follows that $H_i(N) = 0$ for $i > 0$. This is condition (2a′) of the statement.

We have $H_0(N) \simeq H_0(1H^M_0(N))$, and the latter is 0-presented because $1H^M_0(N)$ is a mixed 1-motive by Remark 4.6(2b). This is condition (2b′) of the statement.

We clearly have $H_{-1}(N) \simeq H_{-1}(1\tau_{\geq -1}(N))$. Using the distinguished triangle

$$1H^M_0(N) \longrightarrow 1\tau^M_{\geq -1}(N) \longrightarrow 1H^M_{-1}(N)[-1] \longrightarrow 1H^M_0(N)[1],$$

we deduce an exact sequence of $\mathcal{H}$-sheaves

$$0 \longrightarrow H_{-1}(1H^M_0(N)) \longrightarrow H_{-1}(N) \longrightarrow H_0(1H^M_{-1}(N)). \tag{11}$$

This shows that the $\mathcal{H}$-sheaf $H_{-1}(N)$ is 1-presented. This is condition (2c′) of the statement.

**Part D.** Let $N \in \mathbf{DM}_{\text{eff}}(k)$ satisfying conditions (2a′), (2b′) and (2c′). Then $N$ is $2^t\mathcal{M}$-negative. Indeed, $N \in \mathcal{T}^M_{\geq 0}(k)$. On the other hand, from (11), we deduce that $1H^M_0(N)$ is a mixed 1-motive. This finishes the proof of the proposition. □

The next lemma shows that we may replace condition (1c′) of Proposition 4.8 by a more concrete condition.

**Lemma 4.9.** Let $P \in \mathbf{DM}_{\text{eff}}(k)$ be an object satisfying (1a′) and (1b′) of Proposition 4.8. Then the following conditions are equivalent:

(c′) For every 0-motive $L$, $\text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(P, L[-1]) = 0$,

(c′′) For every 0-motive $L$, $\text{ext}^1_{\mathcal{H}(k)}(H_{-2}(P), L) = 0$ and $L[-1]$ is not a direct summand of $P$ unless $L = 0$.

**Proof.** First, assume that $P$ satisfies condition (c′). Then clearly, $L[-1]$ cannot be a direct summand of $P$ unless $L = 0$.

On the other hand, from the distinguished triangle

$$\tau_{\geq -1}(P)[1] \longrightarrow P[1] \longrightarrow H_{-2}(P)[-1] \longrightarrow \tau_{\geq -1}(P)[2]. \tag{12}$$

we deduce an exact sequence

$$\text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(\tau_{\geq -1}(P)[2], L) \longrightarrow \text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(H_{-2}(P)[-1], L) \longrightarrow \text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(P[1], L).$$
Both extremal terms are zero; the left one is zero for degree reasons and the right one is zero by assumption. We are done as the middle term is isomorphic to $\text{ext}_H^1(H_{-2}(P), L)$. Indeed, both groups classify $H$-sheaves which are extensions of $H_{-2}(P)$ by $L$.

Conversely, assume that $P$ satisfies condition (c’’). We argue by contradiction. Thus, let $\alpha : P[1] \to L$ be a non-zero morphism. As $L$ is a direct sum of indecomposable 0-motives, we may assume that $L$ is itself indecomposable.

By the long exact sequence associated to the distinguished triangle (12), the obvious homomorphism

$$\text{hom}_{\text{DM}_\text{eff}(k)}(P[1], L) \to \text{hom}_{\text{DM}_\text{eff}(k)}(\tau_{\geq -1}(P)[1], L)$$

is injective. (Here again, we use that $\text{hom}_{\text{DM}_\text{eff}(k)}(\tau_{\geq -1}(P)[2], L)$ is zero for degree reasons.) In other words, the composition

$$\alpha' : \tau_{\geq -1}(P)[1] \to P[1] \xrightarrow{\alpha} L$$

is also non-zero. Moreover, $\alpha'$ uniquely factors through $\pi_0(H_{-1}(P))$ yielding a commutative diagram

$$
\begin{array}{ccc}
\tau_{\geq -1}(P)[1] & \xrightarrow{\alpha'} & P[1] \\
\downarrow \alpha & & \downarrow \alpha \\
H_{-1}(P) & \xrightarrow{\pi_0(H_{-1}(P))} & L.
\end{array}
$$

As $\alpha''$ is non-zero and $L$ is indecomposable, we deduce that $\alpha''$ is surjective. It follows that $H_{-1}(P) \to L$ is also surjective. Hence, given a distinguished triangle

$$Q \to \tau_{\geq -1}(P)[1] \xrightarrow{\alpha'} L \to Q[1],$$

the object $Q$ is $t$-positive and thus the morphism $L \to Q[1]$ is zero. This shows that $\alpha'$ has a section $\beta' : L \to \tau_{\geq -1}(P)[1]$. Clearly, the composition

$$\beta : L \xrightarrow{\beta'} \tau_{\geq -1}(P)[1] \to P[1]$$

is a section to $\alpha$. We have proven that $L$ is a non-trivial direct summand of $P[1]$ which is a 0-motive. This contradicts (c’’).

\begin{corollary}
An object $M \in \text{DM}_\text{eff}(k)$ is a $(2, H)$-sheaf if and only if it satisfies:

\begin{enumerate}
\item[(a')] $H_i(M) = 0$ for $i \notin \{0, -1, -2\}$;
\item[(b')] $H_0(M)$ is a 0-presented $H$-sheaf;
\item[(c')] $H_{-1}(M)$ is a 1-presented $H$-sheaf;
\item[(d')] $H_{-2}(M)$ is a 1-connected $H$-sheaf;
\end{enumerate}
\end{corollary}
(e′) \( M[1] \) does not have any non-trivial direct summand which is a 0-motive;

(f′) If \( L \) is a 0-presented \( \mathcal{H} \)-sheaf, then \( \text{ext}^1_{\mathcal{H}(k)}(H_2(M), L) = 0 \), i.e., every extension of \( H_2(M) \) by \( L \) splits.

4.3. \( n \)-Presented \( (2, \mathcal{H}) \)-sheaves and mixed 2-motives

Proposition 4.11. Let \( n \geq 1 \) be an integer. Then Hypothesis 2.8 is satisfied for the 1-motivic \( t \)-structure with \( \mathcal{A} = M_1(k) \) and \( \mathcal{B} = \mathcal{H}^{\leq n}(k) \).

Proof. By Corollary 3.15, we have a right adjoint \( 1Q_n : \mathcal{H}^1(k) \to \mathcal{H}^{\leq n}(k) \) to the obvious inclusion. Next, consider a short exact sequence of \( (1, \mathcal{H}) \)-sheaves

\[
0 \to M' \to M \to M'' \to 0,
\]

where \( M'' \) is a mixed 1-motive. We need to show that \( 1Q_n(M') \to 1Q_n(M) \) is injective. This is easily seen to be equivalent to the condition that

\[
U = \text{ker}\{1Q_n(M) \to M''\}
\]

is \( n \)-presented. Consider the exact sequence of \( \mathcal{H} \)-sheaves

\[
0 \to \text{coker}\{H_0(M) \to H_0(M'')\} \to H_{-1}(U) \to H_{-1}(1Q_n(M)) \to H_{-1}(M'') \to 0.
\]

By Lemma 4.12 below, \( \text{ker}[H_{-1}(1Q_n(M)) \to H_{-1}(M'')] \) is \( n \)-presented. It follows from [2, Lem. 1.1.22] that \( H_{-1}(U) \) is also \( n \)-presented. This implies that \( U \) is an \( n \)-presented \( (1, \mathcal{H}) \)-sheaf. \( \Box \)

Lemma 4.12. Consider a short exact sequence of \( \mathcal{H} \)-sheaves:

\[
0 \to F' \to F \to F'' \to 0.
\]

Assume that \( F \) is \( n \)-presented (with \( n \geq 1 \)) and \( F'' \) is 1-presented. Then \( F' \) is \( n \)-presented.

Proof. Clearly, we can write \( F \) as a filtered colimit as follows:

\[
F = \text{colim}_{i \in I} F_i, \quad \text{where } F_i = \text{coker}\{\alpha_i : h_0(Y_i) \to h_0(X_i)\}.
\]

Above, \( I \) is a filtered ordered set, \( Y_i \) and \( X_i \) are smooth \( k \)-schemes of dimension at most \( n \), and \( \alpha_i \) is a morphism of sheaves. Let \( F_i'' = \text{im}\{F_i \to F''\} \) and \( F_i' = \text{ker}\{F_i \to F''\} \). Then \( F' = \text{colim}_{i \in I} F_i' \) (use that filtered colimits are exact) and we have short exact sequences

\[
0 \to F_i' \to F_i \to F_i'' \to 0.
\]
Clearly, it suffices to show that each $F'_i$ is $n$-presented. As $F''$ is a subsheaf of $F''$, it is 1-presented. Hence, the exact sequence (13) satisfies to the conditions of the statement. In other words, we may assume that

$$F = \text{coker}\{ \alpha : h_0(Y) \to h_0(X) \}$$

with $X$ and $Y$ of dimension at most $n$.

Given a smooth $k$-variety $V$, we denote by $h_0^{\geq 2}(V) = \ker\{ h_0(V) \to \text{Alb}(V) \}$. We also set

$$F^{\geq 2} = \text{coker}\{ h_0^{\geq 2}(Y) \to h_0^{\geq 2}(X) \}.$$

As $F''$ is 1-presented and $F^{\geq 2}$ is 1-connected, the composition $F^{\geq 2} \to F \to F''$ is zero. It follows that $F^{\geq 2} \to F$ factors through $F'$, yielding a morphism

$$\beta : F^{\geq 2} \to F'.$$

The kernel and cokernel of $\beta$ are 1-presented $\mathcal{H}$-sheaves. Indeed, $\ker(\beta)$ is a subquotient of $\text{Alb}(Y)$ and $\text{coker}(\beta)$ is a subquotient of $\text{Alb}(X)$. Using [2, Lem. 1.1.22], we are reduced to show that $F^{\geq 2}$ is $n$-presented. By a second application of [2, Lem. 1.1.22], we are further reduced to check that $h_0^{\geq 2}(V)$ is $n$-presented for $V$ a $k$-smooth scheme of dimension at most $n$.

One can find a smooth curve $C \subset V$ such that the composition

$$h_0(C) \to h_0(V) \to \text{Alb}(V)$$

is surjective. Let $E = \text{coker}\{ h_0(C) \to h_0(V) \}$. This is an $n$-presented $\mathcal{H}$-sheaf. The morphism $\gamma : h_0^{\geq 2}(V) \to E$ is clearly surjective and its kernel is a subquotient of $h_0(C)$, and hence is 1-presented. We use again [2, Lem. 1.1.22] to conclude. □

**Definition 4.13.** Let $n \geq 1$ be an integer. We denote by $\mathcal{H}^M_{\leq n}(k) \subset \mathcal{H}^M(k)$ the $\mathcal{M}_1(k)$-perverted subcategory associated to $\mathcal{H}_{\leq n}^M(k) \subset \mathcal{H}^M(k)$. A $(2, \mathcal{H})$-sheaf which is in $\mathcal{H}^M_{\leq n}(k)$ is called $n$-presented.

Clearly, a 1-presented $(2, \mathcal{H})$-sheaf is simply a mixed 1-motive, i.e., $\mathcal{H}^M_{\leq 1}(k) = \mathcal{M}_1(k)$. By convention, a 0-presented $(2, \mathcal{H})$-sheaf is a 0-motive and we set $\mathcal{H}^M_{\leq 0}(k) = \mathcal{M}_0(k)$.

**Lemma 4.14.** Let $n \geq 1$ be an integer. A $(2, \mathcal{H})$-sheaf $M$ is $n$-presented if and only if the $\mathcal{H}$-sheaf $H_{-2}(M)$ is $n$-presented.

**Proof.** Indeed, $M$ is $n$-presented if and only if the $(1, \mathcal{H})$-sheaf $H_{-1}^M(M)$ is $n$-presented which is equivalent to $H_{-1}^M(M) \simeq H_{-2}(M)$ being an $n$-presented $\mathcal{H}$-sheaf. □

**Proposition 4.15.** $\mathcal{H}^M_{\leq n}(k)$ is an Abelian category and there is a functor

$$\mathcal{Q}_n : \mathcal{H}^M(k) \to \mathcal{H}^M_{\leq n}(k),$$
which is a right adjoint to the obvious inclusion.

We now come to our definition of mixed 2-motives.

**Definition 4.16.** An object $M \in DM_{\text{eff}}(k)$ is a mixed 2-motive if it satisfies the following conditions:

(a) $H_i(M) = 0$ for $i \notin \{0, -1, -2\}$;
(b) $H_0(M)$ is a 0-presented $\mathcal{H}$-sheaf;
(c) $H_{-1}(M)$ is a 1-presented $\mathcal{H}$-sheaf;
(d) $H_{-2}(M)$ is a 1-connected and 2-presented $\mathcal{H}$-sheaf;
(e) $M[1]$ does not have any non-trivial summand which is a 0-motive;
(f) If $L$ is a 0-presented $\mathcal{H}$-sheaf, then $\text{ext}^1_{\text{Hilb}(k)}(H_2(M), L) = 0$, i.e., every extension of $H_{-2}(M)$ by $L$ splits.

We denote by $M_2(k)$ the full subcategory of mixed 2-motives.

Obviously, mixed 2-motives are exactly the 2-presented $(2, \mathcal{H})$-sheaves, i.e., $M_2(k) = 2\mathcal{H}_{\leq 2}(k)$. In particular, $M_2(k)$ is an Abelian category.

**4.4. Mixed 2-motives associated to surfaces**

Ideally, we should have the following.

**Conjecture 4.17.** Let $S$ be a $k$-surface (possibly singular). Then $2H_i^M(M(S))$ is a mixed 2-motive for all $i \in \mathbb{Z}$.

In fact, $2H_i^M(M(S))$ is expected to vanish for $i \notin [0, 4]$ and more precisely, whenever the $\ell$-adic homology group $H_i^\ell(S \otimes k^3, Q_\ell)$ vanishes (here, $k^3$ is a separable closure of the base field $k$ and $\ell$ is a prime which is invertible in $k$). Unfortunately, Conjecture 4.17 seems out of reach of the actual techniques. However, it is possible to attach to $S$ a sequence $\mathcal{H}_i^M(S)$ of mixed 2-motives which hopefully coincide with those in Conjecture 4.17.

For simplicity, we assume that $S$ is smooth and, except for the next result, we consider the cases where $S$ is affine or $S$ is projective.

**Lemma 4.18.** Let $S$ be a smooth surface. Then $2H_i^M(M(S))$ is a mixed $n$-motive for $n \in \{0, 1, 2\}$.

**Proof.** Consider the distinguished triangle

$$ M_{\geq 1}(S) \longrightarrow M(S) \longrightarrow M(\pi_0(S)) \longrightarrow M_{\geq 1}(S)[1]. $$

Clearly, $M(\pi_0(S))$ is $2\mathcal{H}^M$-negative, whereas $M_{\geq 1}(S)$ is strictly $2\mathcal{H}^M$-positive. This shows that $2H_0^M(M(S)) \simeq M(\pi_0(S))$ is a 0-motive.

Similarly, consider the distinguished triangle

$$ \tilde{M}_{\geq 2}(S)[-1] \longrightarrow M_{\geq 1}(S)[-1] \longrightarrow (\text{Alb}^0(S) \otimes \mathbb{Q})[-1] \longrightarrow \tilde{M}_{\geq 2}(S). $$
with \( \text{Alb}^0(S) \) the connected component of the Albanese scheme of \( S \). Clearly, \((\text{Alb}^0(S) \otimes \mathbb{Q})[-1]\) is \(2^\ell M\)-negative, whereas \( M_{\geq 2}[-1] \) is strictly \(2^\ell M\)-positive. This shows that \( 2H^1_M(M(S)) \cong (\text{Alb}^0(S) \otimes \mathbb{Q})[-1] \) is a mixed 1-motive.

Finally, to prove that \( 2H^2_M(M(S)) \) is a mixed 2-motives, it suffices to show that \( \ker(h_0(S) \to \text{Alb}(S)) \). We conclude using Lemma 4.12.

Now, assume that \( S \) is affine. Then \( 2H^i_M(M(S)) \) are expected to be zero for \( i \in \{0, 1, 2\} \). Thus, we can make the following definition.

**Definition 4.19.** For \( i \in \{0, 2\} \), we set \( \,^2H^i_M(S) = 2H^i_M(M(S)). \) These are the (possibly non-zero) mixed 2-motives associated to \( S \).

Next, assume that \( S \) is projective. It is classical that the Chow motive of \( S \) admits a Künneth decomposition (see for example [6]). As the category of Chow motives is embedded into \( \text{DM}_{\text{eff}}(k) \), we deduce a decomposition \( M(S) = \bigoplus_{i=0}^{4} M_i(S)[i] \) such that \( M_i(S) \) corresponds under the \( \ell \)-adic realization to \( H^i(S \otimes_k k^\ell, \mathbb{Q}^\ell) \). We know that \( M_0(S) = M(\pi_0(S)) \) and that

\[
M_4(S) = \text{Hom}(M_0(S), \mathbb{Q}(2)) \cong M_0(S)^{\vee}(2),
\]

where \( M_0(S)^{\vee} = \text{Hom}(M_0(S), \mathbb{Q}(0)) \) is the dual 0-motive to \( M_0(S) \). Also, \( M_1(S) \) is a pure 1-motive given by the complex \((\text{Alb}^0(S) \otimes \mathbb{Q})[-1] \) with \( \text{Alb}^0(S) \) the connected component of the Albanese scheme of \( S \). Moreover,

\[
M_3(S) = \text{Hom}(M_1(S), \mathbb{Q}(2)) \cong M_1(S)^{\vee}(1),
\]

with \( M_1(S)^{\vee} = \text{Hom}(M_1(S), \mathbb{Q}(1)) \), the Cartier dual of the 1-motive \( M_1(S) \).

**Lemma 4.20.** \( M_0(S) \) is a 0-motive and \( M_1(S) \) is a 1-motive. Moreover, for \( i \in \{2, 3, 4\} \), \( 2^\ell M_i(M(S)) \) is a mixed 2-motive.

**Proof.** The first statement is obvious. To prove the second statement, it suffices to show that for \( i \in \{2, 3, 4\} \), the \( H \)-sheaf

\[
H_{-2}(2^\ell M_i(M(S))) \cong H_{-2}(M_i(S))
\]

is 2-presented. By an easy inspection, we see that:

\[
H_{-2}(M_2(S)) = \ker\{h_0(S) \to \text{Alb}(S)\}, \quad H_{-2}(M_3(S)) = K^M_1 \otimes \text{Alb}^0(S)^{\vee}
\]

and \( H_{-2}(M_4(S)) = K^M_2 \otimes \mathbb{Q}_{tr}(\pi_0(S))^{\vee}. \)
In fact, we expect more generally that the \( n \)-th Milnor \( K \)-theory sheaf and the tensor product is taken in \( \text{HI}(k) \). This proves the lemma. \( \square \)

**Definition 4.21.** Under the above hypothesis, we set \( \mathcal{H}_i^M(S) = 2^i \mathcal{H}_0^M(\text{M}_i(S)) \) for \( i \in \mathbb{Z} \). These are the (possibly non-zero) mixed 2-motives attached to \( S \).

**Remark 4.22.** Let us consider the simplest projective surface, i.e., the projective plane \( \mathbb{P}_k^2 \). We have \( \mathcal{M}(\mathbb{P}_k^2) = \mathbb{Z}(0) \oplus \mathbb{Z}(1)[2] \oplus \mathbb{Z}(2)[4] \). Then clearly, \( \mathcal{H}_i^M(\mathbb{P}_k^2) = 0 \) for odd \( i \in [0, 4] \). Also, \( \mathcal{H}_0^M(\mathbb{P}_k^2) = \mathbb{Z}(0) \) and \( \mathcal{H}_2^M(\mathbb{P}_k^2) = \mathbb{Z}(1) \). For \( i = 4 \), we would like to write: “\( \mathcal{H}_4^M(\mathbb{P}_k^2) = \mathbb{Z}(2) \)”, but unfortunately we can’t. Indeed, this requires to prove that \( \mathbb{Z}(2) \) is a mixed 2-motive and, in particular, that the following properties hold true:

1. \( \mathcal{H}_i(\mathbb{Z}(2)) = 0 \) for \( i > 0 \),
2. \( \mathcal{H}_0(\mathbb{Z}(2)) \) is a 0-presented \( \mathcal{H} \)-sheaf,
3. \( \mathcal{H}_{-1}(\mathbb{Z}(2)) \) is a 1-presented \( \mathcal{H} \)-sheaf.

In fact, these properties are sufficient for showing that \( \mathbb{Z}(2) \) is a mixed 2-motive as it follows easily from Definition 4.16. Note also that (1) is a reformulation of the Beilinson–Soulé vanishing conjecture for the motivic cohomology groups \( \mathcal{H}^n(\mathbb{Z}(2)) \) with \( n < 0 \).

### 4.5. Beyond the case \( n = 2 \)

From what we have learned in this paper, it is natural to expect that the 3-motivic \( t \)-structure \( (\mathcal{T}_{\geq 0}^{\mathcal{M}}(k), \mathcal{T}_{\leq 0}^{\mathcal{M}}(k)) \) is the \( \mathcal{M}_2(k) \)-perverted \( t \)-structure associated to the 2-motivic \( t \)-structure. In fact, we expect more generally that the \( n \)-motivic \( t \)-structure is obtained by perverting the \( (n - 1) \)-motivic \( t \)-structure with respect to a well-chosen category \( \mathcal{M}_{n-1}(k) \) of mixed \( (n - 1) \)-motives, and this for all \( n \in \mathbb{N} \). Unfortunately, even for \( n = 3 \), we have to assume some outstanding conjectures to ensure that \( \mathcal{M}_3(k) \) satisfies the conditions in Hypothesis 2.1 which would enable us to pervert the 2-motivic \( t \)-structure as we did in case \( n = 1 \) and \( n = 2 \).

For instance, we need to know that \( \mathcal{M}_2(k) \) is a thick Abelian subcategory of \( \mathcal{H}^{\mathcal{M}}(k) \). However, to prove this along the lines of Proposition 3.9, it seems necessary to assume that \( \text{HI}_{\leq 2}(k) \) is a thick Abelian subcategory of \( \text{HI}(k) \). This is conjecturally true by [2, Cor. 1.4.5] (which relies on Conj. 1.4.1 of [2]). Also, if we want to construct a left adjoint to the inclusion \( \mathcal{M}_2(k) \subset 2^1 \mathcal{H}^{\mathcal{M}}(k) \), it is certainly useful to have at our disposal a left adjoint to the inclusion \( \text{HI}_{\leq 2}(k) \subset \text{HI}(k) \). Again, such an adjoint exists by [2, Prop. 1.4.6] assuming Conj. 1.4.1 of [2]. (It is worth noting here that a similar left adjoint on the level of triangulated categories does not exist by [2, §2.5] as was claimed by Voevodsky in [10, §3.4].)

In any case, it is an interesting problem to give a conditional construction of all the \( n \)-motivic \( t \)-structures using Conj. 1.4.1 of [2]. We will not pursue this goal in this paper.

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I wish to thank Luca Barbieri-Viale and Bruno Kahn for their encouragement and interest in this work. An incomplete definition of mixed 2-motives, very similar to the one proposed in this paper, appeared to me sometimes ago. However, I never dared to take it seriously. Especially, the problem of showing that these mixed 2-motives form an Abelian category seemed, at a first sight, out of reach. After I shared my definition with Luca, he kept telling me that I should try to do
something with it, till one evening (during the “Algebraic K-Theory and Motivic Cohomology” workshop in Oberwolfach (June 28th–July 4th, 2009)), I decided to follow his advise. I then realized that some progress was possible, and this paper was conceived.

References