ELSEVIER Journal of Pure and Applied Algebra 105 (1995) 277–291

JOURNAL OF PURE AND APPLIED ALGEBRA

Limits in free coproduct completions

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Communicated by G.M. Kelly; received 19 January 1994; revised 6 February 1994

Abstract

For a small category $\boldsymbol{C}$ with multilimits for finite diagrams, a conceptual description of its free coproduct completion $\sum(\boldsymbol{C})$ is given as the category of those set-valued functors of a finitely accessible category with connected limits which preserve these limits and filtered colimits. In this way we recognize the free coproduct completion as a finitely complete category and show that $\sum(\boldsymbol{C})$ is universal with respect to existence of finite limits and of small coproducts which are disjoint and stable under pullback.

1991 Math. Subj. Class.: 18A35, 18B40, 18C99, 03630

0. Introduction

In recent years there has been considerable interest in distributive categories (see, for example, [8, 20, 26, 28]). The paper [6] by Carboni et al. gives a good overview of the various approaches and analyses in particular the properties of disjointness and pullback-stability of finite coproducts (see also [4]). They point out that (finite) coproducts in the free completion of a category under (finite) coproducts have the said properties, including the existence of very particular limits: pullbacks along coproduct injections.

In this paper we consider the free completion $\sum(\boldsymbol{C})$ of a small category $\boldsymbol{C}$ under all small coproducts (which has enjoyed recent attention too, see [7, 19]) and solve the following problems:

(1) When does $\sum(\boldsymbol{C})$ have all finite limits?

(2) For finitely complete $\sum(\boldsymbol{C})$, when does the coproduct-preserving extension $F_!:\sum(\boldsymbol{C}) \to \boldsymbol{B}$ of a functor $F:\boldsymbol{C} \to \boldsymbol{B}$ into a category $\boldsymbol{B}$ with coproducts preserve these finite limits?

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SSDI 0022-4049(94)00153-7

(3) Is there a conceptual description of $\sum(\boldsymbol{C})$ as a "double-dual", in the spirit of [13]?

The quite surprising answer to (3) is that for a familially finitely complete category $\boldsymbol{C}$ (so that $\boldsymbol{C}$ has multilimits in the sense of Diers [9] for all finite diagrams), the free coproduct completion $\sum(\boldsymbol{C})$ is equivalent to the category

$$ConnFilt(\boldsymbol{C}^*, \mathbf{Set})$$

of functors $\boldsymbol{C}^* \to \mathbf{Set}$ that preserve all small connected limits and filtered colimits, with $\boldsymbol{C}^* = Flat(\boldsymbol{C})$ the category of flat functors $\boldsymbol{C} \to \mathbf{Set}$ (see Theorem 2.4). The categories of type $\boldsymbol{C}^*$ are known to be finitely accessible (see [23]), and with $\boldsymbol{C}$ familially finitely complete, they are exactly the finitely accessible categories with small connected limits or, equivalently, the locally finitely multi-presentable categories in the sense of Diers [10] (see Theorem 1.2). Limit–colimit commutation in **Set** enables us to show that $ConnFilt(\boldsymbol{C}^*, \mathbf{Set})$ is a category with finite limits and all small coproducts. More precisely, every object in this category is coproduct of coprime objects (i.e., of objects whose representables preserve coproducts; see [5]), and these are exactly the objects of $\boldsymbol{C}$ (when embedded into $ConnFilt(\boldsymbol{C}^*, \mathbf{Set})$).

In particular, in showing (3) we also obtain a complete answer to question (1), since the sufficient condition of familial finite completeness of $\boldsymbol{C}$ is easily seen to be also necessary for the finite completeness of $\sum(\boldsymbol{C})$.

For the answer to problem (2), disjointness and pullbacks stability of coproducts turn out to be the needed characteristic properties. More precisely, coproducts in $\boldsymbol{B}$ need to satisfy these properties in order for us to show that every functor $F: \boldsymbol{C} \to \boldsymbol{B}$ merging multilimits of finite diagrams has a left Kan extension $F_!: \sum(\boldsymbol{C}) \to \boldsymbol{B}$ which preserves finite limits. The proof is quite tedious.

In all of the above, "finite" may be traded for "less than κ", with any infinite regular cardinal number κ. We can then summarize our answer to (1) and (2) as in Theorem 3.6, which describes $\sum(\boldsymbol{C})$ as the free *κ-∞-lextensive* completion of the small familially κ-multicomplete category $\boldsymbol{C}$. Here we extend a terminology used in [6] where lextensive (lex-extensive) categories are described as categories with finite limits and finite coproducts which are disjoint and stable under pullback; we have traded "finite limits" for "κ-limits", "finite coproducts" for "small coproducts", and "pullback" for "κ-wide pullback". Stability of coproducts under κ-wide pullbacks seems to be a new notion which entails an infinite distributive law for product–coproduct commutation in categories. For lattices it amounts to complete distributivity (when generalized over all κ).

1. Accessible categories with connected limits

Let κ be an infinite regular cardinal. Recall from [11] that an object A of a category $\boldsymbol{A}$ is *κ-presentable* if the representable functor $\boldsymbol{A}(A,-): \boldsymbol{A} \to \mathbf{Set}$ preserves κ-filtered colimits. $\boldsymbol{A}$ is *κ-accessible* if $\boldsymbol{A}$ has κ-filtered colimits and if there is a small subcategory

$\boldsymbol{C}$ of $\boldsymbol{A}$ consisting of κ-presentable objects such that every object of $\boldsymbol{A}$ is a κ-filtered colimit of a diagram of objects in $\boldsymbol{C}$. A category is accessible if it is κ-accessible for some κ (see [12, 23]). A functor between accessible categories is accessible if it preserves κ-filtered colimits for some κ.

Recall that for a small category $\boldsymbol{C}$, a functor $F:\boldsymbol{C} \to \mathbf{Set}$ is *κ-flat*, if it is a κ-filtered colimit of representable functors (see [17, 23]). As shown in [23], a category $\boldsymbol{A}$ is κ-accessible iff it is equivalent to a category of the form κ-$Flat(\boldsymbol{C})$ with $\boldsymbol{C}$ small; here κ-$Flat(\boldsymbol{C})$ is the category of κ-flat functors from $\boldsymbol{C}$ to $\mathbf{Set}$.

Finally, recall from [9] that a diagram $D:\boldsymbol{D} \to \boldsymbol{A}$ of an arbitrary category $\boldsymbol{A}$ is said to have a *multicolimit* if the functor $\boldsymbol{A} \to \mathbf{Set}$ which assigns to every object A the set of cocones on D, with vertex A, is isomorphic to a small coproduct of representable functors. The corresponding representing family of $\boldsymbol{A}$-objects is then called the multicolimit of D.

The *multilimit* of D in $\boldsymbol{A}$ is simply a multicolimit of D^{op} in $\boldsymbol{A}^{op}$. Hence it is given by a small family of cones

$$\lambda_i\colon \Delta L_i \to D \quad (i \in I)$$

such that any cone $\alpha: \Delta A \to D$ factors though a unique λ_i by a unique morphism $A \to L_i$. $\boldsymbol{A}$ is said to be *familially κ-complete* if every diagram $D:\boldsymbol{D} \to \boldsymbol{A}$ with $\#\boldsymbol{D} < \kappa$ has a multilimit.

It is easy to see that when $\boldsymbol{A}$ has small coproducts, every multilimit must actually be a limit, that is, the indexing system I must be a singleton set. Consequently, for a functor $F:\boldsymbol{A} \to \boldsymbol{B}$ into any category $\boldsymbol{B}$ with small coproducts, one cannot expect the application of F to the multilimit in $\boldsymbol{A}$ to yield a multilimit in $\boldsymbol{B}$, unless the multilimit in $\boldsymbol{A}$ was actually a limit. Mere multilimit preservation is therefore a concept of limited importance. The following notion, however, turns out to be useful: the functor $F:\boldsymbol{A} \to \boldsymbol{B}$ *merges the multilimit* of D if the coproduct $L = \coprod_{i \in I} F(L_i)$ exists in $\boldsymbol{B}$ and the induced cone $\lambda\colon \Delta L \to F \circ D$ is a limit cone in $\boldsymbol{B}$. Note that if F merges multilimits for some type $\boldsymbol{D}$, then it preserves in particular $\boldsymbol{D}$-limits. We say that F *merges κ-multilimits* or briefly, is *κ-merging* if F merges all multilimits of diagrams of size less than κ (for the case $\boldsymbol{B} = \mathbf{Set}$, see [10]).

The following proposition is crucial:

Proposition 1.1. *For a small familially κ-complete category $\boldsymbol{C}$, a functor $F:\boldsymbol{C} \to \mathbf{Set}$ merges κ-multilimits if and only if it is κ-flat.*

Proof. With $el(F) = 1/F$ denoting the element category of F (see [17]), the functor F is κ-flat if and only if $(el(F))^{op}$ is κ-filtered, that is, if every diagram $G:\boldsymbol{D} \to el(F)$ with $\#\boldsymbol{D} < \kappa$ admits a cone. This property follows immediately when F merges κ-multilimits. One simply forms the multilimit of $D = U \circ G$, with $U: el(F) \to \boldsymbol{C}$ the canonical functor. The canonical natural transformation $t\colon \Delta 1 \to F \circ U$ then yields an element $x: 1 \to L = \coprod_{i \in I} F(L_i)$ with $\lambda \circ \Delta x = t \circ G$. Hence, for a uniquely determined $i \in I$, the cone $\lambda_i\colon \Delta L_i \to U \circ G$ can be lifted to a cone $\Delta(L_i, x) \to G$.

Conversely, let F be κ-flat and therefore a κ-filtered colimit

$$F \cong \operatorname*{colim}_{j \in J} \boldsymbol{C}(C_j, -)$$

of representable functors. Since in **Set** such colimits commute with both coproducts and κ-limits, the fact that representable functors merge κ-multilimits gives the same property for F:

$$\coprod_i F(L_i) \cong \coprod_i \operatorname{colim}_j \boldsymbol{C}(C_j, L_i) \cong \operatorname{colim}_j \coprod_i \boldsymbol{C}(C_j, L_i)$$

$$\cong \operatorname{colim}_j \lim \boldsymbol{C}(C_j, -) \circ D \cong \lim(\operatorname{colim}_j \boldsymbol{C}(C_j, -) \circ D) \cong \lim F \circ D. \quad \square$$

For $\boldsymbol{C}$ with κ-multilimits, Proposition 1.1 says that the κ-accessible category κ-$Flat(\boldsymbol{C})$ contains exactly the set-valued κ-merging functors on $\boldsymbol{C}$. Since small connected limits commute with coproducts in **Set**, this implies the existence of small connected limits in κ-$Flat(\boldsymbol{C})$. Hence we have shown half of the following characterization theorem for accessible categories with small connected limits.

Theorem 1.2. *For an infinite regular cardinal κ, a category $\boldsymbol{A}$ is κ-accessible and has small connected limits if and only if $\boldsymbol{A}$ is equivalent to the category κ-$Flat(\boldsymbol{C})$, for some small familially κ-complete category $\boldsymbol{C}$.*

In order to show that other half of the theorem, first we study the category

$ConnFilt_\kappa(\boldsymbol{A}, \mathbf{Set})$

of all set-valued functors on a κ-accessible category $\boldsymbol{A}$ with small connected limits which preserve these and κ-filtered colimits. Since κ-limits and coproducts commute with connected limits and κ-filtered colimits in **Set**, this category is closed under κ-limits and coproducts in the functor category $(\boldsymbol{A}, \mathbf{Set})$. As a κ-accessible functor, each $F \in ConnFilt_\kappa(\boldsymbol{A}, \mathbf{Set})$ satisfies the solution-set condition (see [23]). Now recall Diers' [9] General Multiadjoint Functor Theorem: for any (locally small) category $\boldsymbol{A}$ with small connected limits, a functor $G: \boldsymbol{A} \to \boldsymbol{B}$ has a left multiadjoint if and only if G satisfies the solution-set condition and preserves connected limits. In case $\boldsymbol{B} = \mathbf{Set}$, such a functor is in particular multirepresentable (or, familially representable, in the recently more popular terminology of [24, 14, 7]). This means that every $F \in ConnFilt_\kappa(\boldsymbol{A}, \mathbf{Set})$ for $\boldsymbol{A}$ κ-accessible with small connected limits is a small coproduct of representable functors $\boldsymbol{A}(A_i, -)$, $i \in I$. Furthermore, since F preserves κ-filtered colimits, the same holds true for each representable $\boldsymbol{A}(A_i, -)$, i.e., A_i is κ-presentable.

We use the terminology of [3, 5] to formulate these facts conveniently. An object B of a category $\boldsymbol{B}$ is called *coprime* if $\boldsymbol{B}(B, -): \boldsymbol{B} \to \mathbf{Set}$ preserves small coproducts (the term "coproduct presentable" was used in [13]). It is easy to see that any coproduct that is coprime must actually be isomorphic to one of its summands. One says that a category $\boldsymbol{B}$ with coproducts is *based* if every object of $\boldsymbol{B}$ is a coproduct of coprime objects. With this terminology we have shown:

Proposition 1.3. *For a κ-accessible category $\mathbf{A}$ with small connected limits, the category $ConnFilt_\kappa(\mathbf{A}, \mathbf{Set})$ is a based category with κ-limits. Its full subcategory of coprime objects is equivalent to $\mathbf{A}_\kappa^{op}$, the opposite of the full subcategory of κ-presentable objects of $\mathbf{A}$.*

Now it is easy to complete the proof of Theorem 1.2. Since one already has the equivalence

$$\mathbf{A} \cong \kappa\text{-}Flat(\mathbf{A}_\kappa^{op})$$

for $\mathbf{A}$ κ-accessible (see [23]), it suffices to show the existence of κ-multicolimits in $\mathbf{A}_\kappa$ when $\mathbf{A}$ has small connected limits. Hence we consider a diagram $D : \mathbf{D} \to \mathbf{A}_\kappa$ of size less than κ. The restriction $Y : \mathbf{A}_\kappa^{op} \to (\mathbf{A}, \mathbf{Set})$ of the Yoneda embedding of $\mathbf{A}^{op}$ factors through the category $\mathbf{B} = ConnFilt_\kappa(\mathbf{A}, \mathbf{Set})$ which has κ-limits. Hence we can form the limit

$$F = \lim Y \circ D^{op} : \mathbf{D}^{op} \to \mathbf{B}.$$

By the Yoneda lemma, $F(A)$ is isomorphic to the set of natural transformations $Y(A) \to F$ for each $A \in \mathbf{A}_\kappa$, hence it is isomorphic to the set of cones $\Delta(Y(A)) \to Y \circ D^{op}$, which is isomorphic to the set of cocones on D in $\mathbf{A}_\kappa$ with vertex A. Proposition 1.3 gives a presentation of the restriction of F to $\mathbf{A}_\kappa$ as a coproduct of representables, hence the existence of a multilimit of D. □

Diers [10] called a κ-accessible category with small multicolimits *locally κ-multipresentable* and showed that each categories are exactly the categories equivalent to the category of κ-merging set-valued functors on $\mathbf{C}$, for some small $\mathbf{C}$ with κ-multilimits. Hence, with Proposition 1.1 one obtains

Corollary 1.4 [23, Theorem 6.1.7]. *Diers' locally κ-multipresentable categories are exactly the κ-accessible categories with connected limits.*

2. A conceptual construction of the free coproduct completion

Recall that the free coproduct completion $\sum(\mathbf{C})$ (also denoted by $Fam(\mathbf{C})$, see [19, 23]) of a category $\mathbf{C}$ can be constructed as follows: its objects are small families $\langle X_i \rangle_{i \in I}$ of $\mathbf{C}$, and a morphism $\langle X_i \rangle_{i \in I} \to \langle Y_j \rangle_{j \in J}$ is given by a function $t : I \to J$ and morphisms $f_i : X_i \to Y_{t(i)}$ in $\mathbf{C}$, with the obvious composition rule. Now $\sum(\mathbf{C})$ has all small coproducts, and the canonical embedding $\mathbf{C} \to \sum(\mathbf{C})$ has the expected universal property: every functor $F : \mathbf{C} \to \mathbf{B}$ into a category with coproducts extends essentially uniquely to a coproduct-preserving functor $F_! : \sum(\mathbf{C}) \to \mathbf{B}$; $F_!$ is actually the left Kan extension of F along the canonical embedding. We may describe this property more precisely by the following proposition which is indeed just a (very) special case of Kelly's theorem 5.35 of [17].

We denote by

$$\coprod\left(\sum(\boldsymbol{C}), \boldsymbol{B}\right)$$

the category of coproduct-preserving functors $\sum(\boldsymbol{C}) \to \boldsymbol{B}$.

Proposition 2.1. *The restriction functor*

$$R: \coprod\left(\sum(\boldsymbol{C}), \boldsymbol{B}\right) \to (\boldsymbol{C}, \boldsymbol{B})$$

is an equivalence of categories, for every category $\boldsymbol{B}$ with small coproducts. Its quasi-inverse takes every functor $F: \boldsymbol{C} \to \boldsymbol{B}$ to its (coproduct-preserving) left Kan extension $F_!$.

Remark 2.2. All of the above can be modified by trading "small coproducts" for "λ-coproducts", that is, coproducts of families of less than λ-many objects, with λ an infinite regular cardinal. For $\boldsymbol{B}$ with λ-coproducts, Proposition 2.1 then yields an equivalence

$$\coprod_{\lambda}\left(\sum_{\lambda}(\boldsymbol{C}), \boldsymbol{B}\right) \to (\boldsymbol{C}, \boldsymbol{B}),$$

with the obvious meaning of the left-hand-side category.

Let us now return to the setting of Section 1 with $\boldsymbol{C}$ small and familially κ-complete. Our aim is to give an alternative description of $\sum(\boldsymbol{C})$, using the ingredients of Section 1. First recall (see [17, 23]) that the category

$$\boldsymbol{C}^* = \kappa\text{-}Flat(\boldsymbol{C})$$

is indeed the free completion of $\boldsymbol{C}^{\mathrm{op}}$ under κ-filtered colimits. More precisely, composition with the (restricted) Yoneda embdding $\boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{C}^*$ yields an equivalence of categories

$$I: Filt_\kappa(\boldsymbol{C}^*, \mathbf{Set}) \to (\boldsymbol{C}^{\mathrm{op}}, \mathbf{Set})$$

with quasi-inverse J. Composition of J with the Yoneda embedding $\boldsymbol{C} \to (\boldsymbol{C}^{\mathrm{op}}, \mathbf{Set})$ leads to a full embedding $\boldsymbol{C} \to Filt_\kappa(\boldsymbol{C}^*, \mathbf{Set})$. Since $J(Y(C)): \boldsymbol{C}^* \to \mathbf{Set}$ preserves connected limits, for every $C \in \boldsymbol{C}$, we actually have a full embedding

$$E: \boldsymbol{C} \to ConnFilt_\kappa(\boldsymbol{C}^*, \mathbf{Set}),$$

which is the *evaluation functor* $C \mapsto (F \mapsto F(C))$. From Theorem 1.2 and Proposition 1.3 we now obtain for $\boldsymbol{A} = \boldsymbol{C}^*$:

Proposition 2.3. *For every small familially κ-complete category $\boldsymbol{C}$, the evaluation functor $E: \boldsymbol{C} \to ConnFilt_\kappa(\boldsymbol{C}^*, \mathbf{Set})$ gives a full, dense and κ-merging embedding of $\boldsymbol{C}$ into a category with small coproducts and κ-limits. Moreover, $ConnFilt_\kappa(\boldsymbol{C}^*, \mathbf{Set})$ is a* *based category whose coprime objects are the objects isomorphic to the (embedded) objects of* $\boldsymbol{C}$.

An easy application of Proposition 2.1 to the functor E now gives the main result of this section:

Theorem 2.4. *For every small familially κ-complete category* $\boldsymbol{C}$, *the left Kan extension*

$$E_!:\sum(\boldsymbol{C}) \to \mathit{ConnFilt}_\kappa(\boldsymbol{C}^*, \mathbf{Set})$$

of E *is an equivalence of categories.*

Proof. $E_!$ takes an object $\langle C_i\rangle_{i\in I}$ to the coproduct $\coprod_{i\in I} E(C_i)$ in $\mathit{ConnFilt}_\kappa(\boldsymbol{C}^*, \mathbf{Set})$. But, according to Proposition 2.3, every $F \in \mathit{ConnFilt}_\kappa(\boldsymbol{C}^*, \mathbf{Set})$ is isomorphic to such a coproduct, hence $E_!$ is essentially surjective on objects. In order to show that $E_!$ is full and faithful, one considers a morphism

$$f: \coprod_{i\in I} E(C_i) \to \coprod_{j\in J} E(D_j)$$

in $\mathit{ConnFilt}_\kappa(\boldsymbol{C}^*, \mathbf{Set})$. For every $i \in I$, coprimity of $E(C_i)$ gives a uniquely determined $t(i)\in J$ and $f_i: C_i \to D_{t(i)}$ with $f\circ p_i = q_{t(i)}\circ E(f_i)$ (with p_i, $q_{t(i)}$ coproduct injections). Hence $f = E_!(t, \langle f_i\rangle_{i\in I})$ for a unique morphism $(t, \langle f_i\rangle_{i\in I})$ in $\sum(\boldsymbol{C})$. □

Corollary 2.5. *The free coproduct completion* $\sum(\boldsymbol{C})$ *of a small category* $\boldsymbol{C}$ *is κ-complete if and only if* $\boldsymbol{C}$ *is familially κ-complete. In this case,* $\boldsymbol{C} \to \sum(\boldsymbol{C})$ *is a κ-merging functor.*

Proof. The "if" part follows from Proposition 2.4, and the "only if" part can be easily checked directly. □

It seems natural now to restrict the functor R of Proposition 2.1 to the category

$$\lim_\kappa \coprod\left(\sum(\boldsymbol{C}), \boldsymbol{B}\right)$$

of functors $\sum(\boldsymbol{C}) \to \boldsymbol{B}$ which preserve κ-limits and coproducts. Here we first consider the case $\boldsymbol{B} = \mathbf{Set}$. □

Proposition 2.6. *For every small familially κ-complete category* $\boldsymbol{C}$, *the restriction functor*

$$R: \lim_\kappa \coprod\left(\sum(\boldsymbol{C}), \mathbf{Set}\right) \to (\boldsymbol{C}, \mathbf{Set})$$

is full and faithful, and its essential image contains exactly the κ-merging functors from $\boldsymbol{C}$ *to* $\mathbf{Set}$.

Proof. According to Propositions 1.1 and 2.1, it suffices to show that the essential image of R is the category κ-$Flat(\boldsymbol{C}) = \boldsymbol{C}^*$. First, for every $M \in \lim_\kappa \coprod(\sum(\boldsymbol{C}), \mathbf{Set})$, $R(M) = M|_C$ is indeed κ-flat, i.e., the category $(el(R(M)))^{\mathrm{op}}$ is κ-filtered. Hence, for every diagram $G: \boldsymbol{D} \to el(R(M))$ with $\#\boldsymbol{D} < \kappa$ we must find a cone. This can be done completely analogously to the first part of the proof of Proposition 1.1, by first forming the limit cone

$$\lambda: \Delta L \to I \circ U \circ G$$

in $\sum(\boldsymbol{C})$ (with $I: \boldsymbol{C} \to \sum(\boldsymbol{C})$ and $U: el(R(M)) \to \boldsymbol{C}$), with $L = \coprod_{i \in I} L_i$ and $L_i \in \boldsymbol{C}$, and then by applying both preservation properties of M.

Conversely, every $F \in \kappa$-$Flat(\boldsymbol{C})$ is a κ-filtered colimit of representables: $F \cong \lim_{j \in J} \boldsymbol{C}(C_j, -)$. Since $\lim_\kappa \coprod(\sum(\boldsymbol{C}), \mathbf{Set})$ has κ-filtered colimits, which are formed pointwise and are therefore preserved by R, one has $R(M) \cong F$ with $M = \lim_{j \in J} \sum(\boldsymbol{C})(C_j, -)$. □

Corollary 2.7. *The category* $\lim_\kappa \coprod(\sum(\boldsymbol{C}), \mathbf{Set})$ *is κ-accessible with connected limits, and its full subcategory of κ-presentable objects is equivalent to* $\boldsymbol{C}^{\mathrm{op}}$.

Proof. $\lim_\kappa \coprod(\sum(\boldsymbol{C}), \mathbf{Set})$ is equivalent to κ-$Flat(\boldsymbol{C})$. □

3. Free κ-∞-lextensive completion

Recall that a coproduct $B = \coprod_{j \in J} B_j$ with coproduct injections $t_j: B_j \to B$ is said to be *stable under pullback* (or *universal*) if for every morphism $p: A \to B$ the pullback diagrams

$$\begin{array}{ccc} A_j & \xrightarrow{s_j} & A \\ \downarrow & & \downarrow{\scriptstyle p} \\ B_j & \xrightarrow{t_j} & B \end{array}$$

exit and describe A as a coproduct of $(A_j)_{j \in J}$. The coproduct of B is *disjoint* if all injections are monic, and if the pullback of (t_j, t_k) for any $j \neq k$ exists and is an initial object of $\boldsymbol{B}$.

In [6], categories with finite limits and finite coproducts which are disjoint and stable under pullback have been described as so-called *lextensive* (= lex-extensive) categories. For an infinite regular cardinal κ, we now wish to trade finite limits for κ-limits and finite coproducts for arbitrary small coproducts which are disjoint and stable under κ-wide pullbacks, as defined below.

First recall that a *κ-wide pullback* (or, *κ-fibred product*) is the limit of a family $(f_i: B_i \to C)_{i \in I}$ of morphisms with $\#I < \kappa$, i.e., a family of commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{p_i} & B_i \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle f_i} & \\ C & & \end{array}$$

with the obvious universal property; equivalently, it is a direct product in the sliced category of morphisms with codomain C. Given an I-indexed family of coproducts $B_i \cong \coprod_{j \in J_i} B_{ij}$ with injections $t_{ij}: B_{ij} \to B_i$, and an I-indexed family of arbitrary morphisms $p_i: A \to B_i$, first for each $i \in I$ and $j \in J_i$ we form the (ordinary) pullback diagram

$$\begin{array}{ccc} A_{ij} & \xrightarrow{s_{ij}} & A \\ {\scriptstyle p_{ij}}\downarrow & & \downarrow{\scriptstyle p_i} \\ B_{ij} & \xrightarrow{t_{ij}} & B_i \end{array}$$

and then, for every $\varphi = (j_i)_{i \in I} \in \prod_{i \in I} J_i$, we form the κ-wide pullback of the family $(s_{ij_i})_{i \in I}$:

$$\begin{array}{ccc} Q_\varphi & \xrightarrow{q_{\varphi i}} & A_{ij_i} \\ {\scriptstyle q_\varphi}\downarrow & \swarrow{\scriptstyle s_{ij_i}} & \\ A & & \end{array}$$

Stability of the coproducts B_i $(i \in I,\ \#I < \kappa)$ *under* κ*-wide pullbacks* means that for every family $(p_i)_{i \in I}$ the (wide) pullbacks A_{ij} and Q_φ exist and that the morphisms q_φ exhibit A as a coproduct

$$A \cong \coprod_{\varphi \in J} Q_\varphi,$$

with $J = \prod_{i \in I} J_i$.

Definition 3.1. A category is called κ-∞-lextensive if it has κ-limits and arbitrary small coproducts which are disjoint and stable under κ-wide pullbacks.

Remark 3.2. (1) To say that an I-indexed family of coproducts with $\#I = 1$ is stable under κ-wide pullbacks means exactly that coproducts are stable under pullback in the ordinary sense. Inductively one shows easily that in this case also every finite family of coproducts (including the case $I = \emptyset$!) is stable under κ-wide pullbacks. Hence, finite families of coproducts are stable under $\aleph_0$-wide pullback if and only if coproducts are stable under pullback in the ordinary sense.

(2) We recall for later reference that in a category with coproducts which are stable under pullback the initial object 0 is necessarily *strict*, that is, any morphism with

codomain 0 is an isomorphism. (Consider the case $\#I = 1$ and $\#J = 0$ in the definition above.)

(3) We remark that when the morphisms p_i are product projections of $A \cong \prod_{i\in I} B_i$, then stability of coproducts under κ-wide pullbacks entails *κ-∞-distributivity*, as a natural extension of the notion of distributive category (cf. [6]). In fact, in this case one has $Q_\varphi \cong \prod_{i\in I} B_{ij_i}$, for every $\varphi = (j_i)_{i\in I} \in J = \prod_{i\in I} J_i$, hence

$$\prod_{i\in I} \coprod_{j\in J_i} B_{ij} \cong \coprod_{\varphi\in J} \prod_{i\in I} B_{i\varphi(i)}$$

It is easy to see that stability of coproducts under κ-wide pullbacks amounts to κ-∞-distributivity of all slices of the category in question.

(4) A complete lattice (when considered a category in the usual way) is κ-∞-distributive if and only if it satisfies the infinite distributive law

$$\bigwedge_{i\in I} \bigvee_{j\in J_i} b_{ij} \cong \bigvee_{\varphi\in J} \bigwedge_{i\in I} b_{i\varphi(i)},$$

with $J = \prod_{i\in I} J_i$ and $\#I < \kappa$. In case $\kappa = \aleph_0$, these are exactly the frames (= complete Heyting algebras). Complete lattices which are κ-∞-distributive for every κ are known as completely-distributive lattices.

(5) The category **Set** is ∞-∞-lextensive, that is, κ-∞-lextensive for every κ. This follows immediately from the fact that power-set lattices satisfy the infinite distributive law of (4) with $\bigwedge = \bigcap$ and $\bigvee = \bigcup$, with the additional observation that disjointness of all unions on the left-hand side implies disjointness of the union on the right-hand side. Consequently, also every presheaf category $(\boldsymbol{C}^{\mathrm{op}}, \mathbf{Set})$ is ∞-∞-lextensive.

It has been observed previously that the free coproduct completion $\sum(\boldsymbol{C})$ (or $\sum_\lambda(\boldsymbol{C})$, see Remark 2.2) has universal and disjoint (λ-) coproducts (see [6]). Here we give a conceptual proof of this fact when $\boldsymbol{C}$ is small and familially κ-complete, by embedding $\sum(\boldsymbol{C})$ into the functor category $(\boldsymbol{C}^{\mathrm{op}}, \mathbf{Set})$, as follows.

Proposition 3.3. *The left Kan extension*

$$Y_! : \textstyle\sum(\boldsymbol{C}) \to (\boldsymbol{C}^{\mathrm{op}}, \mathbf{Set})$$

*of the Yoneda embedding $Y : \boldsymbol{C} \to (\boldsymbol{C}^{\mathrm{op}}, \mathbf{Set})$ is full and faithful and preserves κ-limits and small coproducts. Its essential image contains exactly the so-called coproduct-flat functors $\boldsymbol{C}^{\mathrm{op}} \to$ **Set**, i.e., those functors which are small coproducts of representables.*

Proof. The fullness and faithfulness of $Y_!$ follows from the density of $\boldsymbol{C}$ in $\sum(\boldsymbol{C})$. Coprimity of $C \in \boldsymbol{C}$ in $\sum(\boldsymbol{C})$ shows that the functor $Y_!(-)(C) : \sum(\boldsymbol{C}) \to \mathbf{Set}$ preserves coproducts. Preservation of coproducts by $Y_!$ follows since colimits are computed pointwise in $(\boldsymbol{C}^{\mathrm{op}}, \mathbf{Set})$. Similarly, one can see that $Y_!$ preserves the existing limits of $\sum(\boldsymbol{C})$. □

By embedding Proposition 3.3 with Remark 3.2 (5) one obtains:

Corollary 3.4. *For any small familially κ-complete category $\boldsymbol{C}$, the free coproduct-completion $\sum(\boldsymbol{C})$ is κ-∞-lextensive, and the embedding $\boldsymbol{C} \to \sum(\boldsymbol{C})$ is a κ-merging functor.*

Remark 3.5. Corollaries 2.5 and 3.4 hold for any (not necessarily small) familially κ-complete category $\boldsymbol{C}$. This can be checked by using facts that every object of $\boldsymbol{C}$ is coprime of $\sum(\boldsymbol{C})$ and every object of $\sum(\boldsymbol{C})$ is a coproduct of objects of $\boldsymbol{C}$. The direct proof is straightforward.

The main result of this paper says that $\boldsymbol{C} \to \sum(\boldsymbol{C})$ is universal with respect to the properties mentioned in Corollary 3.4.

Theorem 3.6. *Let $\boldsymbol{C}$ be a familially κ-complete category. Then every κ-merging functor $F:\boldsymbol{C} \to \boldsymbol{B}$ into a κ-∞-lextensive category $\boldsymbol{B}$ factors essentially uniquely through a κ-limit- and coproduct-preserving functor $\sum(\boldsymbol{C}) \to \boldsymbol{B}$. More precisely, restriction of such functors to $\boldsymbol{C}$ defines an equivalence of categories*

$$R:\lim_\kappa \coprod\left(\sum(\boldsymbol{C}), \boldsymbol{B}\right) \to Merg_\kappa(\boldsymbol{C}, \boldsymbol{B}),$$

with $Merg_\kappa(\boldsymbol{C}, \boldsymbol{B})$ the category of κ-merging functors from $\boldsymbol{C}$ to $\boldsymbol{B}$.

Proof. We want to show that the left Kan extension $F_!:\sum(\boldsymbol{C}) \to \boldsymbol{B}$ preserves κ-limits, for any κ-merging functor $F:\boldsymbol{C} \to \boldsymbol{B}$. This is done in three steps.

Step 1: $F_!$ preserves limits of diagrams in $\boldsymbol{C}$ of size less than κ. Indeed, for $D:\boldsymbol{D} \to \boldsymbol{C}$ with $\#\boldsymbol{D} < \kappa$, one has a multilimit $\langle L_j \rangle_{j \in J}$ of D which describes the limit of $I \circ D$ in $\sum(\boldsymbol{C})$, with $I:\boldsymbol{C} \to \sum(\boldsymbol{C})$. That F merges κ-multilimits and coproduct preservation of $F_!$ then yield

$$\lim F_! \circ I \circ D \cong \lim F \circ D \cong \coprod_{j \in J} F(L_j) \cong F_!\left(\coprod_{j \in J} L_j\right) \cong F_!(\lim I \circ D).$$

Since the empty diagram of $\sum(\boldsymbol{C})$ factors through D, $F_!$ preserves in particular terminal objects.

Step 2: $F_!$ preserves pullbacks. Consider a pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle g} \\ A & \underset{f}{\longrightarrow} & C \end{array}$$

in $\sum(\boldsymbol{C})$, with $A \cong \coprod_{m \in M} A_m$, $B \cong \coprod_{n \in N} B_n$ and $C \cong \coprod_{k \in K} C_k$ given as coproducts of coprime objects. If $\#N = 0$ or $\#M = 0$, then $P = 0$, and the pullback is trivially preserved by $F_!$, because of the strictness of initial objects in $\sum(\boldsymbol{C})$ and in $\boldsymbol{B}$ (see Remark 3.2.(2)). If $\#M = \#N = 1$, then f and g are given by $\boldsymbol{C}$-morphisms $f':A \to C_k$ and $g':B \to C_l$. For $k \neq l$, the pullback of $(C_k \to C,\ C_l \to C)$ is 0. Since there is a morphism $P \to 0$ also P is 0, hence the pullback is trivially preserved by $F_!$, as in the previous case. For $k = l$, P is actually the pullback of f', g' (since $C_k \to C$ is monic) and is therefore preserved by $F_!$, according to Step 1. If $\# M > 1$ and $\# N = 1$, then for each $m \in M$ we form the pullback diagrams

$$\begin{array}{ccccc} P_m & \longrightarrow & P & \longrightarrow & B \\ \big\downarrow & & \big\downarrow & & \big\downarrow g \\ A_m & \xrightarrow[r_m]{} & A & \xrightarrow[f]{} & C \end{array}$$

in $\sum(C)$, with coproduct injections r_m. Pullback stability gives $P \cong \coprod P_m$. Both coproducts and (according to the case considered previously) each outer pullback are preserved by $F_!$. Hence, when we form the pullback R of $F_!(f)$ and $F_!(g)$ in $\boldsymbol{B}$, we obtain a canonical morphism $h : F_!(P) \to R$, and each $F_!(P_m)$ is the pullback of $F_!(r_m)$ along the pullback projection $R \to F_!(A)$. Pullback stability of the coproduct $F_!(A) \cong \coprod F_!(A_m)$ therefore gives

$$R \cong \coprod F_!(P_m) \cong F_!(P),$$

i.e., h is an isomorphism, as required. Finally, similarly one reduces the case $\# M > 1$ and $\# N > 1$ to the case just considered.

Step 3: $F_!$ preserves κ-wide pullbacks. We consider a family $(f_i : B_i \to C)_{i \in I}$ in $\sum(C)$ with $\# I < \kappa$, with coproducts $C \cong \coprod_{k \in K} C_k$ and $B_i \cong \coprod_{j \in J_i} B_{ij}$ of coprime objects for every $i \in I$. We then form the (ordinary) pullbacks A_{ij} and the κ-wide pullbacks Q_φ, keeping the same notation as in the definition of stability of coproducts under κ-wide pullbacks. Certainly, if each B_i is coprime (so that $\# J_i = 1$ for all $i \in I$), then the pullback P is preserved by $F_!$, by the same argumentation as in Step 2. Therefore, when in the general case we consider for each $\varphi = (j_i)_{i \in I}$ the κ-wide pullback Q_φ, which is easily seen to give a κ-wide pullback

$$\begin{array}{ccc} Q_\varphi & \xrightarrow{p_{ij_i} \circ q_{\varphi i}} & B_{ij_i} \\ {\scriptstyle p \circ q_\varphi} \big\downarrow & \swarrow {\scriptstyle f_i \circ s_{ij_i}} & \\ C & & \end{array}$$

then this limit is preserved by $F_!$. Furthermore, by step 2 $F_!$ preserves the ordinary pullbacks A_{ij_i}, for every $i \in I$. Similarly to Step 2, we now form the κ-wide pullback

$$\begin{array}{ccc} R & \xrightarrow{u_i} & F_!(B_i) \\ {\scriptstyle u} \big\downarrow & \swarrow {\scriptstyle F_!(f_i)} & \\ F_!(C) & & \end{array}$$

and then, for every $i \in I$, the ordinary pullbacks

$$\begin{array}{ccc} R_i & \longrightarrow & R \\ \big\downarrow & & \big\downarrow{\scriptstyle u_i} \\ F_!(B_{ij_i}) & \xrightarrow[F_!(s_{ij_i})]{} & F_!(B_i) \end{array}$$

We have canonical morphisms $h : F_!(A) \to R$ and $h_i : F_!(A_{ij_i}) \to R_i$ and obtain a κ-wide pullback

$$\begin{array}{ccc} F_!(Q_\varphi) & \xrightarrow{h_i \circ F_!(q_{\varphi i})} & R_i \\ {\scriptstyle h \circ F_!(q_\varphi)}\big\downarrow & \swarrow{\scriptstyle r_i} & \\ R & & \end{array}$$

The stability of the coproducts $F_!(B_i) \cong \coprod_{j \in J_i} F_!(B_{ij_i})$ under κ-wide pullbacks in $\boldsymbol{B}$ therefore yields

$$R \cong \coprod_{\varphi} F_!(Q_\varphi) \cong F_!(A),$$

as desired.

Since $F_!$ preserves terminal objects and κ-wide pullbacks, $F_!$ preserves all κ-limits. □

Corollary 3.7. *For any κ-complete category $\boldsymbol{C}$, a functor $F : \boldsymbol{C} \to \boldsymbol{B}$ into a κ-∞-lextensive category $\boldsymbol{B}$ merges κ-multilimits iff it preserves κ-limits.*

Remark 3.8. (1) In Theorem 3.6, it suffices to assume $\boldsymbol{B}$ to be $\aleph_0$-∞-lextensive with κ-limits and to satisfy the κ-∞-distributive law (see Remark 3.2 (3)). In fact, Step 3 of the proof Theorem 3.6 may be replaced by a shorter argument which shows that $F_!$ preserves κ-products, as follows: if each B_i ($i \in I$, $\# I < \kappa$) is a coproduct $B_i \cong \coprod_{j \in J_i} B_{ij}$ of objects in $\boldsymbol{C}$, then κ-∞-distributivity in $\sum(\boldsymbol{C})$ and in $\boldsymbol{B}$ and preservation by $F_!$ of coproducts and of κ-limits of objects in $\boldsymbol{C}$ show

$$\begin{aligned} F_!\Big(\prod_{i \in I} B_i\Big) &\cong F_!\Big(\coprod_{\varphi \in J} \prod_{i \in I} B_{i\varphi(i)}\Big) \cong \coprod_{\varphi \in J} F_!\Big(\prod_{i \in I} B_{i\varphi(i)}\Big) \\ &\cong \coprod_{\varphi \in J} \prod_{i \in I} F_!(B_{i\varphi(i)}) \cong \prod_{i \in I} \coprod_{j \in J_i} F_!(B_{ij}) \\ &\cong \prod_{i \in I} F_!(B_i). \end{aligned}$$

(2) An advantage of the argumentation given in Step 3 of the proof of Theorem 3.6 is that it does not rely on the existence of a terminal object in $\sum(\boldsymbol{C})$ but yields results also when we restrict ourselves to considering functors preserving certain connected limits. (Readers interested in these are particularly referred to [24].) More precisely, we have shown that, for a small familially κ-complete category $\boldsymbol{C}$ and for every functor

$F: \boldsymbol{C} \to \boldsymbol{B}$ taking κ-wide multipullbacks of $\boldsymbol{C}$ into κ-wide pullbacks of $\boldsymbol{B}$, its extension $F_!: \sum(\boldsymbol{C}) \to \boldsymbol{B}$ preserves κ-wide pullbacks, provided $\boldsymbol{B}$ has κ-wide pullbacks and coproducts which are disjoint and stable under κ-wide pullbacks.

(3) When we restrict our attention to the 2-category κ-**Lex** of categories with κ-limits, whose 1-arrows are functors preserving κ-limits and 2-arrows are all natural transformations between latter, the universal property of Theorem 3.6 becomes part of the left 2-adjoint to the inclusion

$$\kappa\text{-}\infty\text{-}\mathbf{Lex} \longrightarrow \kappa\text{-}\mathbf{Lex}$$

here κ-∞-**Lex** is the 2-category of κ-∞-lextensive categories with the obvious meaning of 1-arrows and 2-arrows. The unit of the 2-adjunction at a category $\boldsymbol{C}$ of κ-**Lex** is just the inclusion $\boldsymbol{C} \to \sum(\boldsymbol{C})$.

(4) The 2-categorical description above cannot be applied for familially κ-complete categories. Indeed, assuming that we have such a 2-adjunction with the same unit above. For a familially κ-complete category $\boldsymbol{C}$ and a category $\boldsymbol{B}$ of κ-∞-**Lex**, the 1-arrows from $\sum(\boldsymbol{C})$ into $\boldsymbol{B}$ of κ-∞-**Lex** must correspond to the 1-arrows between $\boldsymbol{C}$ and $\boldsymbol{B}$, i.e., κ-merging functors from $\boldsymbol{C}$ into $\boldsymbol{B}$. Therefore, the composition of κ-merging functors must be a κ-merging functor, but this is not true in general. For instance, taking a category $\boldsymbol{C}$ without terminal object, then the composition of the inclusions $\boldsymbol{C} \to \sum(\boldsymbol{C})$ and $\sum(\boldsymbol{C}) \to \sum(\sum(\boldsymbol{C}))$ is not a κ-merging functor. This can be seen as follows. Let $\langle C_i \rangle_{i \in I}$ be a multi-terminal family of $\boldsymbol{C}$, and let B be the coproduct of C_i in $\sum(\boldsymbol{C})$. If the composition above is a κ-merging functor, then B is the coproduct of C_i in $\sum(\sum(\boldsymbol{C}))$. But the identity arrow of B must factor through a unique C_i, since B is coprime in $\sum(\sum(\boldsymbol{C}))$. Hence, B is isomorphic to C_i. This is contrary to the assumption that $\boldsymbol{C}$ has no terminal object.

Remark 3.9. (1) Further to Remark 3.8 (3) we refer the reader to the general theory of completions with respect to a class of colimits as presented in [1, 18].

(2) Categorical considerations of infinite distributive laws are to be found in [25]. It is clear that the results of [6, 4] allow for "κ-λ-fications" which we plan to outline in a separate paper.

Acknowledgements

The first author acknowledges financial assistance from a special research grant of the Faculty of Arts at York University. Work by the second author is partially supported by an NSERC operating grant and a NATO collaborative research grant.

We also acknowledge helpful comments on the subject of this paper which we received from Jiri Adámek concerning the non-significance of the smallness of $\boldsymbol{C}$, from Reinhard Börger concerning the role of coprimity, and from Max Kelly and the referee concerning 2-categorical aspects and the non-compositivity of κ-merging functors.

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