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# On a planar Liouville-type problem in the study of selfgravitating strings

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ABSTRACT

Motivated by the construction of selfgravitating strings (cf. Yang, 2001, 1994 [22,23]), we analyze a Liouville-type equation on the plane, derived in Yang (1994) [23]. We establish sharp existence and uniqueness properties for the corresponding radial solutions. We investigate also when the problem allows for non-radial solutions.



## 1. Introduction

In this paper we analyze an elliptic problem of Liouville type in $\mathbb{R}^2$, whose solutions yield to selfgravitating strings for a massive W-boson model coupled to Einstein theory in account of gravitational effects (cf. [22]).

To handle the analytical difficulties posed by the corresponding string's equations, Y. Yang in [23] introduced a set of ansatz so that the corresponding string configuration obeyed to a system of Bogomolnyi-type (selfdual) first order equations coupled with Einstein's equation.

Such a construction was inspired by the work of Ambjorn and Olesen in [1–4]. It gives rise to (selfgravitating) strings that are parallel in the $x_3$-direction and whose cross section (with respect to the plane: $x_3 = 0$) is localized around some given points $p_1, \dots, p_N \in \mathbb{R}^2$ (repeated according to the assigned multiplicity).

Consistently, the gravitational metric can be chosen to be conformally equivalent to the flat $\mathbb{R}^2$-metric.

As a consequence, the full string's problem can be reduced to an elliptic system involving two unknown (real) functions $(u, \eta)$, with $\eta$ the conformal factor and $e^u$ the "strength" of the W-boson

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field. The location of the string at the points $p_1, \dots, p_N$, requires the following "singular" behavior of $u$:

$$u(x) = \ln\big(|x - p_j|^2\big) + O(1) \quad \text{as } x \to p_j \tag{1.1}$$

for any $j \in \{1, \dots, N\}$.

Thus, the governing string's system takes the form:

$$\begin{cases} -\Delta u = 2m_W^2 e^{\eta} + 4b^2 e^u - 4\pi \displaystyle\sum_{j=1}^{N} \delta_{p_j}, \\ -\dfrac{\Delta \eta}{8\pi G} = \dfrac{2m_W^4}{b^2} e^{\eta} + 4m_W^2 e^u \end{cases} \tag{1.2}$$

with $m_W =$ boson's mass, $-b =$ electron charge, $G =$ gravitational constant. The details of the derivation of (1.2) can be found in [22] and [23]. In the planar case, problem (1.2) must be equipped with the (boundary) conditions:

$$e^u, e^{\eta} \in L^1\big(\mathbb{R}^2\big) \tag{1.3}$$

in order to ensure *finite* (total) energy and (total) curvature. Notice that (1.3) implies that both $u$ and $\eta$ must admit a logarithmic growth at infinity (e.g. see [14,15,13]). Thus, by (1.2), we find that the function

$$w := u - \frac{b^2}{m_W^2 8\pi G}\eta - \sum_{j=1}^{N} \ln\big(|x - p_j|^2\big) \tag{1.4}$$

defines an entire harmonic function with logarithmic growth at infinity. So $w$ must be *constant*, say $w \equiv C$ with $C \in \mathbb{R}$. Therefore the system (1.2) can be further reduced to a single equation in terms of the unknown function

$$v = \frac{b^2}{m_W^2 8\pi G}\eta + C + \ln\big(4b^2\big) \tag{1.5}$$

given as follows:

$$-\Delta v = \lambda e^{av} + \prod_{j=1}^{N} |x - p_j|^2 e^v \tag{1.6}$$

where

$$a = \frac{m_W^2 8\pi G}{b^2} \gg 1, \qquad \lambda = 2m_W^2 e^{-a\mu} \quad \text{and} \quad \mu = 4b^2 e^C. \tag{1.7}$$

Moreover, the (boundary) conditions (1.3), can be restated in terms of $v$, by requiring that the right-hand side of (1.6) belongs to $L^1(\mathbb{R}^2)$.

To investigate (1.6) we use its "natural" scaling property. For instance if we set:

$$v_\varepsilon(x) = v(x/\varepsilon) + 2\max\big\{1/a, (N+1)\big\}\ln(1/\varepsilon) \tag{1.8}$$

with $v$ that solves (1.6), then

(i) for $a > 1/(N+1)$, $v_\varepsilon$ satisfies the equation:

$$-\Delta v = \lambda \varepsilon^{2((N+1)a-1)} e^{av} + \prod_{j=1}^{N} |x - \varepsilon p_j|^2 e^v. \tag{1.9}$$

Formally, as $\varepsilon \to 0$ we can interpret (1.9) as a "perturbation" of the (singular) Liouville equation:

$$\begin{cases} -\Delta v = |x|^{2N} e^v \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |x|^{2N} e^v \, dx < +\infty. \end{cases} \tag{1.10}$$

Solutions of (1.10) have been completely classified in [19], and in particular they satisfy

$$\int_{\mathbb{R}^2} |x|^{2N} e^v \, dx = 8\pi(N+1). \tag{1.11}$$

In this situation, Chae in [9] has been able to exploit such a "perturbation" property to obtain (as in [11]) a family of solutions $V_\varepsilon$ for (1.6) such that

$$\int_{\mathbb{R}^2} \left\{ \lambda e^{aV_\varepsilon} + \prod_{j=1}^{N} |x - p_j|^2 e^{V_\varepsilon} \right\} dx \to 8\pi(N+1), \quad \text{as } \varepsilon \to 0.$$

(ii) For $0 < a < 1/(N+1)$, $v_\varepsilon$ satisfies the equation:

$$-\Delta v = \lambda e^{av} + \varepsilon^{2(1-(N+1)a)/a} \prod_{j=1}^{N} |x - \varepsilon p_j|^2 e^v \tag{1.12}$$

that instead can be interpreted as a "perturbation" of the (classical) Liouville equation:

$$\begin{cases} -\Delta v = \lambda e^{av} \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{av} \, dx < +\infty \end{cases} \tag{1.13}$$

whose solutions have been completely classified in [15] and they satisfy:

$$\lambda \int_{\mathbb{R}^2} e^{av} \, dx = \frac{8\pi}{a}. \tag{1.14}$$

In principle an analogous perturbation argument as in [9] (see also [10,12,13]) could be used to obtain a family of solutions $V_\varepsilon$ such that

$$\int_{\mathbb{R}^2} \left\{ \lambda e^{aV_\varepsilon} + \prod_{j=1}^{N} |x - p_j|^2 e^{V_\varepsilon} \right\} dx \to \frac{8\pi}{a}, \quad \text{as } \varepsilon \to 0.$$

The case $a = 1/(N+1)$ enters in this analysis as a "special" case. Indeed (1.9) and (1.12) coincide and problem (1.6) becomes, "essentially" scale invariant. It can be reduced to a "perturbation" of the $\varepsilon = 0$ problem (i.e. $p_1 = p_2 = \cdots = p_N = 0$), given as follows

$$\begin{cases} -\Delta v = \lambda e^{av} + |x|^{2N} e^v \quad \text{in } \mathbb{R}^2, \\ \lambda \int\limits_{\mathbb{R}^2} e^{av}\, dx + \int\limits_{\mathbb{R}^2} |x|^{2N} e^v\, dx < +\infty. \end{cases} \tag{1.15}$$

It is interesting to note that, when $a = 1/(N+1)$, problem (1.15) shares many properties with the "singular" Liouville problem (1.10), corresponding to the case $\lambda = 0$ in (1.15).

Indeed, as established in [8] and [19], we have that, if $\lambda \geqslant 0$, $N > -1$, $a = 1/(N+1)$ and $v$ is a solution of (1.15), then

(i) $\lambda \int_{\mathbb{R}^2} e^{av}\, dx + \int_{\mathbb{R}^2} |x|^{2N} e^v\, dx = 8\pi(N+1)$,
(ii) $v_\tau(x) = v(\tau x) + 2(N+1)\ln(\tau)$ and $\hat{v}(x) = v(x/|x|^2) + 2(N+1)\ln(1/|x|^2)$ are also solutions for (1.15),
(iii) $v(x) = \hat{v}_\tau(x)$, with $\tau = e^{\frac{v(0)-\hat{v}(0)}{2(N+1)}}$.

When $N = 0$ and $a = 1$, then problem (1.15) reduces to the "classical" Liouville equation, and the property above can be checked directly from the explicit solutions, see [14].

Explicit solutions are also known for the "singular" Liouville problem (i.e. $\lambda = 0$), but to check (iii) in this case is less obvious.

Explicit solutions for (1.15) are not available, when $\lambda > 0$. Even the radial solutions can be expressed only in terms of some elliptic integrals, that can be computed explicitly only when $N = 1$ and $a = 1/2$ (see (2.15) below). So far, when $a = 1/(N+1)$ and $\lambda > 0$, we have no information concerning the existence of non-radial solutions for (1.15). By keeping in mind that for $\lambda = 0$, the corresponding "singular" Liouville problem admits non-radial solutions *if and only if* $N$ is an integer (see [19]), it is an interesting open problem to determine whether an analogous phenomenon occurs also when $\lambda > 0$.

The aim of this paper is to investigate problem (1.15) when $0 < a \neq 1/(N+1)$, which relates to the $N$-string problem, when all the strings are superimposed at the origin. In this respect, it is relevant to identify the exact range of $\beta$'s for which problem (1.15) can be solved by a solution $v$ satisfying:

$$\beta = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(\lambda e^{av}\, dx + |x|^{2N} e^v\right) dx. \tag{1.16}$$

We are able to answer this question in the *radial* case as follows:

**Theorem 1.1** *(Existence). Let $\lambda > 0$, $N > -1$ and $0 < a \neq 1/(N+1)$. Problem* (1.15)–(1.16) *admits a* radial *solution if and only if*:

$$\text{(i)} \quad \beta \in \left(\max\left\{4(N+1), \frac{4}{a} - 4(N+1)\right\}, \frac{4}{a}\right) \quad \textit{when } 0 < a < \frac{1}{N+1}, \tag{1.17}$$

$$\text{(ii)} \quad \beta \in \left(\max\left\{\frac{4}{a}, 4(N+1) - \frac{4}{a}\right\}, 4(N+1)\right) \quad \textit{when } a > \frac{1}{N+1}. \tag{1.18}$$

To illustrate Theorem 1.1 we notice that

- if $\frac{1}{N+1} < a \leqslant \frac{2}{N+1}$ then $\max\{\frac{4}{a}, 4(N+1) - \frac{4}{a}\} = \frac{4}{a}$,
- if $\frac{1}{2(N+1)} \leqslant a < \frac{1}{N+1}$ then $\max\{4(N+1), \frac{4}{a} - 4(N+1)\} = 4(N+1)$

and by combining the above result with some suitable integral identities of Pohozaev type (see (2.9)–(2.10)) we conclude:

**Corollary 1.1.** *Let $\lambda > 0$ and $N > -1$;*

(a) *if $\frac{1}{N+1} < a \leqslant \frac{2}{N+1}$ then problem* (1.15)–(1.16) *admits a solution* (*not necessary radial*) *if and only if $\beta \in (\frac{4}{a}, 4(N+1))$,*
(b) *if $\frac{1}{2(N+1)} \leqslant a < \frac{1}{N+1}$ then problem* (1.15)–(1.16) *admits a solution* (*not necessary radial*) *if and only if $\beta \in (4(N+1), \frac{4}{a})$.*

Concerning the uniqueness issue, we obtain the following

**Theorem 1.2** *(Uniqueness). Let $\lambda > 0$ and $N > 0$.*

(i) *If $a \geqslant \frac{N+2}{2(N+1)}$ and $\beta$ satisfies* (1.18)*, then problem* (1.15)–(1.16) *admits a* unique *radial solution.*
(ii) *If $(\frac{1}{N+1} <)\frac{2}{N+2} < a < \frac{N+2}{2(N+1)}$ and $\beta \in (-\infty, \frac{2N}{1-a}]$ satisfies* (1.18)*, then* (1.15)–(1.16) *admits a* unique *radial solution.*

**Remark 1.1.** (a) We shall show that the claimed *unique* radial solution is also *non-degenerate* (in a suitable sense) in the space of radial functions, but (in some cases) not necessarily so when non-radial functions are also taken into account.

(b) Observe that if $0 < a \neq \frac{1}{N+1} \leqslant \frac{2}{N+2}$ then (1.17) or (1.18) could imply that: $\beta > \frac{2N}{1-a}$, and by our method, no uniqueness property can be claimed in this case. Actually, we suspect that multiplicity may occur in this case.

For $\frac{2}{N+2} < a < \frac{N+2}{2(N+1)}$ the value $\beta = \frac{2N}{1-a}$ satisfies (1.18) and it gives a "special" value in relation to problem (1.15). Indeed, we shall find suitable pairs $(a, N)$ for which problem (1.15)–(1.16) with $\beta = \frac{2N}{1-a}$ admits a branch of *non-radial* solutions bifurcating from the (unique) radial one. In particular, the "linearized" problem around the radial solution admits a nontrivial kernel formed by non-radial functions. See Section 4 for details.

If $-1 < N \leqslant 0$, then by a straightforward application of the moving plane technique (cf. [14]), we see that *every* solution of (1.15) must be radially symmetric. Moreover by exploiting a "natural" duality property of the radial problem (see (2.13), (2.17)–(2.20) below), we can formulate the following uniqueness result:

**Theorem 1.3.** *Let $\lambda > 0$.*

(i) *If $-1 < N < 0$, then every solution of* (1.15) *is radially symmetric about the origin and* (1.17), (1.18) *define necessary and sufficient conditions for the solvability of* (1.15)–(1.16)*. Furthermore, if $0 < a \leqslant \frac{2}{N+2}$ or if $\frac{2}{N+2} < a < \frac{N+2}{2(N+1)} (< \frac{1}{N+1})$ and $\beta \leqslant \frac{2|N|}{a-1}$ then* (1.15)–(1.16) *admits a unique radial solution.*
(ii) *If $N = 0$ and $a \neq 1$ then the problem*:

$$\begin{cases} -\Delta v = \lambda e^{av} + e^{v} \quad \textit{in } \mathbb{R}^2, \\ \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \left(\lambda e^{av} + e^{v}\right) dx = \beta \end{cases} \tag{1.19}$$

*admits a solution if and only if $\beta \in I(a)$, where*:

$$\textit{if } 0 < a < 1 \quad \textit{then} \quad I(a) = \left(\max\left\{4, \frac{4(1-a)}{a}\right\}, \frac{4}{a}\right),$$

$$\textit{if } a > 1 \quad \textit{then} \quad I(a) = \left( \max\left\{ \frac{4}{a}, \frac{4(a-1)}{a} \right\}, 4 \right).$$

*Moreover, for any $\beta \in I(a)$ problem* (1.19) *admits a* unique *radial solution about the origin, and any other solution is obtained by a translation of the radial one.*

Obviously the assumption $a \neq 1$ is essential for the validity of (ii) in the above result. Indeed, for $N = 0$ and $a = 1$, problem (1.19) reduces to the well-known Liouville equation, which is scale invariant (no uniqueness) and, under (1.16), solvable only for $\beta = 4$, see [14,15].

When $N > 0$ and $a = 1$ then (1.15) becomes a particular case of the class of problems:

$$\begin{cases} -\Delta v = K(|x|)e^v & \text{in } \mathbb{R}^2, \\ \beta = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} K(|x|)e^v \, dx \end{cases} \tag{1.20}$$

which has attracted much attention in the context of the prescribed Gauss curvature problem in $\mathbb{R}^2$ (see e.g. [18] and references therein).

According to our results, we have that,

**Corollary 1.2.** *Let $K(|x|) = 1 + |x|^{2N}$, $N > 0$; then problem* (1.20) *admits a radial solution if and only if $\beta \in (4\max\{1, N\}, 4(N+1))$. Moreover for such $\beta$'s the corresponding radial solution is unique. Furthermore, for $0 < N \leqslant 1$ the interval above is optimal for the solvability of* (1.20)*, among also non-radial functions.*

To emphasize the subtle issues connected to such an existence and uniqueness result, let us observe that if we choose instead the weight function $K(|x|) = (1 + |x|^2)^N$, then the nature of problem (1.20) changes dramatically, as $N$ increases.

Indeed, the problem:

$$\begin{cases} -\Delta v = \left(1 + |x|^2\right)^N e^v & \text{in } \mathbb{R}^2, \\ \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \left(1 + |x|^2\right)^N e^v \, dx = \beta \end{cases} \tag{1.21}$$

has been analyzed in [16] in connection with a "singular" mean field equation on the sphere $S^2$ of interest in the study of gauge field vortices (see [21] for details, and [6,17] for "sharp" symmetry results).

When $0 < N \leqslant 1$ or $N > 1$ and $\beta \in (4N, 4(N+1))$, then the existence and uniqueness properties of (1.20) are unaffected by either choice of $K(r) = 1 + r^{2N}$ or $K(r) = (1 + r^2)^N$. Indeed according to a result of C.S. Lin [18] the following holds:

**Theorem 1.4.** *(See [18].) Assume that $K = K(r) > 0$ is a $C^1$-function satisfying*:

$$\text{(a)} \quad \frac{rK'(r)}{K(r)} \textit{ is nondecreasing and not identically constant}, \tag{1.22}$$

$$\text{(b)} \quad \lim_{r \to +\infty} \frac{rK'(r)}{K(r)} = 2N. \tag{1.23}$$

*If $0 < N \leqslant 1$ then problem* (1.20) *admits a solution if and only if $\beta \in (4, 4(N+1))$. Moreover for each of such $\beta$'s, there exists a unique radial solution of* (1.20)*. If $N > 1$ then* (1.20) *admits a* unique *radial solution, for any $\beta \in (4N, 4(N+1))$.*

Such a result applies to our case (where $K(r) = (1 + r^{2N}))$ and gives the statement of Corollary 1.2. It was obtained in [18] by a clever use of Alexandroff–Bol's inequality, as indicated for the first time by Bandle in [5]. This approach however has no chance to be extended to the case $0 < a \neq 1$, considered here. In fact, to obtain Theorem 1.2 we have to device a suitable "comparison principle" (see Proposition 2.1) that allows us to obtain the same information as those derived by Alexandroff–Bol's isoperimetric inequality for the case $a = 1$. In particular it allows us to provide an alternative proof of Theorem 1.4, as shown in Section 3.

Obviously, Lin's result also applies to the weight function $K(r) = (1 + r^2)^N$, and for $N > 1$ establishes a uniqueness result analogous to Corollary 1.2 when $\beta \in (4N, 4(N + 1))$. But now, such an interval is no longer the optimal interval for the existence (and uniqueness) of radial solutions.

Indeed, by summarizing the results in [16], one finds that, for $N > 1$ there exists a value $\beta_N \in (2(N + 1), 4N)$ such that, for $\beta \in (\beta_N, 4N)$ problem (1.21) admits at least *two* radially symmetric solutions (multiplicity). Actually for such $\beta$'s the number of radial (and non-radial) solutions increases as $N$ increases, and an explicit bifurcation diagram can be obtained for the specific value $\beta = 2(N+2)$.

So for $N > 1$ large, the nature of (1.21) is quite different from that described by Corollary 1.2 concerning problem (1.20) with $K(|x|) = 1 + |x|^{2N}$, under exam here. Therefore, we'll have to really exploit the specific structure of (1.15), in order to establish Theorems 1.1 and 1.2.

The paper is organized as follows, in Section 2 we provide some preliminary information about the solutions of (1.15)–(1.16). Section 3 is devoted to the proof of the theorems stated above. While in Section 4 we carry out the construction of non-radial solutions and formulate some open questions and their connections to problems treated in [6] and [16].

## 2. Preliminaries

In this section we collect some known properties about solutions of problem (1.15)–(1.16).

By taking into account the scaling properties in (1.9) and (1.12) we see that,

$$\text{for } a \neq \frac{1}{N+1} \text{ without loss of generality we can take } \lambda = 1. \tag{2.1}$$

Hence, for $N > -1$ and $0 < a \neq 1/(N + 1)$, we consider

$$\begin{cases} -\Delta u = e^{au} + |x|^{2N} e^u \quad \text{in } \mathbb{R}^2, \\ \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \left(e^{au} + |x|^{2N} e^u\right) dx = \beta. \end{cases} \tag{2.2}$$

By following the approach of [15], Chen, Guo and Sprin in [8] obtained the following:

**Lemma 2.1.** *(See [8].) If u is a solution of* (2.2) *then*:

$$\text{(i)} \quad \left|u(x) + \beta \ln\left(|x| + 1\right)\right| \leqslant C \quad \textit{in } \mathbb{R}^2, \tag{2.3}$$

$$\text{(ii)} \quad \int_{\mathbb{R}^2} \left\{2\left(\frac{1}{a} - 1\right)e^{au} + 2N|x|^{2N} e^u\right\} = \pi\beta(\beta - 4). \tag{2.4}$$

More precisely in [8] the authors were able to complete the information in (2.3) and give a more accurate description about the asymptotic behavior of the solution as $|x| \to +\infty$. In particular,

$$u(x) = -\beta \ln\left(|x|\right) + O(1) \quad \text{as } |x| \to +\infty \ \left(\text{see [8]}\right);$$

and by the integral condition in (2.2) we obtain that

$$\beta > \max\left\{\frac{2}{a}, 2(N+1)\right\}. \tag{2.5}$$

Furthermore from (2.4) we easily deduce that,

- if $a = 1/(N+1)$ then

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(e^{au} + |x|^{2N}e^{u}\right)dx = 4(N+1); \tag{2.6}$$

- if $0 < a \neq 1/(N+1)$ then:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{au}\,dx = \frac{\beta a}{4}\cdot\frac{4(N+1)-\beta}{a(N+1)-1}, \tag{2.7}$$

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |x|^{2N}e^{u}\,dx = \frac{\beta a}{4}\cdot\frac{\beta-4/a}{a(N+1)-1}. \tag{2.8}$$

Consequently,

$$\text{if } 0 < a < \frac{1}{N+1} \quad \text{then} \quad 4(N+1) < \beta < \frac{4}{a}, \tag{2.9}$$

$$\text{if } a > \frac{1}{N+1} \quad \text{then} \quad \frac{4}{a} < \beta < 4(N+1). \tag{2.10}$$

We shall see that these bounds on $\beta$ are actually "sharp" for the solvability of (3.2), if and only if $\frac{1}{2(N+1)} < a \neq \frac{1}{N+1} < \frac{2}{N+1}$.

In the study of (2.2) it is useful to introduce the change of variable $r = e^t$ with $r = |x|$, and so consider the new unknown function,

$$v(t,\theta) := u\left(e^t\cos(\theta), e^t\sin(\theta)\right) \tag{2.11}$$

which satisfies

$$\begin{cases} -\left(\partial^2_{tt}v + \partial^2_{\theta\theta}v\right) = \exp(2t+av) + \exp\left(2(N+1)t+v\right) \quad \text{for } t\in\mathbb{R},\ \theta\in[-\pi,\pi], \\ v(t,\cdot) \text{ is } 2\pi\text{-periodic } \forall t\in\mathbb{R}, \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\int_{\mathbb{R}}\left(\exp(2t+av) + \exp\left(2(N+1)t+v\right)\right)dt\,d\theta = \beta. \end{cases} \tag{2.12}$$

In particular *radial* solutions of (2.2) can be described through the solutions $v = v(t)$ of the boundary value problem

$$\begin{cases} v_{tt} + \exp(2t+av) + \exp\left(2(N+1)t+v\right) = 0 \quad \text{for } t\in\mathbb{R}, \\ v_t(-\infty) = 0, \quad v_t(+\infty) = -\beta. \end{cases} \tag{2.13}$$

When $a=1/(N+1)$ then $\beta=4(N+1)$ (see (2.6)) and we can use the transformation: $w(t)=v(t)+2(N+1)t$ to arrive at the autonomous problem

$$\begin{cases} w_{tt}+\exp\bigl(w(t)/(N+1)\bigr)+\exp\bigl(w(t)\bigr)=0 \quad \text{for } t\in\mathbb{R}, \\ w_t(-\infty)=2(N+1), \quad w_t(+\infty)=-2(N+1). \end{cases} \tag{2.14}$$

Thus, for the *unique* $\bar{t}$: $w_t(\bar{t})=0$ we find $(\bar{t}-t)w_t(t)>0$ $\forall t\neq\bar{t}$ and $w(\bar{t}+t)=w(\bar{t}-t)$. We can use those properties together with the energy identity: $w_t^2/2+(N+1)\exp(w/(N+1))+\exp(w)=2(N+1)^2$, to obtain an explicit expression for $w(t)$ in terms of suitable elliptic integrals. When $N=1$ and $a=1/2$, we can actually explicitly derive $w(t)$. Thus, we leave to the reader to check that, after some calculations, one obtains the following family of *radial* solutions:

$$u(x)=2\ln\left(\frac{\tau^2}{1+\frac{\lambda}{8}|\tau x|^2+\frac{1}{32}(1+\frac{\lambda^2}{8})|\tau x|^4}\right) \tag{2.15}$$

for the problem:

$$\begin{cases} -\Delta u=\lambda e^{u/2}+|x|^2e^u \quad \text{in } \mathbb{R}^2, \\ \lambda e^{u/2}+|x|^2e^u\in L^1\bigl(\mathbb{R}^2\bigr). \end{cases}$$

Notice that (2.15) reduces to the well-known radial solution for the singular Liouville problem (1.10) when $\lambda=0$ and $N=1$ (see [19]).

In order to identify the range of $\beta$'s for which (2.13) is solvable, we point out the following modified versions of the “energy identity”.

**Lemma 2.2.** *Let $N>-1$, $0<a\neq 1/(N+1)$ and $v$ be a solution of* (2.13)*. There holds*

$$\text{(a)}\quad \frac{d}{dt}\left(\frac{1}{2}v_t\left(v_t+\frac{4}{a}\right)+\frac{1}{a}\exp(2t+av)+\exp\bigl(2(N+1)t+v\bigr)\right)$$
$$=\frac{2}{a}\bigl((N+1)a-1\bigr)\exp\bigl(2(N+1)t+v\bigr),$$

$$\text{(b)}\quad \frac{d}{dt}\left(\frac{1}{2}v_t\bigl(v_t+4(N+1)\bigr)+\frac{1}{a}\exp(2t+av)+\exp\bigl(2(N+1)t+v\bigr)\right)$$
$$=-\frac{2}{a}\bigl((N+1)a-1\bigr)\exp(2t+av).$$

**Proof.** Multiplying the equation in (2.13) by $v_t$ we obtain:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}v_t^2+\frac{1}{a}\exp(2t+av)+\exp\bigl(2(N+1)t+v\bigr)\right) \\ &\quad=\frac{2}{a}\exp(2t+av)+2(N+1)\exp\bigl(2(N+1)t+v\bigr) \\ &\quad=-\frac{2}{a}v_{tt}+2\left((N+1)-\frac{1}{a}\right)\exp\bigl(2(N+1)t+v\bigr) \end{aligned}$$

from which we deduce (a).

Similarly by observing that

$$\frac{2}{a}\exp(2t+av)+2(N+1)\exp\bigl(2(N+1)t+v\bigr)$$
$$=-2(N+1)v_{tt}+2\left(\frac{1}{a}-(N+1)\right)\exp(2t+av)$$

we obtain (b). □

We have already noticed how, in the analysis of (2.13), we need to distinguish between the cases:

$$0<a<\frac{1}{N+1}\quad\text{or}\quad a>\frac{1}{N+1}\quad(N>-1).$$

As matter of fact, those describe two dual situations, and we can go from one to the other via transformation:

$$v(t)\to\hat{v}(t)=av\left(\frac{t}{N+1}+\tau\right)+\ln(\mu) \tag{2.16}$$

with

$$\tau=\frac{1-a}{2((N+1)a-1)}\ln\left(\frac{a}{(N+1)^2}\right)\quad\text{and}\quad\mu=\left(\frac{a}{(N+1)^2}\right)^{\frac{Na}{(N+1)a-1}}.$$

Indeed, it can be easily checked, that if $v$ satisfies (2.13) then $\hat{v}$ solves the analogous problem:

$$\begin{cases}\hat{v}_{tt}+\exp(2t+\hat{a}\hat{v})+\exp\bigl(2(\hat{N}+1)t+\hat{v}\bigr)=0 & \text{for } t\in\mathbb{R},\\ \hat{v}_t(-\infty)=0,\quad \hat{v}_t(+\infty)=-\hat{\beta}\end{cases} \tag{2.17}$$

with

$$\hat{a}=\frac{1}{a},\qquad \hat{N}=-\frac{N}{N+1}>-1\quad\text{and}\quad\hat{\beta}=\frac{a\beta}{N+1}. \tag{2.18}$$

Moreover, the following transformation rules hold:

$$a=\frac{1}{N+1}\quad\Leftrightarrow\quad\hat{a}=\frac{1}{\hat{N}+1},$$
$$0<a<\frac{1}{N+1}\quad\Leftrightarrow\quad\hat{a}>\frac{1}{\hat{N}+1},$$
$$\left(a>\frac{1}{N+1}\quad\Leftrightarrow\quad 0<\hat{a}<\frac{1}{\hat{N}+1}\right). \tag{2.19}$$

For later use, let us observe also that:

$$\frac{1}{N+1}<a<\frac{2}{N+1}\quad\Leftrightarrow\quad\frac{1}{2(\hat{N}+1)}<\hat{a}<\frac{1}{\hat{N}+1},$$
$$a>\frac{2}{N+1}\quad\Leftrightarrow\quad 0<\hat{a}<\frac{1}{2(\hat{N}+1)}. \tag{2.20}$$

So, via the transformation (2.16)–(2.18), without loss of generality, we only have to account for the case:

$$N > -1 \quad \text{and} \quad a > \frac{1}{N+1}. \tag{2.21}$$

We know that for a solution $v$ of (2.13), its derivative $v_t$ decreases from 0 to $-\beta$. Consequently by (2.5) and (2.10), there exist unique values $t_\pm = t_\pm(v)$ such that

$$-\infty < t_- < t_+ < +\infty \quad \text{and} \quad v'(t_-) = -\frac{2}{a}, \quad v'(t_+) = -2(N+1) \quad \text{and}$$
$$\Lambda(v) = \left\{t \in \mathbb{R}:\ -2(N+1) < v'(t) < -\frac{2}{a}\right\} = \big(t_-(v), t_+(v)\big). \tag{2.22}$$

As a consequence of Lemma 2.2 we find:

**Lemma 2.3.** *Assume* (2.21) *and let* $v$ *be a solution of* (2.13)*.*

(a) *If* $s \in \mathbb{R}$ *satisfies* $v_t(s) > -2(N+1)$ *(i.e.* $s < t_+(v)$*), then*

$$\begin{aligned}&\frac{1}{2}v_t(s)\left(v_t(s) + \frac{4}{a}\right) + \frac{1}{a}\exp\big(2s + av(s)\big) + \exp\big(2(N+1)s + v(s)\big)\\ &\quad < \frac{2}{a}\big((N+1)a - 1\big)\frac{\exp(2(N+1)s + v(s))}{2(N+1) + v_t(s)}.\end{aligned}$$

(b) *If* $s \in \mathbb{R}$ *satisfies* $v_t(s) < -2/a$ *(i.e.* $s > t_-(v)$*), then*

$$\begin{aligned}&\frac{1}{2}v_t(s)\big(v_t(s) + 4(N+1)\big) + \frac{1}{a}\exp\big(2s + av(s)\big) + \exp\big(2(N+1)s + v(s)\big) + \frac{1}{2}\beta\big(4(N+1) - \beta\big)\\ &\quad < \frac{2}{a}\big((N+1)a - 1\big)\frac{\exp(2s + av(s))}{|2 + av_t(s)|}.\end{aligned}$$

**Proof.** We start to observe that, if $v_t(s) > -2(N+1)$, then

$$\begin{aligned}\int_{-\infty}^{s} \exp\big(2(N+1)t + v(t)\big)\,dt &= \int_{-\infty}^{s} \exp\big(\big(2(N+1) + v_t(s)\big)t\big)\exp\big(v(t) - v_t(s)t\big)\,dt\\ &< \exp\big(v(s) - v_t(s)s\big)\int_{-\infty}^{s} \exp\big(\big(2(N+1) + v_t(s)\big)t\big)\,dt\\ &= \frac{\exp(2(N+1)s + v(s))}{2(N+1) + v_t(s)}\end{aligned} \tag{2.23}$$

and the inequality above follows by observing that the function $v(t) - v_t(s)t$ attains its strict maximum value at $t = s$.

Similarly, if $v_t(s) < -2/a$ we find:

$$\int_s^{+\infty} \exp\bigl(2t + av(t)\bigr)\,dt = \int_s^{+\infty} \exp\bigl(\bigl(2 + av_t(s)\bigr)t\bigr)\exp\bigl(a\bigl(v(t) - v_t(s)t\bigr)\bigr)\,dt$$
$$< \exp\bigl(a\bigl(v(s) - v_t(s)s\bigr)\bigr) \int_s^{+\infty} \exp\bigl(\bigl(2 + av_t(s)\bigr)t\bigr)\,dt$$
$$= -\frac{\exp(2s + av(s))}{2 + av_t(s)} = \frac{\exp(2s + av(s))}{|2 + av_t(s)|}. \tag{2.24}$$

At this point, inequality (a) follows by integrating the identity (a) of Lemma 2.2 in $(-\infty, s]$ and by using (2.23). While inequality (b) follows by integrating the identity (b) of Lemma 2.2 in $[s, +\infty)$ and by using (2.24). □

From Lemma 2.3 we get:

**Corollary 2.1.** *For any $s \in \Lambda(v)$ we have*:

$$\beta\bigl(4(N+1) - \beta\bigr) < \frac{2(v_t(s))^2((N+1)a - 1)}{|v_t(s)|a - 2}. \tag{2.25}$$

**Proof.** We obtain (2.25) by using together the inequalities (a) and (b) of Lemma 2.3.

Indeed, for $s \in \Lambda(v)$, we can rewrite (a) equivalently as follows

$$\frac{1}{2} v_t(s)\bigl(av_t(s) + 4\bigr)\bigl(2(N+1) + v_t(s)\bigr) + \bigl(2(N+1) + v_t(s)\bigr)\exp\bigl(2s + av(s)\bigr)$$
$$+ \bigl(2 + av_t(s)\bigr)\exp\bigl(2(N+1)s + v(s)\bigr) < 0. \tag{2.26}$$

While (b) takes the form:

$$\frac{1}{2}\beta\bigl(4(N+1) - \beta\bigr)\bigl(2 + av_t(s)\bigr) + \frac{1}{2} v_t(s)\bigl(v_t(s) + 4(N+1)\bigr)\bigl(2 + av_t(s)\bigr)$$
$$+ \bigl(2(N+1) + v_t(s)\bigr)\exp\bigl(2s + av(s)\bigr) + \bigl(2 + av_t(s)\bigr)\exp\bigl(2(N+1)s + v(s)\bigr) > 0. \tag{2.27}$$

So, for $s \in \Lambda(v)$, we can subtract (2.26) from (2.27) to deduce

$$2\bigl(v_t(s)\bigr)^2\bigl(1 - a(N+1)\bigr) < \beta\bigl(4(N+1) - \beta\bigr)\bigl(2 + av_t(s)\bigr)$$

from which (2.25) easily follows. □

**Corollary 2.2.** *Let $N > -1$.*

(i) *If $a > \frac{2}{N+1}$ then $\beta > 4(N+1) - \frac{4}{a}$ is a necessary condition for the solvability of* (2.13).
(ii) *If $0 < a < \frac{1}{2(N+1)}$ then $\beta > \frac{4}{a} - 4(N+1)$ is a necessary condition for the solvability of* (2.13).

**Remark 2.1.** Notice that $4(N+1) - \frac{4}{a} > \frac{4}{a} \Leftrightarrow a > \frac{2}{N+1}$; and similarly $\frac{4}{a} - 4(N+1) > 4(N+1) \Leftrightarrow 0 < a < \frac{1}{2(N+1)}$. Therefore, at least in the radial case, the lower bounds on $\beta$ provided by Corollary 2.2, improve those in (2.9) and (2.10), and we can conclude the following:

**Corollary 2.3.** *Let* $N > -1$.

(i) *If* $a > \frac{1}{N+1}$ *then a necessary condition for the solvability of* (2.13) *is that*

$$\beta \in \left(\max\left\{4/a, 4(N+1) - 4/a\right\}, 4(N+1)\right).$$

(ii) *If* $0 < a < \frac{1}{N+1}$ *then a necessary condition for the solvability of* (2.13) *is that*

$$\beta \in \left(\max\left\{4(N+1), 4/a - 4(N+1)\right\}, 4/a\right).$$

**Proof of Corollary 2.2.** We start to establish (i). To this purpose we use (2.25), and in order to estimate its right-hand side, we consider the function:

$$f(x) = 2\big((N+1)a - 1\big)\frac{x^2}{ax - 2}, \quad x \in \big(2/a, 2(N+1)\big).$$

We see that $f$ attains its minimum value at $x_0 = 4/a$; and for $a > \frac{2}{N+1}$ we find that $x_0 = 4/a \in (2/a, 2(N+1))$. Therefore, from (2.25), we obtain

$$\beta\big(4(N+1) - \beta\big) < f(4/a) = \big((N+1)a - 1\big)\left(\frac{4}{a}\right)^2. \tag{2.28}$$

At this point we deduce (i) by using (2.28) together with the fact that $\beta > 2(N+1)$.

To obtain (ii) we use simply the duality (2.16)–(2.18). Namely, by (2.20) we can apply (i) to $\beta$ in order to check that (ii) holds for $\hat{\beta}$. Indeed, $\hat{\beta} = \frac{a\beta}{N+1} > 4a - \frac{4}{N+1} = \frac{4}{\hat{a}} - 4(\hat{N}+1)$. □

The information of Corollary 2.2 will be crucial to establish Theorem 1.1. To proceed further, we need to link the boundary value problem (2.13) to the Cauchy problem:

$$\begin{cases} v_{tt}(t) + \exp\big(2t + av(t)\big) + \exp\big(2(N+1)t + v(t)\big) = 0 \quad \text{for } t \in \mathbb{R}, \\ \lim\limits_{t\to-\infty} v(t) = \alpha, \quad \lim\limits_{t\to-\infty} v_t(t) = 0. \end{cases} \tag{2.29}$$

It is not difficult to check that $\forall\alpha \in \mathbb{R}$, problem (2.29) admits a *unique* solution $v_\alpha$ globally defined, and such that $v'_\alpha := \frac{dv_\alpha}{dt}$ admits a finite limit as $t \to +\infty$, see [23] and [8] for details. So, $v_\alpha$ also satisfies (2.13) with suitable

$$\beta(\alpha) := -\lim_{t\to+\infty} v'_\alpha(t). \tag{2.30}$$

Clearly $\beta(\alpha)$ defines a smooth function of $\alpha$. Similarly, it is not difficult to check that:

$$w_\alpha = \frac{\partial}{\partial\alpha}(v_\alpha) \tag{2.31}$$

is well defined and identifies an element of the kernel of the linearized operator around $v = v_\alpha$. In other words, if we consider the problem

$$-w_{tt} = \big(a\exp(2t + av) + \exp\big(2(N+1)t + v\big)\big)\cdot w \quad \text{for } t \in \mathbb{R} \tag{2.32}$$

then $w_\alpha$ satisfies (2.32) with $v = v_\alpha$, together with the boundary conditions:

$$\lim_{t\to-\infty} w'_\alpha(t) = 0, \qquad \lim_{t\to+\infty} w'_\alpha(t) = -\beta'(\alpha). \tag{2.33}$$

As expected, the linearized problem (2.32) will enter in a crucial way in the analysis of the *uniqueness* issue. To this purpose, let

$$Q(t) = a\exp\bigl(2t + av(t)\bigr) + \exp\bigl(2(N+1)t + v(t)\bigr). \tag{2.34}$$

Then for $v = v_\alpha$ and $y(t) = w'_\alpha(t)$ we have:

$$\frac{d}{dt}\left(\frac{1}{Q(t)}y'(t)\right) + y(t) = 0 \quad \text{for } t \in \mathbb{R}, \tag{2.35}$$

$$\lim_{t\to-\infty} y(t) = 0, \qquad \lim_{t\to+\infty} y(t) = -\beta'(\alpha). \tag{2.36}$$

We shall control the "nodal" regions of $y = y(t)$, by means of the following "comparison" principle.

**Proposition 2.1.** *Let $I := (a,b) \subseteq \mathbb{R}$, with $-\infty \leqslant a < b \leqslant +\infty$, $U(t) \in C^1(I)$ with $U > 0$ in $I$, and $V \in C(I)$. Suppose that $y(t) \in C^2(I)$ satisfies*

$$\begin{cases} \dfrac{d}{dt}\bigl(U(t)y'(t)\bigr) + V(t)y(t) \geqslant 0 \quad \forall t \in I, \\ \lim\limits_{t\to a^+} y(t) = 0 = \lim\limits_{t\to b^-} y(t), \\ y(t) > 0 \quad \forall t \in I. \end{cases}$$

*If there exists a function $z = z(t) \in C^2(I)$ such that*

$$\begin{cases} \dfrac{d}{dt}\bigl(U(t)z'(t)\bigr) + V(t)z(t) \leqslant 0 \quad \forall t \in I, \\ U(t)z'(t) \in L^\infty(I) \end{cases}$$

*then one of the following holds*:

(i) *$z(t) \equiv Cy(t)$ $\forall t \in I$ for a suitable constant $C \in \mathbb{R}$.*
(ii) *$\exists t_0 \in I$: $z(t_0) < 0$.*

**Proof.** Assume that $z(t) \geqslant 0$ for every $t \in I$, then we will prove that $z(t) \equiv Cy(t)$. Indeed, since $y(t) > 0$ and $z(t) \geqslant 0$ for every $t \in I$, we have

$$\frac{d}{dt}\bigl\{U(t)\bigl(z(t)y'(t) - y(t)z'(t)\bigr)\bigr\} \geqslant 0 \quad \forall t \in I; \tag{2.37}$$

and

$$\lim_{t\to a^+} U(t)z'(t)y(t) = \lim_{t\to b^-} U(t)z'(t)y(t) = 0. \tag{2.38}$$

Furthermore, there exist a sequence $\{a_n\}_{n=1}^{+\infty} \subset I$ such that $\lim_{n\to\infty} a_n = a$ and $y'(a_n) > 0$; and a sequence $\{b_n\}_{n=1}^{+\infty} \subset I$ such that $\lim_{n\to\infty} b_n = b$ and $y'(b_n) < 0$. Thus,

$$U(a_n)z(a_n)y'(a_n) \geqslant 0 \quad \text{and} \quad U(b_n)z(b_n)y'(b_n) \leqslant 0. \tag{2.39}$$

Plugging (2.39) into (2.38) we obtain

$$\underline{\lim}_{n\to\infty} U(a_n)\big(z(a_n)y'(a_n) - y(a_n)z'(a_n)\big) \geqslant 0 \quad \text{and}$$
$$\overline{\lim}_{n\to\infty} U(b_n)\big(z(b_n)y'(b_n) - y(b_n)z'(b_n)\big) \leqslant 0. \tag{2.40}$$

Therefore,

$$\begin{aligned}
&\overline{\lim}_{n\to\infty}\int_{a_n}^{b_n} \frac{d}{dt}\big\{U(t)\big(z(t)y'(t) - y(t)z'(t)\big)\big\}\,dt \\
&= \overline{\lim}_{n\to\infty}\big\{U(b_n)\big(z(b_n)y'(b_n) - y(b_n)z'(b_n)\big) - U(a_n)\big(z(a_n)y'(a_n) - y(a_n)z'(a_n)\big)\big\} \\
&\leqslant \overline{\lim}_{n\to\infty} U(b_n)\big(z(b_n)y'(b_n) - y(b_n)z'(b_n)\big) - \underline{\lim}_{n\to\infty} U(a_n)\big(z(a_n)y'(a_n) - y(a_n)z'(a_n)\big) \\
&\leqslant 0.
\end{aligned} \tag{2.41}$$

Thus, by (2.41) and (2.37) we conclude that

$$\int_I \frac{d}{dt}\big\{U(t)\big(z(t)y'(t) - y(t)z'(t)\big)\big\}\,dt = 0. \tag{2.42}$$

Using again (2.37), we deduce that the function $U(t)(z(t)y'(t) - y(t)z'(t))$ must be a constant, and by (2.40), we find that necessarily,

$$U(t)\big(z(t)y'(t) - y(t)z'(t)\big) \equiv 0, \quad \forall t \in I. \tag{2.43}$$

Since $U(t) > 0$ and $y(t) > 0$ for all $t \in I$ we conclude:

$$\frac{d}{dt}\left(\frac{z(t)}{y(t)}\right) = \frac{1}{y^2(t)}\big(y(t)z'(t) - z(t)y'(t)\big) \equiv 0 \quad \forall t \in I,$$

and so, for a suitable constant $C$, $z(t) \equiv Cy(t)$ for all $t \in I$, as claimed. □

As an application of the above comparison principle, and to illustrate also the ideas of our uniqueness result, we provide a crucial estimate that yields to an alternative proof of Lin's uniqueness result, as stated in Theorem 1.4. Radial solution of (1.20) corresponds (with the change of variable $r = e^t$) to solution of the problem:

$$\begin{cases}
-\dfrac{d^2v}{dt^2}(t) = G(t)\exp\big(v(t)\big) \quad \text{for } t \in \mathbb{R}, \\
v'(-\infty) := \lim_{t\to-\infty} v'(t) = 0, \\
\dfrac{dv}{dt}(+\infty) := \lim_{t\to+\infty} v'(t) = -\beta,
\end{cases} \tag{2.44}$$

with $G(t) = e^{2t} K(e^t)$. We consider the following boundary value problem, related to the "linearization" of (2.44):

$$\begin{cases} -\dfrac{d^2 w}{dt^2}(t) = \big(G(t)\exp\big(v(t)\big)\big)w(t) \quad \text{for } t \in \mathbb{R}, \\ w'(-\infty) := \lim\limits_{t\to-\infty} w'(t) = 0, \\ \dfrac{dw}{dt}(+\infty) := \lim\limits_{t\to+\infty} w'(t) = 0. \end{cases} \tag{2.45}$$

By recalling (1.22) and (1.23) we give below an alternative proof of the crucial Lemma 3.3 in [18]. Such lemma was proved in [18] by means of an improved Alexandroff–Bol's isoperimetric inequality, valid for radial functions.

**Proposition 2.2.** *Let $G \in C^2(\mathbb{R})$ be such that $G(t) > 0$ for every $t \in \mathbb{R}$. Suppose that the function $F(t) := G'(t)/G(t)$ is nondecreasing and satisfies $F(-\infty) := \lim_{t\to-\infty} F(t) = 2$ and $F(+\infty) := \lim_{t\to+\infty} = 2(N+1)$ for some $N > 0$. Let $v$ be a solution to (2.44) and assume that (2.45) has a nontrivial solution $w \neq 0$. Then there exists $t_0 \in \mathbb{R}$ such that $w'(t) \neq 0$ $\forall t \in (-\infty, t_0)$, $w'(t_0) = 0$, and $v'(t_0) < -4$.*

**Proof.** Let

$$Q(t) := G(t)\exp\big(v(t)\big) > 0, \quad \forall t \in \mathbb{R}; \tag{2.46}$$

so that $Q(-\infty) = Q(+\infty) = 0$ and (2.44) reads as follows:

$$\begin{cases} \dfrac{d^2 v}{dt^2}(t) + Q(t) = 0 \quad \text{for } t \in \mathbb{R}, \\ v'(-\infty) = 0, \\ v'(+\infty) = -\beta. \end{cases} \tag{2.47}$$

Moreover, letting

$$Y(t) := w'(t) \quad \forall t \in \mathbb{R}, \tag{2.48}$$

from (2.45), we find:

$$\frac{d}{dt}\left(\frac{1}{Q(t)}\frac{dY}{dt}(t)\right) + Y(t) = 0 \quad \text{for } t \in \mathbb{R}, \qquad Y(-\infty) = 0, \qquad Y(+\infty) = 0. \tag{2.49}$$

Define

$$Z(t) := -4v'(t) - \big(v'(t)\big)^2 = -v'(t)\big(v'(t) + 4\big). \tag{2.50}$$

By (2.47) we see that

$$\frac{dZ}{dt}(t) := 2Q(t)\big(v'(t) + 2\big) \tag{2.51}$$

and consequently,

$$\frac{d}{dt}\left(\frac{1}{Q(t)}\frac{dZ}{dt}(t)\right) + Z(t) = -2Q(t) - 4\cdot v'(t) - \big(v'(t)\big)^2 := -D(t), \tag{2.52}$$

where

$$D(t) = 2G(t)\exp\bigl(v(t)\bigr) + 4\cdot v'(t) + \bigl(v'(t)\bigr)^2 \quad \forall t \in \mathbb{R} \quad \text{and} \quad D(-\infty) := \lim_{t\to-\infty} D(t) = 0. \tag{2.53}$$

Moreover, by straightforward calculations we find:

$$\frac{dD}{dt}(t) = 2G(t)\exp\bigl(v(t)\bigr)\left(\frac{1}{G(t)}\frac{dG}{dt}(t) - 2\right) = 2G(t)\exp\bigl(v(t)\bigr)\bigl(F(t)-2\bigr) \geqslant (\neq)\, 0 \quad \forall t \in \mathbb{R}, \tag{2.54}$$

since by assumption, $G > 0$ and $F(t) := G'(t)/G(t) \geqslant (\neq)\, 2$. Recalling that $D(-\infty) = 0$, we find: $D(t) \geqslant (\neq)\, 0$ $\forall t \in \mathbb{R}$, and from of (2.52), we conclude:

$$\frac{d}{dt}\left(\frac{1}{Q(t)}\frac{dZ}{dt}(t)\right) + Z(t) \leqslant (\neq)\, 0 \quad \forall t \in \mathbb{R}. \tag{2.55}$$

Next let $X(t) := Q(t)$, $\forall t \in \mathbb{R}$. We calculate:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{Q(t)}\frac{dX}{dt}(t)\right) + X(t) \\
&\quad = \frac{d}{dt}\left(\frac{1}{G(t)\exp(v(t))}\left(G(t)\exp\bigl(v(t)\bigr)\cdot\frac{dv}{dt}(t) + \frac{dG}{dt}(t)\cdot\exp\bigl(v(t)\bigr)\right)\right) - \frac{d^2v}{dt}(t) \\
&\quad = \frac{d}{dt}\left(\frac{1}{G(t)}\frac{dG}{dt}(t)\right) = F'(t) \geqslant 0 \quad \forall t \in \mathbb{R}.
\end{aligned} \tag{2.56}$$

Notice that $F'$ cannot be identically zero, since, by assumption, the image of $F$ must cover the interval $(2, 2(N+1))$. Furthermore, $X(t) > 0$ for every $t \in \mathbb{R}$ and $X(-\infty) = X(+\infty) = 0$. So, we can apply Proposition 2.1, first with $I = \mathbb{R}$, $y(t) = X(t)$ and $z(t) = \pm Y(t)$ to conclude that $\exists t_0 \in \mathbb{R}$: $w'(t) = Y(t) \neq 0$, $\forall t \in (-\infty, t_0)$ and $w'(t_0) = Y(t_0) = 0$. At this point, we apply again Proposition 2.1, now with $I = (-\infty, t_0)$, $y(t) = |Y(t)|$ and $z(t) = Z(t) = -v'(4+v')$, and arrive at the desired conclusion by observing that $v'$ is negative and decreasing. □

## 3. The proofs

We shall focus first with the case

$$a > \frac{1}{N+1}, \qquad N > -1 \tag{3.1}$$

and by recalling (2.29) and (2.30), we denote by $u_\alpha = u_\alpha(|x|)$ the unique *radial* solution satisfying:

$$\begin{cases} -\Delta u = e^{au} + |x|^{2N}e^u \quad \text{in } \mathbb{R}^2, \\ u(0) = \max\limits_{\mathbb{R}^2}(u) = \alpha, \\ \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2} \bigl(e^{au} + |x|^{2N}e^u\bigr)\,dx = \beta(\alpha), \end{cases} \tag{3.2}$$

see [23]. Our first task will be to determine the limit values of $\beta(\alpha)$ as $\alpha \to \pm\infty$.

First of all recall that, when (3.1) holds, then

$$\beta(\alpha) \in \left(\max\left\{4/a, 4(N+1) - 4/a\right\}, 4(N+1)\right), \quad \forall \alpha \in \mathbb{R} \tag{3.3}$$

(see Corollary 2.3).

**Proposition 3.1.** *Assume* (3.1)*, and let* $\beta(\alpha)$ *be defined in* (3.2)*. We have*:

$$\lim_{\alpha \to -\infty} \beta(\alpha) = 4(N+1), \qquad \lim_{\alpha \to +\infty} \beta(\alpha) = \max\left\{4/a, 4(N+1) - 4/a\right\}. \tag{3.4}$$

**Proof.** We shall show that (3.4) holds along any sequence $\alpha_n \to \pm\infty$. To this purpose, set $u_n = u_{\alpha_n}$ and $\beta_n = \beta(\alpha_n)$.

**Claim 1.**

$$\text{If } \alpha_n \to -\infty \quad \text{then} \quad \beta_n \to 4(N+1). \tag{3.5}$$

To establish the claim, let

$$\tau_n = e^{-\frac{\alpha_n}{2(N+1)}} \to +\infty \quad \text{and} \quad v_n(x) = u_n(\tau_n x) - \alpha_n. \tag{3.6}$$

Then $v_n$ defines a blow-down of $u_n$ and satisfies:

$$\begin{cases} -\Delta v_n = e^{(a-1/(N+1))\alpha_n} e^{a v_n} + |x|^{2N} e^{v_n} \quad \text{in } \mathbb{R}^2, \\ v_n(0) = \max\limits_{\mathbb{R}^2} (v_n) = 0, \\ \dfrac{1}{2\pi} \displaystyle\int\limits_{\mathbb{R}^2} \left\{e^{(a-1/(N+1))\alpha_n} e^{a v_n} + |x|^{2N} e^{v_n}\right\} dx = \beta_n. \end{cases}$$

As in [7], we use well-known Harnack-type inequalities, (e.g. see [20, Corollary 5.2.9]) together with elliptic estimates and a diagonalization process, in order to find a function $V$ such that (along a subsequence):

$$v_n \to V \quad \text{in } C^{2,\gamma}_{\text{loc}}\left(\mathbb{R}^2\right)$$

and $V$ satisfies:

$$\begin{cases} -\Delta V = |x|^{2N} e^V \quad \text{in } \mathbb{R}^2, \\ V(0) = \max\limits_{\mathbb{R}^2} (V) = 0, \quad \displaystyle\int\limits_{\mathbb{R}^2} |x|^{2N} e^V \, dx < +\infty. \end{cases} \tag{3.7}$$

As already mentioned, solutions of (3.7) satisfy: $\int_{\mathbb{R}^2} |x|^{2N} \exp(V)\, dx = 8\pi(N+1)$ (cf. [19]). Consequently, by Fatou's Lemma and (3.3) we find:

$$4(N+1) \leqslant \varliminf_{n \to +\infty} \beta_n \leqslant \varlimsup_{n \to +\infty} \beta_n \leqslant 4(N+1)$$

and Claim 1 follows.

**Claim 2.**

$$\text{If } \alpha_n \to +\infty \quad \text{then} \quad \beta_n \to \max\{4/a, 4(N+1) - 4/a\}. \tag{3.8}$$

To establish Claim 2 we use a blow-up argument and let

$$\sigma_n = \exp(-a\alpha_n/2) \to 0 \quad \text{as } n \to +\infty \quad \text{and} \quad w_n(x) = u_n(\sigma_n x) - \alpha_n.$$

Then $w_n$ satisfies:

$$\begin{cases} -\Delta w_n = e^{aw_n} + e^{-((N+1)a-1)\alpha_n}|x|^{2N}e^{w_n} \quad \text{in } \mathbb{R}^2, \\ w_n(0) = \max\limits_{\mathbb{R}^2}(w_n) = 0, \\ \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2}\{e^{aw_n} + e^{-((N+1)a-1)\alpha_n}|x|^{2N}e^{w_n}\}\,dx = \beta_n. \end{cases}$$

As above, we see that (along a subsequence),

$$w_n \to W \quad \text{in } C^{2,\gamma}_{\text{loc}}(\mathbb{R}^2)$$

with $W$ satisfying:

$$\begin{cases} -\Delta W = e^{aW} \quad \text{in } \mathbb{R}^2, \\ W(0) = \max\limits_{\mathbb{R}^2}\{W\} = 0, \quad \displaystyle\int_{\mathbb{R}^2} e^{aW}\,dx < +\infty. \end{cases}$$

In particular, $\int_{\mathbb{R}^2}\exp(aW) = 8\pi/a$ (cf. [14,15]). By the uniform convergence of $w_n \to W$ on compact set, we obtain that:

$$\text{for every } \varepsilon > 0, \text{ there exist } R_\varepsilon \gg 1 \text{ and } n_\varepsilon \in \mathbb{N}:$$
$$\int_{\{y\in\mathbb{R}^2:\ |y|\leqslant R_\varepsilon\}}\{e^{aw_n(y)} + e^{-((N+1)a-1)\alpha_n}\cdot|y|^{2N}e^{w_n(y)}\}\,dy \geqslant \frac{8\pi}{a} - \varepsilon, \quad \forall n \geqslant n_\varepsilon.$$

Equivalently,

$$\int_{\{x\in\mathbb{R}^2:\ |x|\leqslant\sigma_n R_\varepsilon\}}\{e^{au_n(x)} + |x|^{2N}e^{u_n(x)}\}\,dx \geqslant \frac{8\pi}{a} - \varepsilon, \quad \forall n \geqslant n_\varepsilon.$$

Argue by contradiction and, in account of (3.3), assume that (along a subsequence):

$$\beta_n \to \bar\beta > \max\{4/a, 4(N+1) - 4/a\}. \tag{3.9}$$

Observe that

$$\sup_{\{x\in\mathbb{R}^2:\ |x|\leqslant\sigma_n R_\varepsilon\}}\left\{u_n(x) + \frac{2}{a}\ln(|x|)\right\} \leqslant \sup_{\{y\in\mathbb{R}^2:\ |y|\leqslant R_\varepsilon\}}\left\{w_n(y) + \frac{2}{a}\ln(|y|)\right\} \leqslant \frac{2}{a}\ln(R_\varepsilon)$$

and therefore,

$$\int_{\{x\in\mathbb{R}^2:\ |x|\leqslant\sigma_n R_\varepsilon\}} |x|^{2N} e^{u_n}\,dx \leqslant R_\varepsilon^{2/a} \int_{\{x\in\mathbb{R}^2:\ |x|\leqslant\sigma_n R_\varepsilon\}} |x|^{2N-2/a}\,dx$$
$$= \frac{\pi a}{((N+1)a-1)} R_\varepsilon^{2(N+1)} \sigma_n^{\frac{2}{a}((N+1)a-1)}.$$

Thus,

$$\int_{\{x\in\mathbb{R}^2:\ |x|\leqslant\sigma_n R_\varepsilon\}} e^{au_n}\,dx \geqslant \frac{8\pi}{a} - \varepsilon - \frac{\pi a}{((N+1)a-1)} R_\varepsilon^{2(N+1)} \sigma_n^{\frac{2}{a}((N+1)a-1)}.$$

By recalling (2.7), we obtain:

$$0 \leqslant \int_{\{x\in\mathbb{R}^2:\ |x|\geqslant\sigma_n R_\varepsilon\}} e^{au_n}\,dx \leqslant \int_{\mathbb{R}^2} e^{au_n}\,dx - \frac{8\pi}{a} + \varepsilon + \frac{\pi a}{((N+1)a-1)} R_\varepsilon^{2(N+1)} \sigma_n^{\frac{2}{a}((N+1)a-1)}$$
$$= \frac{\pi}{2} a\beta_n \cdot \frac{4(N+1)-\beta_n}{a(N+1)-1} - \frac{8\pi}{a} + \varepsilon + \frac{\pi a}{((N+1)a-1)} R_\varepsilon^{2(N+1)} \sigma_n^{\frac{2}{a}((N+1)a-1)}.$$

Hence, by passing to the limit first, as $n \to +\infty$, and then as $\varepsilon \to 0$, we arrive at the desired contradiction as follows:

$$0 \leqslant \frac{\pi}{2} a\bar\beta \cdot \frac{4(N+1)-\bar\beta}{a(N+1)-1} - \frac{8\pi}{a} = -\frac{\pi a}{2(a(N+1)-1)} \left(\bar\beta - \frac{4}{a}\right)\left(\bar\beta - \left(4(N+1) - \frac{4}{a}\right)\right) < 0.$$

Thus also Claim 2 is established. Since both Claim 1 and Claim 2 hold along any sequence, we conclude (3.4). □

**Proof of Theorem 1.1 and Corollary 1.1.** When $a > 1/(N+1)$, then the statement of Theorem 1.1 and Corollary 1.1 readily follows by the continuity of $\beta(\alpha)$, Proposition 3.1 and Corollary 2.3.

When $0 < a < 1/(N+1)$, then we use the duality properties (2.16)–(2.19), and apply the result already established to $\hat a = 1/a > 1/(\hat N + 1)$ and $\hat\beta = a\beta/(N+1)$, to deduce the desired statement for $\beta$. □

Next we turn to analyze the *uniqueness* issue. The goal is to show that under the given assumptions, the function $\beta(\alpha)$ is strictly monotone decreasing. To this purpose, we need to locate the possible zeros of $\beta'$.

By recalling (2.31), (2.32) and (2.33), we see that, if there exists $\bar\alpha \in \mathbb{R}$: $\beta'(\bar\alpha) = 0$, then $\bar w = \frac{\partial v_\alpha}{\partial \alpha}|_{\alpha=\bar\alpha}$ will be a *bounded* solution of the linearized equation (2.32) with $v = v_{\bar\alpha}$. As a consequence, $Y(t) = \bar w'(t)$ will define a *nontrivial* solution for the problem:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{Q(t)} Y'(t)\right) + Y(t) = 0 \quad \text{for } t \in \mathbb{R}, \\ Y(-\infty) := \lim\limits_{t\to-\infty} Y(t) = 0, \quad Y(+\infty) := \lim\limits_{t\to+\infty} Y(t) = 0, \end{cases} \tag{3.10}$$

with

$$Q(t) = a\exp\big(2t + av(t)\big) + \exp\big(2(N+1)t + v(t)\big), \tag{3.11}$$

and $v = v_{\bar\alpha}$.

To show that this is *impossible* we start by showing the following:

**Proposition 3.2.** *Let $v$ be a solution of* (2.13) *with $a > 1/(N+1)$, $N > -1$, and $Y = Y(t) \neq 0$ satisfy* (3.10)*, with $Q = Q(t)$ defined in* (3.11)*. Then $Y(t)$* cannot *change sign in $\mathbb{R}$.*

**Proof.** We introduce the following notations:

$$A(t) := \exp\bigl(2t + av(t)\bigr) \quad \text{and} \quad B(t) := \exp\bigl(2(N+1)t + v(t)\bigr) \tag{3.12}$$

and consider the functions:

$$R(t) := -v'(t)\left(\frac{4}{a} + v'(t)\right) - \frac{2(1-a)A(t)}{a}, \tag{3.13}$$

$$Z(t) := -\bigl(\beta + v'(t)\bigr)\bigl(4(N+1) + v'(t) - \beta\bigr) + \frac{2(1-a)B(t)}{a}. \tag{3.14}$$

Then, we can express (2.26) and (2.27) in terms of the functions $R = R(t)$ and $Z = Z(t)$ as follows:

$$\forall s \in \Lambda(v) = \bigl\{s \in \mathbb{R}\colon 2/a < \bigl|v'(s)\bigr| = -v'(s) < 2(N+1)\bigr\}, \quad \text{there holds:}$$
$$-R(s) + 2A(s) + \frac{2}{a}\frac{(2 + av_t(s))}{(2(N+1) + v_t(s))}B(s) \leqslant 0,$$
$$-Z(s) + 2\frac{(2(N+1) + v_t(s))}{(2 + av_t(s))}A(s) + \frac{2}{a}B(s) \leqslant 0.$$

From the above inequality we deduce that:

$$\forall s \in \Lambda(v) \quad \Rightarrow \quad -R(s)\bigl(2(N+1) + v_t(s)\bigr) + Z(s)\bigl(2 + av_t(s)\bigr) \leqslant 0 \tag{3.15}$$

which imply in particular that $R$ and $Z$ *cannot* be simultaneously negative at a point $s \in \Lambda(v)$.

Moreover, concerning $R(t)$ and $Z(t)$ we observe that, by straightforward calculation, the following holds. Firstly,

$$\frac{dR}{dt}(t) = 2Q(t)\left(v'(t) + \frac{2}{a}\right), \qquad R(-\infty) = \lim_{t\to-\infty} R(t) = 0,$$
$$R(+\infty) = \lim_{t\to+\infty} R(t) = -\beta\left(\beta - \frac{4}{a}\right) < 0 \tag{3.16}$$

and

$$\frac{d}{dt}\left(\frac{1}{Q(t)}R'(t)\right) + R(t) = -\left(\frac{2}{a}A(t) + 2B(t) + v'(t)\left(v'(t) + \frac{4}{a}\right)\right) := -\Psi(t).$$

Since $\Psi(-\infty) = \lim_{t\to-\infty}\Psi(t) = 0$ and $\Psi'(t) = 4((N+1) - 1/a)B(t) > 0$, then $\Psi(t) > 0$ $\forall t \in \mathbb{R}$ and consequently:

$$\frac{d}{dt}\left(\frac{1}{Q(t)}R'(t)\right) + R(t) < 0, \quad \forall t \in \mathbb{R}. \tag{3.17}$$

Furthermore, from (3.16), we check that $R(t)$ changes sign exactly once, and more precisely, there exists $s_0 \in \mathbb{R}$ such that:

$$s_0 > t_- = t_-(v)\colon \quad v'(t_-) = -\frac{2}{a} \quad \text{and} \quad R(t) < 0 \quad \Leftrightarrow \quad t > s_0. \tag{3.18}$$

Furthermore,

$$\frac{dZ}{dt}(t) = \frac{2}{a} Q(t)\big(v'(t) + 2(N+1)\big), \qquad Z(-\infty) = -\beta\big(4(N+1) - \beta\big) < 0, \qquad Z(+\infty) = 0 \tag{3.19}$$

and

$$\frac{d}{dt}\left(\frac{1}{Q(t)} Z'(t)\right) + Z(t) = -\left(\frac{2}{a}A(t) + 2B(t) + \big(\beta + v'(t)\big)\big(v'(t) + 4(N+1) - \beta\big)\right) := -\Phi(t).$$

Since $\Phi(+\infty) = \lim_{t\to+\infty} \Phi(t) = 0$ and $\Phi'(t) = -4((N+1) - 1/a)A(t) < 0$ we find that $\Phi(t) > 0$ $\forall t \in \mathbb{R}$, and consequently:

$$\frac{d}{dt}\left(\frac{1}{Q(t)} Z'(t)\right) + Z(t) < 0, \quad \forall t \in \mathbb{R}. \tag{3.20}$$

Moreover, from (3.19), we see that

$$Z(t) > 0 \quad \forall t > t_+ = t_+(v): \quad v'(t_+) = -2(N+1). \tag{3.21}$$

Those information about $R(t)$ and $Z(t)$, allow us to show that $Y(t)$ cannot change sign in $\mathbb{R}$. Indeed, if by contradiction we assume that there exist values $t_1 \leqslant t_2$ such that

$$\big|Y(t)\big| > 0, \quad \forall t \in (-\infty, t_1) \cup (t_2, +\infty), \quad \text{and} \quad Y(t_1) = Y(t_2) = 0$$

then we can apply Proposition 2.1 in the interval $I_1 = (-\infty, t_1)$ with $y(t) = |Y(t)|$ and $z(t) = R(t)$ to obtain $s_1 \in I_1$: $R(s_1) < 0$. Thus from (3.18) we deduce:

$$t_- < s_1 < t_1 \quad \text{and} \quad R(s) < 0, \quad \forall s \geqslant s_1. \tag{3.22}$$

On the other hand, if we apply Proposition 2.1 in the interval $I_2 = (t_2, +\infty)$ with $y(t) = |Y(t)|$ and $z(t) = Z(t)$, we find $s_2 \in I_2$: $Z(s_2) < 0$. By (3.21) and (3.22) we have that $t_- < s_1 < t_1 \leqslant t_2 < s_2 < t_+$ and $R(s_2) < 0$, $Z(s_2) < 0$. In other words $s_2 \in (t_-, t_+) \equiv \Lambda(v)$ and both $R$ and $Z$ assume negative values at $s_2$, in contradiction with (3.15). So $Y(t)$ cannot change sign, as claimed. □

**Proposition 3.3.** *Let $N \geqslant 0$, $a > 1/(N+1)$ and $v = v(t)$ be a solution of* (2.13) *such that*

$$2N + (a-1)\beta \geqslant 0. \tag{3.23}$$

*Then problem* (3.10) *with $Q$ in* (3.11) *admits only the* trivial *solution $Y(t) \equiv 0$.*

**Proof.** Let

$$\tau = \frac{a(4(N+1) - \beta)}{4((N+1)a - 1)} \in (0, 1),$$

and define:

$$X(t) = \tau R(t) + (1-\tau) Z(t) + \frac{a\beta(\beta - 4/a)(4(N+1) - \beta)}{4((N+1)a - 1)}. \tag{3.24}$$

In view of (3.16) and (3.19), we easily check that

$$\frac{dX}{dt}(t)=\frac{2}{a}Q(t)\big(\tau\big(2+av'(t)\big)+(1-\tau)\big(2(N+1)+v'(t)\big)\big),$$
$$X(-\infty)=\lim_{t\to-\infty}X(t)=0,\qquad X(+\infty)=\lim_{t\to+\infty}X(t)=0.$$

So $X(t)$ admits exactly one critical point, say $t_0$, is increasing in $(-\infty,t_0)$ and decreasing in $(t_0,+\infty)$. In particular, $X(t)>0$, $\forall t\in\mathbb{R}$. Furthermore,

$$\frac{d}{dt}\left(\frac{1}{Q(t)}X'(t)\right)+X(t)=-\left(\frac{2}{a}A(t)+2B(t)+v'(t)\big(\beta+v'(t)\big)\right):=-\Lambda(t).$$

We check that assumption (3.23) implies that $\Lambda(t)<0$, $\forall t\in\mathbb{R}$. Indeed by straightforward calculations, we find:

$$\frac{d\Lambda}{dt}(t)=\big(4(N+1)-\beta\big)B(t)-\left(\beta-\frac{4}{a}\right)A(t),\qquad \Lambda(-\infty)=0=\Lambda(+\infty).$$

So, if $\bar{t}$ is a critical point of $\Lambda$ then it satisfies:

$$\big(4(N+1)-\beta\big)B(\bar{t})=\left(\beta-\frac{4}{a}\right)A(\bar{t}):=\bar{c}>0;$$

and

$$\frac{d^2\Lambda}{dt^2}(t)=\big(4(N+1)-\beta\big)B(t)\big(2(N+1)+v'(t)\big)-\left(\beta-\frac{4}{a}\right)A(t)\big(2+av'(t)\big)$$
$$=\bar{c}\big(2N+(a-1)\big|v'(t)\big|\big).$$

Clearly (3.23) implies that $\frac{d^2\Lambda}{dt^2}(\bar{t})>0$. Indeed, this is obviously the case when $a\geqslant 1$ (the condition $a>1/(N+1)$ rules out the possibility that simultaneously: $a=1$ and $N=0$). When $1/(N+1)<a<1$ then $2N+(a-1)|v'(t)|>2N+(a-1)\beta\geqslant 0$.

Consequently, $\Lambda$ can only admit a *unique* strict minimum and so $\Lambda(t)<0$, $\forall t\in\mathbb{R}$. In other words, under the given assumption:

$$\begin{cases}\dfrac{d}{dt}\left(\dfrac{1}{Q(t)}X'(t)\right)+X(t)>0,\\ X(t)>0,\quad X(-\infty)=0=X(+\infty).\end{cases}$$

Thus, by virtue of Propositions 2.1 and 3.2, we conclude that problem (3.10) with $Q(t)$ in (3.11) can only admit the trivial solution $Y(t)\equiv 0$. □

**Remark 3.1.** We observe that if $\beta$ satisfies (1.18), then (3.23) always holds when $N>0$ and $a\geqslant\frac{N+2}{2(N+1)}$ or $N=0$ and $a>1$.

**Proof of Theorems 1.2 and 1.3.** Under the assumptions (i) of Theorem 1.2, we see by Remark 3.1, that (3.23) holds. So, by recalling (2.32)–(2.36), we must have that necessarily $\beta'(\alpha)\neq 0$ $\forall t\in\mathbb{R}$, with $\beta(\alpha)$ the (smooth) function defined in (3.2). Therefore, by virtue of Proposition 3.1, it follows that $\beta'<0$ and the function $\beta(\alpha)$ is strictly monotone decreasing in $\mathbb{R}$. The desired conclusion in part (i) of Theorem 1.2 then follows by the uniqueness of (3.2), and the fact that the range of the function $\beta(\alpha)$ covers exactly once the range of $\beta$ in (1.18).

In the exact same way, uniqueness follows when $N=0$ and $a>1$ (see Remark 3.1).

Concerning part (ii) of Theorem 1.2, we see that the given assumptions on $a$ imply that $\beta_0 := \frac{2N}{1-a}$ satisfies (1.18) and (3.23). In view of Proposition 3.3 this implies that the equation $\beta(\alpha) = \beta_0$ admits a *unique* solution $\alpha_0 \in \mathbb{R}$, and $\beta(\alpha)$ is strictly decreasing in the interval $(\alpha_0, +\infty)$. Then, as above, we can assure the uniqueness of the radial solution of (2.2), for every $\beta \leqslant \beta_0$ satisfying (1.18).

At this point, we can use the duality properties (2.17), (2.18) and (2.19) in order to establish part (i) and part (ii) with $0 < a < 1$, of Theorem 1.3. □

It is a challenging open problem to see whether the uniqueness of radial solutions remains valid without the restriction $\beta \leqslant \frac{2N}{1-a}$.

On the other hand, the value of $\beta = \frac{2N}{1-a}$ assumes a special role in the solvability of (2.13). This fact emerged already in [8] (see Theorem 1.3). Indeed, in the following section we show that problem (2.2) with $\beta = \frac{2N}{1-a}$ admits a one-parameter family of non-radial solutions bifurcating from the (unique) radial one.

**Proof of Theorem 1.4, i.e. Theorem 1.5 of [18].** In the radial setting, problem (1.20), (1.22) and (1.23), reduces to problem (2.44) with $G(t) = e^{2t}K(e^t)$ satisfying the assumption of Proposition 2.2. Actually Proposition 2.2 provides the crucial information, as it corresponds to Lemma 3.3 of [18]. At this point, one arrives at the desired conclusion by following the arguments of [18]. □

## 4. Non-radial solutions

We are going to identify suitable pairs: $(a, N) \in (0, 1) \times (1, +\infty)$ such that problem (1.15) admits *non-radial* solutions satisfying (1.16) with $\beta = \frac{2N}{1-a}$.

To this purpose, let $0 < a < 1$, and for every $\beta_0 \in I(a) := (\max\{4, 4(1-a)/a\}, 4/a)$ denote by $u_0 = u_0(r)$ the *unique* radial solution for the problem:

$$\begin{cases} -\Delta v = \exp(av) + \exp(v) \quad \text{in } \mathbb{R}^2, \\ \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \left(e^{av} + e^{v}\right) dx = \beta_0, \end{cases} \tag{4.1}$$

see Theorems 1.2 and 1.3.

We use complex notation, and for $x = (x_1, x_2) \in \mathbb{R}^2$ we set $z = x_1 + ix_2$ and $u_0 = u_0(|z|)$.

Any other solution $u$ of (4.1) must satisfy: $u(z) = u_0(|z + \xi|)$ with $\xi \in \mathbb{C}$. For any $m \in \mathbb{N}$, we consider:

$$U(z) = u_0\left(\left|z^{m+1} + \xi\right|\right), \quad \xi \in \mathbb{C}; \tag{4.2}$$

then for $\xi \neq 0$, $U$ is *not* radially symmetric about any point and it satisfies:

$$\begin{cases} -\Delta U = (m+1)^2 |z|^{2m} \left(e^{aU} + e^{U}\right), \\ \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} (m+1)^2 |z|^{2m} \left(e^{aU} + e^{U}\right) = (m+1)\beta_0. \end{cases}$$

In turn, if we let

$$v(z) = U\left(\frac{z}{|z|^2}\right) + (m+1)\beta_0 \ln\left(\frac{1}{|z|}\right)$$

then $v$ can be extended smoothly at the origin, to satisfy:

$$\begin{cases} -\Delta v = (m+1)^2\left(|z|^{a(m+1)\beta_0-2(m+2)}e^{av} + |z|^{(m+1)\beta_0-2(m+2)}e^{v}\right), \\ \frac{1}{2\pi}(m+1)^2\int\limits_{\mathbb{R}^2}\left(|z|^{a(m+1)\beta_0-2(m+2)}e^{av} + |z|^{(m+1)\beta_0-2(m+2)}e^{v}\right) = (m+1)\beta_0. \end{cases}$$

For the particular choice of $\beta_0 = \frac{2(m+2)}{(m+1)a}$ we obtain a (non-radial) solution for the problem:

$$\begin{cases} -\Delta v = (m+1)^2\left(e^{av} + |z|^{2N}e^{v}\right), \\ \frac{1}{2\pi}(m+1)^2\int\limits_{\mathbb{R}^2}\left(e^{av} + |z|^{2N}e^{v}\right) = \frac{2N}{1-a} \end{cases} \tag{4.3}$$

with

$$N = N(m,a) = \frac{(m+2)(1-a)}{a}. \tag{4.4}$$

Hence, for all possible choices of:

$$0 < a < 1 \quad \text{and} \quad m \in \mathbb{N}\colon \frac{2(m+2)}{(m+1)a} \in I(a) \tag{4.5}$$

we obtain a 1-parameter family of non-radial solution of (4.3), with $N > 0$ given in (4.4). Notice also that in this situation, the linearized problem around the *unique* radial solution (corresponding to the choice $\xi = 0$ in (4.2)), admits a nontrivial kernel of bounded $\theta$-depending functions given by: $u_0'(r)\cos(\theta)$ and $u_0'(r)\sin(\theta)$.

Concerning the validity of (4.5), we see that it holds if

$$a = 1/2 \quad \text{and} \quad \forall m \in \mathbb{N}, \quad \text{so that} \quad N = m+2 \in \mathbb{N}\cap[3,+\infty), \tag{4.6}$$

$$0 < a < 1/2 \quad \text{and} \quad 1 \leqslant m < \frac{2a}{1-2a}, \tag{4.7}$$

$$1/2 < a < 1 \quad \text{and} \quad 1 \leqslant m < \frac{2(1-a)}{2a-1}; \tag{4.8}$$

and the conditions (4.7) and (4.8) are related via the transformation: $a \to (1-a)$.

From a direct inspection of (4.7) and (4.8) we derive that: for $1/4 < a \neq 1/2 < 3/4$, there exists $m_a \in \mathbb{N}$: $m_a = m_{1-a}$, $m_a \to +\infty$ as $a \to 1/2$, and (4.5) holds for every $m \in \{1,\dots,m_a\}$.

Consequently, for $a = 1/2$, problem (4.3) exhibits a symmetry breaking phenomenon for every $N \in \mathbb{N}$, $N \geqslant 3$; quite similar to what occurs for problem (1.10) when $N \in \mathbb{N}$, see [19].

While for $1/4 < a \neq 1/2 < 3/4$, such a brake of symmetry can occur only for finite values of $N$ which are given by (4.4), with $m = 1,\dots,m_a$. Notice in particular that such "admissible" $N$'s, are always larger than 1, and can be made as close to 1 as wanted by letting $a \to 3/4$.

This leads us to formulate the following conjecture:

$$\begin{aligned} &\text{if } 0 < N \leqslant 1 \quad \text{or} \quad N > 0 \quad \text{and} \quad a \geqslant 1, \\ &\text{then every solution of problem (1.15) is radially symmetric about the origin.} \end{aligned} \tag{4.9}$$

A positive answer to (4.9), would imply in particular that, for $N > 0$ and $a \geqslant 1$, the solvability of problem (1.15)–(1.16) is fully described by Theorems 1.2 and 1.3.

To support the above conjecture, let us consider the case $a = 1 = N$, given by the problem

$$\begin{cases} -\Delta v = \left(1+|z|^2\right)e^v \quad \text{in } \mathbb{R}^2, \\ \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2} \left(1+|z|^2\right)e^v = \beta, \end{cases} \tag{4.10}$$

with $\beta \in (4,8)$. By direct calculations one can check that the function: $u_*(x) = u_*(|x|) = \ln(12/(1+|x|^2)^3)$ defines the *unique* radial solution of problem (4.10) with $\beta = 6$. A recent result of Ghoussoub and Lin in [17], shows that indeed $u_*$ is the *unique* solution for (4.10), when $\beta = 6$. While in [6] it is shown that there exists $\beta_0 \in (4,8)$ such that, for any $\beta \in (\beta_0, 8)$ problem (4.10) admits only radially symmetric solutions.

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