ENUMERATION OF HIGHER-DIMENSIONAL PATHS UNDER RESTRICTIONS

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The enumeration of lattice paths lying between two boundaries in two dimensional space has been done and the explicit expression is a determinant. By considering a natural extension of the boundaries in higher dimensional space, a generalized recurrence relation is established, the solution of which gives the number of paths in higher dimension not crossing the boundaries, in a determinant form.

1. Introduction

It is known that the number of 2-dimensional lattice paths lying between any two boundaries can be explicitly expressed in the form of a determinant. The proof in [7] (one boundary case in [6]) utilizes the elementary property of determinants as suggested in [10]. Interestingly, an effective use of such determinantal properties leads to a more general counting result on Young chains [1, p. 69; 4, p. 64]. Never-the-less, one may not find in a natural way, an analogous approach to count the number of higher dimensional paths between two boundaries.

The present paper deals with the problem of counting lattice paths from the origin to a point \((n_0, n_1, \ldots, n_k)\), under restrictions. The number of such paths without restrictions is the multinomial coefficient

\[
\binom{\sum_{i=0}^{k} n_i}{n_0, n_1, \ldots, n_k} = \frac{(\sum_{i=0}^{k} n_i)!}{\prod_{i=0}^{k} n_i!},
\]

which we denote by \(\binom{\sum_{i=0}^{k} n_1}{n_1, \ldots, n_k}\).

In [3, 7], the expression for the number of paths from the origin to \((n_0, n_1, \ldots, n_k)\), \(n_0 \geq \sum_{i=1}^{k} \mu_i n_i\) (\(\mu_i\)'s being non-negative integers), not crossing the hyperplane \(x_0 = \sum_{i=1}^{k} \mu_i x_i\) is obtained as

\[
\frac{a + 1 - \sum_{i=1}^{k} \mu_i n_i}{a + 1 + \sum_{i=1}^{k} n_i} \binom{a + 1 + \sum_{i=1}^{k} n_i}{n_1, \ldots, n_k}.
\]
Here, we define more general boundaries that paths are not allowed to cross and establish a recurrence relation which gives rise to an orthogonal relation between two triangular matrices and enables us to obtain a determinantal expression for the number of higher dimensional paths with the above restrictions. For 2-dimensional paths, this technique is suggested in [2], which also appears in [8].

2. Higher dimensional paths

In \((k + 1)\)-dimensional space with axes \(X_0, X_1, \ldots, X_k\), consider lattice paths from the origin to \((n_0, n_1, \ldots, n_k)\), \(n_i \geq 0\) for all \(i\). By the \(r = (r_1, \ldots, r_k)\)th level, where \(r_i = 0, 1, \ldots, n_i, n_i \geq 0, i = 1, \ldots, k\), we mean the set of points \(\{(x_0, n_1 - r_1, \ldots, n_k - r_k): 0 \leq x_0 \leq n_0\}\). We say that a path reaches the \(r\)th level if it passes through one of the points in that level. Note that a path may or may not reach a particular level. Also if it reaches the \(r\)th level it may pass through several of its points before reaching either one of the levels \((r_1, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_k)\), \(i = 1, \ldots, k\). Let

\[
x(r) = \begin{cases} 
    n_0 - \min \{x_0\} & \text{if the path reaches the } r \text{th level} \\
    \text{and } (x_0, n_1 - r_1, \ldots, n_k - r_k) \text{ is a point on the path,} \\
    0 & \text{otherwise.}
\end{cases}
\]

Clearly \(0 \leq x(r) \leq n_0\), and \(x(n) = n_0\).

The representation of a path is done inductively. We know that [9] a 2-dimensional path from the origin to \((n_0, n_1)\) has a non-decreasing vector representation which will be slightly modified due to the new notation. The modified vector notation representation is \((x(0), x(1), \ldots, x(n_1))\). Now consider 3-dimensional paths. Since such a path from the origin to \((n_0, n_1, n_2)\) intersects with the plane \(X_i = i, i = 0, 1, \ldots, n_i\) in a 2-dimensional path (including a degenerate one, viz. a point), it consists of a string of \((n_i + 1)\) 2-dimensional path segments successively lying in planes \(X_i = i, i = 0, 1, \ldots, n_i\), such that the initial point of the path segment on each of the planes is joined by the terminal point of the path segment in the preceding plane by a step along \(X_i\)-axis. Thus any path can be uniquely represented by the matrix

\[
\begin{pmatrix}
x(0, 0), x(0, 1), \ldots, x(0, j_1) \\
x(1, j_1), x(1, j_1 + 1), \ldots, x(1, j_2) \\
\vdots \\
x(n_1, j_n), x(n_1, j_n + 1), \ldots, x(n_1, n)
\end{pmatrix}
\]

with \(x(n_1, n_2) = n_n, 0 = j_0 \leq j_1 \leq \cdots \leq j_n = j_{n+1} = n_2\), where the vector \((x(i, j_i), x(i, j_i + 1), \ldots, x(i, j_{i+1}))\) in the \((i + 1)\)th row corresponds to the 2-dimensional path segment on the plane \(X_i = i\). Alternatively, we observe that to a path there corresponds a non-decreasing vector with \((n_1 + n_2 + 1)\) non-negative
integer elements arranged along a path from \((0, 0)\) to \((n_1, n_2)\). By an extension of the above representation, it is not difficult to see that to each path from the origin to \((n_0, n_1, \ldots, n_k)\), there corresponds a unique non-decreasing vector with \((n_1 + \cdots + n_k + 1)\) non-negative integer elements arranged along a unique path from the origin to \((n_1, \ldots, n_k)\). If the path in \((k + 1)\)-dimension passes through the \(r\)th level, then the \((r_1 + \cdots + r_k + 1)\)th element is \(x(r)\).

To consider restricted paths, we denote by \(a(r)\) and \(b(r)\) the upper and lower restrictions respectively at the \(r\)th level such that the path at the \(r\)th level can pass through only points in the set

\[
\{(x_0, n_1 - r_1, \ldots, n_k - r_k) : 0 \leq b(r) \leq n_0 - x_0 \leq a(r) \leq n_0\}.
\]

The sets

\[
A(n) = \{a(r) : 0 \leq r_i \leq n_i, i = 1, \ldots, k\}
\]

and

\[
B(n) = \{b(r) : 0 \leq r_i \leq n_i, i = 1, \ldots, k\}
\]

are respectively called the upper and lower restrictions on the path. Obviously, we have to set \(a(n) \geq n_0\) and \(b(0) = 0\), where \(0\) is the \(k\)-vector with all zeros. In the following, we assume that the restrictions on a path are such that for every \(r\), \(a(r)\) and \(b(r)\) are non-decreasing in any coordinate, i.e.,

\[
a(r_1, \ldots, r_k) \leq a(r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_k)
\]

and

\[
b(r_1, \ldots, r_k) \leq b(r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_k)
\]

for \(j = 1, \ldots, k\).

3. Main results

For simplicity and clarity, we only consider paths with upper restriction as we would see that the case with both restrictions would not entail any special difficulty. Observe that \(0 \leq x(r) \leq a(r)\) in this case. Denote by \(g_k(A(n))\) the number of paths from the origin to \(n\) with the upper restriction or simply the restrictions \(A(n)\). In the next theorem, we give a recurrence relation on \(g_k(A(n))\) which is vital for the enumeration of paths.

Theorem 1.

\[
\sum_{0 \leq r \leq n} (-1)^{(a(r) + 1)_1} \binom{a(r) + 1}{n - r} g_k(A(r)) = \delta_0^n,
\]

(3)
where \( \mathbf{1} \) is the \( k \)-vector with all ones,

\[
\begin{pmatrix}
(a(r)+1) \\
n - r
\end{pmatrix} = \begin{pmatrix}
(a(r)+1) \\
n_1 - r_1, \ldots, n_k - r_k
\end{pmatrix} = \frac{(a(r)+1)(a(r)) \cdots (a(r)-\sum_{i=1}^{k} (n_i + r_i) + 2)}{\prod_{i=1}^{n} (n_i - j_i)!},
\]

\[
\delta_m^n = \begin{cases}
1 & \text{when } m = n, \\
0 & \text{otherwise,}
\end{cases}
\]

and the vector ordering \( x \leq y \) means \( x_i \leq y_i \) for every \( i \).

**Proof.** We give the proof for \( k = 3 \) as the general proof is analogous, except that it would involve complicated notations. Recall that \( g_3(A(n_1, n_2, n_3)) \), represents the number of non-decreasing \((n_1 + n_2 + n_3 + 1)\)-vectors arranged along 3-dimensional paths from the origin to \((n_1, n_2, n_3)\), subject to the restriction that

\[0 \leq x(r_1, r_2, r_3) \leq a(r_1, r_2, r_3), \quad r_i = 0, 1, \ldots, n_i, \quad i = 1, 2, 3.\]

For a given arrangement in a 3-dimensional path, the number of non-decreasing \((n_1 + n_2 + n_3 + 1)\)-vectors is \( g_s(A(n_1 + n_2 + n_3 + 1)) \), where

\[A(n_1 + n_2 + n_3 + 1) = \{a(s, t, u) : s = 0, 1, \ldots, n_1, t = i_s, i_s + 1, \ldots, i_{s+3}, u = j_s + 1, j_s + 1 + 1, \ldots, j_s + 1 + 1 + 1, j_s + 1 + 1 + 1 + 1 = i_3\}.\]

Note that by fixing \((i_1, \ldots, i_{n_1})\) and \((j_1, \ldots, j_{n_1+n_3})\), we get the arrangement on a particular 3-dimensional path. Thus, when we vary both sets \((i_1, \ldots, i_{n_1})\) and \((j_1, \ldots, j_{n_1+n_3})\) such that \(0 \leq i_1 \leq \cdots \leq i_{n_1} \leq n_2\) and \(0 \leq j_1 \leq \cdots \leq j_{n_1+n_3} \leq n_3\), we get all arrangements of \((n_1 + n_2 + n_3 + 1)\)-vectors along 3-dimensional paths. The above discussion leads to

\[g_s(A(n_1, n_2, n_3)) = \sum_{R_1} \sum_{R_2} g_s(A(n_1 + n_2 + n_3 + 1)), \quad (4)\]

where

\[R_1 = \{(i_1, \ldots, i_{n_1}) : 0 \leq i_1 \leq \cdots \leq i_{n_1} \leq n_2\},\]

and

\[R_2 = \{(j_1, \ldots, j_{n_1+n_3}) : 0 \leq j_1 \leq \cdots \leq j_{n_1+n_3} \leq n_3\}.
\]

But for \( k = 1 \), it is known [6] from the determinantal solution or otherwise that

\[g_s(a(0), a(1), \ldots, a(n)) = \sum_{r=0}^{n-1} (-1)^{n-r-1} \binom{a(r)+1}{n-r} g_s(a(0), a(1), \ldots, a(r)). \quad (5)\]
Combining (4) and (5), we have

\[ g_3(A(n_1, n_2, n_3)) = \]

\[ = \sum_{R_1} \sum_{R_2} \sum_{s=0}^{n_1} \sum_{t=0}^{n_2} \sum_{u=0}^{n_3} \left[ (-1)^{n_1+n_2+n_3-s-t-u} \left( a(s, t, u) + 1 \right) \frac{a(s, t, u) + 1}{n_1 + n_2 + n_3 - s - t - u} \right] \]

\[ \times [g_1(a(0, 0, 0), \ldots, a(s, t, u))]. \] \tag{6}

Consider the summation \( \sum_{R_1} \sum_{u} \) which is equivalent to

\[ \sum_{u=0}^{n_3} \sum_{0 \leq j_1 \leq \ldots \leq j_{n_2+1} \leq u} \sum_{u=0}^{n_2-1} \sum_{u=0}^{n_3-1} \sum_{s+t+u=0}^{n_1+n_2+n_3} \]

Since the term under summation is independent of \( j_5 + t + 1, \ldots, j_{n_1+n_3} \), the summation over \( R_2 \) and \( u \) becomes

\[ \sum_{u=0}^{n_3} \sum_{0 \leq j_1 \leq \ldots \leq j_{n_2+1} \leq u} \left[ (-1)^{n_1+n_2+n_3-s-t-u} \left( a(s, t, u) + 1 \right) \frac{a(s, t, u) + 1}{n_1 + n_2 + n_3 - s - t - u} \right] \]

\[ \times g_1(a(0, 0, 0), \ldots, a(s, t, u)) \left[ \sum_{u=0}^{n_3} \sum_{0 \leq j_1 \leq \ldots \leq j_{n_2+1} \leq u} \right]. \] \tag{7}

But the last factor simplifies to

\[ \left( \frac{n_1 + n_2 + n_3 - s - t - u}{n_1 + n_2 - s - t} \right). \]

Similar simplification can be made for the summation over \( R_1 \) and \( t \). On implementing these simplifications in (6), we get

\[ \sum_{s=0}^{n_1} \sum_{t=0}^{n_2} \sum_{u=0}^{n_3} \left[ (-1)^{n_1+n_2+n_3-s-t-u} \left( a(s, t, u) + 1 \right) \frac{a(s, t, u) + 1}{n_1 + n_2 + n_3 - s - t - u} \right] \]

\[ \times \left[ \sum_{0 \leq i_1 \leq \ldots \leq i_{n_3+1} \leq u} \sum_{s+t+u=0}^{n_1+n_2+n_3} g_1(a(0, 0, 0), \ldots, a(s, t, u)) \right], \tag{7} \]

because

\[ \left( \frac{a(s, t, u) + 1}{n_1 + n_2 + n_3 - s - t - u} \right) \left( \frac{n_1 + n_2 + n_3 - s - t - u}{n_1 + n_2 - s - t} \right) \left( \frac{n_1 + n_2 - s - t}{n_1 - s} \right) = \left( \frac{a(s, t, u) + 1}{n_1 - s, n_2 - t, n_3 - u} \right). \]

However,

\[ \sum_{0 \leq i_1 \leq \ldots \leq i_{n_3+1} \leq u} q_i(a(0, 0, 0), \ldots, a(s, t, u)) = g_3(A(s, t, u)), \tag{8} \]

by using (4). Now we can readily check that from (6), (7), and (8) follows (3) for \( k = 3 \). The general proof is similar and this completes the proof.

Let

\[ A \left( \mathbf{m} \right) = \{ a(r) : m_i \leq r_i \leq n_i, \ i = 1, \ldots, k \}. \]
Let $g_k(A(n))$ denote the paths from the origin to the point $(n_0, n_1 - m_1, \ldots, n_k - m_k)$ with the restriction \{a(m + r): 0 \leq r \leq n_i - m_i, i = 1, \ldots, k\}. In this notation, (3) is equivalent to

$$\sum_{r \leq r \leq n} (-1)^{n-r} \binom{a(r)+1}{n-r} g_k \left( A \left( \binom{m}{r} \right) \right) = \delta_{n}^{n}. \tag{9}$$

We introduce an ordering $\preceq$ on vectors different from $\leq$, in which $x \preceq y$ means that for at least one $i, x_i < y_i, i = 1, \ldots, k$. Thus, it is possible that simultaneously $x \preceq y$ and $y \preceq x$.

Consider the set $\{r: 0 \leq r \leq n\}$ of $k$-vectors. Denote by $d$ its cardinality. Obviously $d = \prod_{i=1}^{k} (n_i + 1)$. Let $u_1, \ldots, u_d$ be an arrangement of this set of vectors such that $0 = u_1 u_2 \cdots u_d = n$. There are several ways of ordering the set. Denote by $M_k$ and $G_k$ two upper triangular matrices of order $d$ as given below:

$$G_k = (-1)^{u_i - u_i} \cdot g_k \left( A \left( \binom{u_i}{u_i} \right) \right) \quad \text{and} \quad M_k = \left( a(u_i) + 1 \right) \frac{u_i}{u_i - u_i} \tag{10}$$

where the $(i, j)$th element of each matrix is given, $i, j = 1, \ldots, d$.

**Corollary 1.**

$$G_k M_k = I. \tag{11}$$

**Proof.** The $(i, j)$th element of $G_k M_k$ is given by

$$\sum_{r=1}^{d} (-1)^{u_i - u_i} g_k \left( A \left( \binom{u_i}{u_i} \right) \right) \left( a(u_i) + 1 \right) \frac{u_i}{u_i - u_i}. \tag{12}$$

Observing that

$$\begin{pmatrix} a(u_i) + 1 \\ u_i - u_i \end{pmatrix} = 0$$

unless $u_i \preceq u_i$, and

$$g_k \left( A \left( \binom{u_i}{u_i} \right) \right) = 0$$

unless $u_i \preceq u_i$ and using (9) in (12), we get

$$\sum_{u_i, u_i, u_i \preceq u_i} (-1)^{u_i - u_i} g_k \left( A \left( \binom{u_i}{u_i} \right) \right) \left( a(u_i) + 1 \right) \frac{u_i}{u_i - u_i} = \delta_{u_i}^{u_i}. \tag{13}$$

The proof is complete, when we note that $u_i = u_i$ if and only if $i = j$.

The next corollary gives $\varepsilon$: explicit expression for $g_k(A(n))$.

**Corollary 2.**

$$g_k(A(n)) = (-1)^{d - \sum_{i=1}^{d} n_i - 1} \det \left| \frac{a(u_i) + 1}{u_{i+1} - u_i} \right|, \quad i, j = 1, \ldots, d - 1. \tag{13}$$
Proof. Clearly $g_k(A(n)) = g_k(A(\emptyset))$. From Corollary 1, we see that

$$(-1)^{l_1 - m} g_k\left(A\left(\emptyset^l\right)\right)$$

is equal to the $(1, d)$th element of $M_k^1$. Some minor simplification helps us to complete the proof.

Taking into account both restrictions, let $g_k(A(n) \mid B(n))$ be the number of paths with upper restrictions $A(n)$ and lower restriction $B(n)$. We know that $x(r) = a(r)$. However, the non-decreasing condition on $b(r)$ leads us to $x(r) \geq b(r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_k)$ if the path has moved from the $(r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_k)$th level $(i = 1, \ldots, k)$ to the $r$th level. Also, $x(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k) \geq b(n)$ for every $i$. For $k = 1$, it is easy to see that $b(r + 1) \leq x(r) \leq a(r), r = 0, 1, \ldots, n - 1$.

From the determinantal solution [7], we note that

$$g_1(a(0), a(1), \ldots, a(n) \mid b(0), b(1), \ldots, b(n)) = \sum_{r=0}^{n-1} (-1)^{n-r-1}
\times \binom{a(r) - b(n) + 1}{n-r} g_1(a(0), a(1), \ldots, a(r) \mid b(0), b(1), \ldots, b(r))$$

(14)

where $(\cdot)_{+}$ is the usual binomial coefficient except that $(\cdot) = 0$ if $y < z$.

Proceeding similarly as in the case of upper restriction and using (14), we obtain the following generalization of (3).

Theorem 2.

$$\sum_{0 \leq r \leq n} (-1)^{(n-r)-1} \binom{a(r) - b(n) + 1}{n-r} g_k(A(r) \mid B(r)) = \delta_0^n.$$  

(15)

If we introduce triangular matrices analogous to $G_k$ and $M_k$ with both lower and upper restrictions, the orthogonality of (11) can be established which in turn leads to the expression for $g_k(A(n) \mid B(n))$.

Corollary 3.

$$g_k(A(n) \mid B(n)) = (-1)^{d-\sum n_i-1} \det \left[ \binom{a(u_i)-b(u_{i+1})+1}{u_{i+1}-u_i} \right]_{i=1}^n.$$ 

(16)

4. Computation and a special case

One way of choosing $u_1, \ldots, u_d$ is to let

$$u_i + (n_1 + 1)i_2 + (n_1 + 1)(n_2 + 1)i_3 + \cdots + (n_1 + 1) \cdots (n_{i-1} + 1)i_i = (i_1, i_2, \ldots, i_k)$$
for \(0 \leq i \leq n, r = 1, \ldots, k\). (When \(k = 1\), the natural ordering of \((0, 1, \ldots, n)\) determines \(u_1, \ldots, u_n\), viz. \(u_1 = 0, u_2 = 1, \ldots, u_{n+1} = n\).) Using this ordering let us find the number of lattice paths from \((0, 0, 0)\) to \((5, 1, 2)\), not crossing the surfaces determined by the equations \(x = 3.7y\) and \(x = z^2 - 1\). Here the lower restriction \(B(1, 2)\) on the path does not exist which is equivalent to saying \(b(i, j) = 0\) for \(i = 0, 1\) and \(j = 0, 1, 2\), whereas the upper restriction \(A(1, 2)\) on the path is given by

\[
A(1, 2) = \begin{pmatrix}
a(0, 0) = 1 & a(0, 1) = 1 & a(0, 2) = 1 \\
(a(1, 0) = 2 & a(1, 1) = 5 & a(1, 2) = 5
\end{pmatrix}.
\]

We take \(u_1 = (0, 0), u_2 = (1, 0), u_3 = (0, 1), u_4 = (1, 1), u_5 = (0, 2),\) and \(u_6 = (1, 2)\). From (13), we have

\[
g_2(A(1, 2)) = (-1)^{6-3-1} \det \left| \begin{array}{cccc}
(a(u_i) + 1) & (a(u_i) + 1) & (a(u_i) + 1) & (a(u_i) + 1) \\
1 & 0 & 1 & 1, 2 \\
1 & 0 & 1 & 2 \\
0 & 1 & 0 & 1, 0 \\
0 & 0 & 1 & 0, 1 \\
0 & 0 & 1 & 2
\end{array} \right| = 44.
\]

The derivation of (1) has been done either by the usual generating function method [3] or by a combinatorial argument [7]. Let us rederive it from (13). First of all, we note that

\[
a(r) = n_0 - \sum_{i=1}^{k} \mu_i(n_i - r_i), \quad 0 \leq r \leq n,
\]

and therefore from (13) the required number is

\[
(-1)^{d-\sum_{i=1}^{k} n_i-1} \det \begin{vmatrix}
(n_0 + 1 - \mathbf{u} \cdot (n - u_i)) \\
u_{i+1} - u_i
\end{vmatrix}, \quad (17)
\]
which can be simplified by using the convolution formula (10) in [5] which can be written as the following:

\[
\sum_{0\leq k\leq n} \frac{\alpha}{\alpha + \beta \cdot (n-k)} \left( \frac{\alpha + \beta \cdot (n-k)}{n-k} \right) \left( \gamma + \beta \cdot k \right) = \left( \alpha + \gamma + \beta \cdot n \right) \tag{18}
\]

for any \(\alpha, \beta\) and any vector \(\beta\). Multiply the \(i\)th row of the determinant in (17) by

\[
\frac{a+1-\mu \cdot n}{a+1-\mu \cdot n + \mu \cdot u_i} \left( -(a+1)+\mu \cdot (n-u_i) \right)
\]

\(i = 2, \ldots, d-1\) and add to the first row. Then the element in the first row and \(j\)th column, \(j=1, \ldots, d-2\) becomes

\[
\frac{a+1-\mu \cdot n}{a+1} \sum_{i=1}^{d-1} \frac{a+1}{a+1-\mu \cdot (n-u_i)} \left( a+1-\mu \cdot (n-u_i) \right) u_{i+1} - u_i \times \left( -(a+1)+\mu \cdot n - \mu \cdot u_i \right).
\tag{19}
\]

In view of the fact that

\[
\left( a+1-\mu \cdot (n-u_i) \right) = 0
\]

unless \(u_i \leq u_{i+1}, \alpha u_d-1\) and that in the sequence \((u_1, \ldots, u_n)\) every \(u_i\) precedes \(u_{i+1}\) if \(u_i \leq u_{i+1}\), we find \(\Sigma_{i=1}^{d-1}\) simplifies to \(\Sigma_{0\leq u_i \leq u_{i+1}}\). Now by using (18), it can be checked that the summation in (19) equals zero. Next, we consider the element in the first row and the last column, which can be written as

\[
\frac{a+1-\mu \cdot n}{a+1} \sum_{i=1}^{d} \frac{a+1}{a+1-\mu \cdot (n-u_i)} \left( a+1-\mu \cdot (n-u_i) \right) u_{i+1} - u_i \times \left( -(a+1)+\mu \cdot n - \mu \cdot u_i \right) \frac{a+1-\mu \cdot n}{a+1} \left( -a-1 \right).
\tag{20}
\]

Note that \(\Sigma_{i=1}^{d}\) is the same as \(\Sigma_{0\leq u_i \leq n}\). Again by using (18), the summation part in (20) is zero. Thus the first row of the determinant in (17) gets simplified to

\[
\left( 0, 0, \ldots, 0, \frac{a+1-\mu \cdot n}{a+1} \left( -a-1 \right) \right).
\]

Expanding the determinant by the first row and the last column, we find (17) is reduced to

\[
(-1)^{\Sigma_{i=1}^{d}} \frac{a+1-\mu \cdot n}{a+1} \left( -a-1 \right) = \frac{a+1-\mu \cdot n}{a+1+n \cdot 1} \left( a+1+n \cdot 1 \right)
\]

which checks with (1).
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