The Pseudo-Integral of a System of Differential Equations*

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I. INTRODUCTION

First we shall briefly review what is meant by saying that a function is an integral for a system of differential equations. Next we shall introduce a generalization of this idea by modifying two conditions on the function to obtain what in this paper shall be called the pseudo-integral. Finally we shall apply this new concept to some problems in celestial mechanics to demonstrate its usefulness in obtaining information about the solution of a system of differential equations.

II. INTEGRAL

Let \( \Gamma \) be the system of differential equations

\[
\dot{x} = f(x, t)
\]

where \( x \) belongs to \( E^n \), an \( n \)-dimensional Euclidean space. \( t \) belongs to \( T \), a one-dimensional Euclidean space. \( f^i \) and \( \frac{\partial f^i}{\partial x^j} \) for \( i, j = 1, 2 ... n \) are continuous in a domain \( D' \) of \( E^n \times T \) containing the point \( (\dot{x}, \dot{t}) \), and the dot represents a derivative with respect to \( t \).

From the Theory of Differential Equations there exists domains \( S \) in \( E^n \) and \( I \) in \( T \) containing \( \dot{x} \) and \( \dot{t} \) respectively, and a function \( x(w, t) \) defined in \( S \times I \) such that

(i) The set of points \( \{x(w, t), t\} \) for all \( (w, t) \) in \( S \times I \) is a domain \( D \) contained in \( D' \).

(ii) \( x = x(w, t) \) is the unique solution to \( \Gamma \) in \( I \) for which \( w = x(w, \dot{t}) \).

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(iii) \( w = w(x, t) \) the inverse of \( x = x(w, t) \) is defined and continuous for all points of \( D \).

The function \( F = F(x, t) \) is called an integral of \( I \) if (i) \( F(x, t) \) has continuous first partial derivatives in \( D \), and (ii) \( F(x(w, t), t) = F(w, t) \) identically in \( S \times I \). Assuming (i), (ii) is equivalent to the following condition:

\[
\frac{dF}{dt} = \frac{\partial F}{\partial x^i}(x(w, t), t) f^i(x(w, t), t) + \frac{\partial F}{\partial t}(x(w, t), t) = 0
\]

identically in \( S \times I \) where \( \frac{\partial F}{\partial t} \) and \( \frac{dF}{dt} \) are the derivatives of \( F \) with respect to \( t \) keeping \( x \) and \( w \) fixed respectively.

Substituting \( w = w(x, t) \) into this expression we obtain

\[
\frac{\partial F}{\partial x^i}(x, t) f^i(x, t) + \frac{\partial F}{\partial t}(x, t) = 0
\]

identically in \( D \). Therefore, if is not necessary to know the general solution of \( I \) in order to determine whether a given function is an integral. This coupled with the fact that for the particular solution \( x = x(w_0, t) \), where \( w_0 \) belongs to \( S \),

\[
F(x, t) = F(w_0, t)
\]

makes this concept very useful.

III. THE PSEUDO-INTEGRAL

Let us consider the function \( F(x, w, t) \) such that (i) for any value of \( w \) in \( S \), \( \frac{\partial F}{\partial x^i} \) and \( \frac{\partial F}{\partial t} \) are continuous in \( D \) and (ii) \( F(x(u, t), u, t) = F(u, u, t) \) identically for all \( t \) in \( I \) and all \( u \) in \( U \), where \( U \) is the set of all \( w \) in \( S \) that satisfy the \( k \) (where \( k \) may be 0) equations of constraints:

\[
h_j(w) = 0 \quad \text{for} \quad j = 1, 2, \ldots, k.
\]

If \( k = 0 \) and \( F \) does not depend explicitly on \( w \) then \( F \) is a pseudo-integral if and only if it is an integral.

For convenience we shall call \( G(x, w, t) \) a normalized pseudo-integral if it is a pseudo-integral and if \( G(u, u, t) = 0 \) identically for all \( u \) in \( U \). Given a pseudo-integral \( F(x, w, t) \) we can obtain a normalized pseudo-integral \( G(x, w, t) \) by defining \( G \) as follows:

\[
G(x, w, t) = F(x, w, t) - F(w, w, t).
\]

Before we consider specific examples, let us indicate in general what our
procedure shall be. For the purpose of this discussion, we shall assume that for any \( w \) in \( S \) all functions of \( x, w, \) and \( t, \) including \( f(x, t), \) have sufficiently many derivatives with respect to \( x \) and \( t \) that are continuous in \( D. \) Given a function \( G(x, w, t) \) let us define \( G_0(x, w, t) = G(x, w, t) \) and

\[
G_l(x, w, t) = \frac{\partial G_{l-1}}{\partial x} f^l + \frac{\partial G_{l-1}}{\partial t}
\]

where \( l \) is any integer greater than 0. Thus

\[
G_l(x(w, t), w, t) = \frac{d^lG}{dt^l} (x(w, t), w, t)
\]

i.e. \( G_l \) is the \( l \)th derivative of \( G \) along the solution curves of \( \Gamma. \) \( G \) is a normalized pseudo-integral of \( \Gamma \) only if \( G_p(x(u, t), u, t) = 0 \) for all \( p \geq 0 \) and all \((u, t)\) in \( U \times I. \) One might expect that the infinite system of equations \( G_p(x, u, t) = 0 \) for all \( q \geq 0 \) would impose too many conditions on the variables \( x, u \) and \( t, \) i.e., would have no solution. However, if there is an integer \( s \) such that\(^1\)

\[
G_s(x, u, t) = \sum_{p=0}^{s-1} a_p(x, u, t) G_p(x, u, t)
\]

and a value of \((x, u, t)\) for which \( G_s(x, u, t) = 0 \) if \( p < s \); then for all \( q \) and the same \((x, u, t), G_p(x, u, t) = 0. \) If in addition \( G_p(u, u, t) = 0 \) identically for \( p < s \) and all \( u \) in \( U, G \) is a normalized pseudo-integral since it satisfies the ordinary linear differential equation

\[
\frac{d^sG}{dt^s} = \sum_{p=0}^{s-1} a_p(x(u, t), u, t) \frac{d^pG}{dt^p}
\]

containing the parameter \( u \) and subject to the initial conditions

\[
\frac{d^pG}{dt^p} (u, u, t) = 0 \quad \text{for} \quad p = 0, 1 \ldots s - 1.
\]

We see that if \( G(x, w, t) \) is a normalized pseudo-integral so is \( G_p(x, w, t) \) for any \( p \) and the same constraints. In general a set of functions \( \{G^a\} \) where \( a \) belongs to some set are called simultaneous pseudo-integrals if they are all pseudo-integrals for the same set of constraints. In practice the equations of constraints shall not usually be given a priori. Rather, if there exists a relationship of the form

\[
G_s(x, w, t) = \sum_{p=0}^{s-1} a_p(x, w, t) G_p(x, w, t)
\]

\(^1\) For example if \( G_{s-1} = H(G_0, G_1, \ldots, G_{s-2}), \) we may take

\[
a_p = \frac{\partial H}{\partial G_{p-1}} (G_0(x, u, t), G_1(x, u, t), \ldots, G_{s-2}(x, u, t))
\]

for \( 1 \leq p \leq s - 1 \) and \( a_0 = 0.\)
then the equations of constraints can be taken to be $G_p(w, w, t) = 0$ for $p < s$, if these equations have a solution. Also in practice we may consider a function $G(x, w, a_1, a_2, \ldots a_m, t)$ where $a_i$ for $i = 1, 2 \ldots m$ is a function of $w$ to be so chosen that if possible $G$ is a pseudo-integral. These ideas shall be made more concrete in the following examples.

IV. Examples

Let

$$\ddot{r}_i = -g(m + m_i) \frac{r_i}{|r_i|^3} - g \sum_{i=1}^{k} \frac{m_i}{|r_i|^3} \left( \frac{|r_i - r_i|}{|r_i|^3} + \frac{r_i}{|r_i|^3} \right)$$

where $r_i$ is a vector in a plane with coordinates $(x_i, y_i)$, and $m, m_i$, and $g$ are greater than 0 for $i = 1, 2 \ldots k$. These are the equations of motion of $k + 1$ bodies in a coordinate system with origin on the body of mass $m$ and axes parallel to an inertial system, under the assumptions that the bodies always lie in a plane and the only forces are the gravitational interaction between the bodies. $r_i$ is the position vector of the body of mass $m_i$.

A. First Example

As our first example we shall consider the case $k = 1$. For this case the equations are

$$\ddot{r} = -g(m + m_1) \frac{r}{|r|^3}$$

where for convenience the subscript on $r_1$ has been dropped since $k = 1$.

Let

$$G = Ax + By + D \frac{r}{|r|^3} - D^2$$

Therefore,

$$\ddot{G} = A\ddot{x} + B\ddot{y} + D \frac{\dot{r} \cdot \ddot{r}}{|r|}$$

$$\ddot{G} = -\frac{g(m + m_1)}{|r|^3} \left[ Ax + By + D \frac{r}{|r|^3} - \frac{D}{g(m + m_1)} (r^2 \dot{r}^2 - (r \cdot \dot{r})^2) \right]$$

$$\ddot{G} = -\frac{g(m + m_1)}{|r|^3} \left[ G + \frac{D}{g(m + m_1)} (r \times \dot{r})^2 \right]$$

$$\ddot{G} = |r|^3 \left[ \frac{d}{dt} \left( \frac{1}{|r|^3} \right) \right] \dot{G} - \frac{g(m + m_1)}{|r|^3} \dot{G}$$
Setting \( G = \dot{G} = \ddot{G} = 0 \) at \( t = \dot{t} \) we obtain
\[
D^2 = \frac{D}{g(m + m_1)} (\dot{r}_0 \times \ddot{r}_0)^2
\]
where we use \( r_0 \) and \( \dot{r}_0 \) with components \((x_0, y_0)\) and \((\dot{x}_0, \dot{y}_0)\) respectively instead of \( w \). Let us set \( D = [1/g(m + m_1)] (\dot{r}_0 \times \ddot{r}_0)^2 \) Since \( A^2 = g(m_1 + m) D \) where \( A \) is the determinant \[
\left| \begin{array}{cc} x_0 & y_0 \\ \dot{x}_0 & \dot{y}_0 \end{array} \right|, \ A, \ B, \text{ and } D \text{ can be be determined in terms of } r_0 \text{ and } \dot{r}_0 \text{ so that } G \text{ is a normalized pseudo-integral for all solutions to the two-body problem, i.e., all solutions lie on a straight line or on (the branch of) a conic section. This is of course a well known fact in celestial mechanics although } G \text{ is not an integral since } A, \ B, \text{ and } D \text{ depend on initial conditions.}
\]

It is interesting to note that if the most general Second order polynomial in \( x \) and \( y \) was chosen as a possible normalized pseudo-integral for this problem instead of the selection made above, the same results could have been obtained in a similar fashion.

\[ \text{B. Second Example} \]

As our second example let us see if there is a solution to the planar three body problem such that the triangle formed by the bodies is always isosceles, but not necessarily congruent to its initial configuration, i.e., if \( G = r_1^2 - r_2^2 \) is a normalized pseudo-integral (in the coordinate system we are using the body of mass \( m \) is at the origin) for the equations
\[
\dot{r}_1 = -g(m + m_1) \frac{r_1}{|r_1|^3} - g m_2 \left[ \frac{r_1 - r_2}{|r_1 - r_2|^3} + \frac{r_2}{|r_2|^3} \right] \\
\dot{r}_2 = -g(m + m_2) \frac{r_2}{|r_2|^3} - g m_1 \left[ \frac{r_2 - r_1}{|r_2 - r_1|^3} + \frac{r_1}{|r_1|^3} \right]
\]

In deriving a differential equation for \( G \) we may make use of relationships such as
\[
\frac{1}{|r_2|} = \frac{1}{|r_1|} + \frac{G}{|r_1| |r_2| (|r_1| + |r_2|)}
\]
However, our work will be simplified if we make use of the fact that \( G \) is a normalized pseudo-integral for the given system of equations only if \( G \) is a normalized pseudo-integral for the following system:
\[
\dot{r}_1 = -g(m + m_1) \frac{r_1}{|r_2|^3} - g m_2 \left[ \frac{r_1 - r_2}{|r_1 - r_2|^3} + \frac{r_2}{|r_2|^3} \right] \\
\dot{r}_2 = -g(m + m_2) \frac{r_2}{|r_2|^3} - g m_1 \left[ \frac{r_2 - r_1}{|r_2 - r_1|^3} + \frac{r_1}{|r_1|^3} \right]
\]
where $|r_1|$ has been replaced by $|r_2|$

$$G = r_1^2 - r_2^2$$

$$G = 2r_1 \cdot \dot{r}_1 - 2r_2 \cdot \dot{r}_2$$

$$\ddot{G} = 2r_1^2 - 2r_2^2 - 2g \left[ \frac{m + m_1}{|r_2|^3} + \frac{m_2}{|r_1 - r_2|^3} \right] G$$

$$- 2g(m_1 - m_2) \left[ \frac{1}{|r_2|^3} - \frac{1}{|r_1 - r_2|^3} \right] (r_2^2 - r_1 \cdot \dot{r}_2)$$

$$G = -2gG \frac{d}{dt} \left[ \frac{m + m_1}{|r_2|^3} + \frac{m_2}{|r_1 - r_2|^3} \right] - 4g \left[ \frac{m + m_1}{|r_2|^3} + \frac{m_2}{|r_1 - r_2|^3} \right] G$$

$$- 2g(m_1 - m_2) \left\{ 2r_2 \cdot \dot{r}_2 \left( \frac{1}{|r_2|^3} - \frac{1}{|r_1 - r_2|^3} \right) + \frac{d}{dt} \left[ \left( \frac{1}{|r_2|^3} - \frac{1}{|r_1 - r_2|^3} \right) (r_2^2 - r_1 \cdot \dot{r}_2) \right] \right\}$$

$$- 4g(m_2 r_2 \cdot \dot{r}_1 - m_1 m_2) \left( \frac{1}{|r_2|^3} - \frac{1}{|r_1 - r_2|^3} \right)$$

Let $H = r_2 \cdot \dot{r}_1 - r_2 \cdot r_1$ and $K = (r_1 - r_2)^2 - r_2^2$. If either $H = 0$ were an integral, and $m_1 = m_2$ or $K = 0$ were an integral the differential equation for $G$ would reduce to

$$\ddot{G} = -2gG \frac{d}{dt} \left[ \frac{m + m_1}{|r_2|^3} + \frac{m_2}{|r_1 - r_2|^3} \right] - 4g \left[ \frac{m + m_1}{|r_2|^3} + \frac{m_2}{|r_1 - r_2|^3} \right] G$$

This suggests that we should (a) set $m_1 = m_2$ and see if the constraints can be chosen so that $G$ and $H$ are simultaneous normalized pseudo-integrals, or (b) see if the constraints can be chosen so that $G$ and $K$ are simultaneous normalized pseudo-integrals.

**Case (a).** For this case

$$\ddot{G} = -2gG \frac{d}{dt} \left[ \frac{m + m_2}{|r_2|^3} + \frac{m_2}{|r_1 - r_2|^3} \right]$$

$$- 4g \left[ \frac{m + m_2}{|r_2|^3} + \frac{m_2}{|r_1 - r_2|^3} \right] \dot{G} - 4g m_2 H \left( \frac{1}{|r_2|^3} - \frac{1}{|r_1 - r_2|^3} \right)$$

$$H = g m_2 G \left( \frac{1}{|r_2|^3} - \frac{1}{|r_1 - r_2|^3} \right)$$
Thus $H$ and $G$ will be simultaneous normalized pseudo integral if at $t = \hat{t}$ (and consequently see above, for all $t$ in a domain)

$$H = G = \dot{\dot{G}} = 0$$

that is, at $t = \hat{t}$,

$$r_0^2 = r_1^2$$
$$r_2^2 = r_1^2$$
$$r_2 \cdot \dot{r}_2 = r_1 \cdot \dot{r}_1$$
$$r_2 \cdot \ddot{r}_1 = r_1 \cdot \ddot{r}_2$$

If we let $(\rho_0, \theta_0)$ and $(\rho_1, \alpha_1)$ be the polar coordinates of $r_1$ and $\dot{r}_1$ at $t = \hat{t}$ and make use of trigonometric identities for

$$\cos (\theta_2 - \alpha_2) = \cos [(\theta_2 - \alpha_2) + (\alpha_2 - \alpha_1)]$$
and

$$\cos (\theta_1 - \alpha_2) = \cos [(\theta_1 - \alpha_1) - (\alpha_2 - \alpha_1)]$$

these constraints are equivalent to (we assume $p_1$ and $p_2$ are not 0 since this is a singularity of the system of differential equations)

(i) $p_2 = p_1$
(ii) $v_2 = v_1$
(iii) $v_1 = 0$ or $\alpha_2 - \alpha_1 = n\pi$ and $\theta_2 - \alpha_2 = +(\theta_1 - \alpha_1) + 2l\pi$ or $\theta_2 - \alpha_2 = -(\theta_1 - \alpha_1) + 2l\pi$ where $n$ and $l$ are integers.

Case (b). If we replace both $|r_1 - r_2|$ and $|r_1|$ by $|r_2|$ in the differential equations of motion these equations become

$$\dot{r}_1 = Lr_1$$
$$\dot{r}_2 = Lr_2$$

where $L = -g[(m + m_1 + m_2)/|r_2|^3]$. As above if $G$ and $K$ are simultaneous normalized pseudo-integrals for this system they are also simultaneous normalized pseudo-integrals for the original system. It readily follows that

$$G = r_1^2 - r_2^2$$
$$K = (r_1 - r_2)^2 + r_2^2 = r_1^2 - 2r_1 \cdot r_2$$
$$\dot{G} = 2r_1 \cdot \dot{r}_1 - 2r_2 \cdot \dot{r}_2$$
$$\dot{K} = 2r_1 \cdot \dot{r}_1 - 2r_2 \cdot \dot{r}_2 - 2r_1 \cdot \ddot{r}_2$$
$$\ddot{G} = 2\dot{r}_1^2 - 2\dot{r}_2^2 + 2LG$$
$$\ddot{K} = 2\dot{r}_1^2 - 4\dot{r}_1 \cdot \dot{r}_2 + 2LK$$
$$\dddot{G} = 4L\dot{G} + 2LG$$
$$\dddot{K} = 4L\dot{K} + 2LK$$
Thus $G$ and $K$ are simultaneous normalized pseudo-integrals if at $t = \hat{t}$

\[
\begin{align*}
\dot{r}_2^0 &= r_1^0 \\
\dot{r}_2^0 &= r_1^0 \\
r_2 \cdot \dot{r}_2 &= r_1 \cdot \dot{r}_1 \\
r_1 \cdot r_2 &= \frac{1}{2} r_1^2 \\
r_1 \cdot \dot{r}_1 &= \frac{1}{2} r_1^2 \\
r_1 \cdot \dot{r}_1 &= r_1 \cdot r_2 + r_1 \cdot \dot{r}_2
\end{align*}
\]

Using the same notation as for case (a) we find that these conditions are equivalent to (again $p_1$ and $p_2$ are not 0)

(i) $p_2 = p_1$

(ii) $v_2 = v_1$

(iii) $\theta_2 - \theta_1 = \pm \pi/3 + 2l\pi$

(iv) $v_1 = 0$ or $\theta_2 - \alpha_2 = \theta_1 - \alpha_1 + 2n\pi$, where $n$ and $l$ are integers.

References


