Abel’s lemma on summation by parts and basic hypergeometric series

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Abstract

Basic hypergeometric series identities are revisited systematically by means of Abel’s lemma on summation by parts. Several new formulae and transformations are also established. The author is convinced that Abel’s lemma on summation by parts is a natural choice in dealing with basic hypergeometric series.

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In 1826, Abel [1] (see Bromwich [7, §20] and Knopp [22, §43] also) found the following ingenious lemma on summation by parts. For two arbitrary sequences \( \{a_k\}_{k \geq 0} \) and \( \{b_k\}_{k \geq 0} \), if we denote the partial sums by

\[
A_n = \sum_{k=0}^{n} a_k \quad \text{where } n = 0, 1, 2, \ldots
\]

then for two natural numbers \( m \) and \( n \) with \( m \leq n \), there holds the relation:

\[
\sum_{k=m}^{n} a_k b_k = A_n b_m - A_m b_n
\]
\[ \sum_{k=m}^{n} a_k b_k = A_n b_{n+1} - A_{m-1} b_m + \sum_{k=m}^{n} A_k (b_k - b_{k+1}). \]  \hspace{1cm} (1)

As a classical analytic instrument, this transformation formula has been fundamental in convergence test of infinite series (cf. [7, §80], [22, §43] and [25, §7.36] for example). However, it has not been utilized hitherto to evaluate finite and infinite summations. For this purpose, we need to reformulate slightly Abel’s transformation formula on summation by parts.

For an arbitrary complex sequence \( \{\tau_k\} \), define the backward and forward difference operators \( \nabla \) and \( \triangle \), respectively, by

\[ \nabla \tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \triangle \tau_k = \tau_k - \tau_{k+1} \]  \hspace{1cm} (2)

where \( \triangle \) is adopted for convenience in the present paper, which differs from the usual operator \( \Delta \) only in the minus sign.

Then Abel’s lemma on summation by parts for unilateral and bilateral series may be reformulated respectively as

\[ \sum_{k=0}^{+\infty} B_k \nabla A_k = [AB]_+ - A_{-1} B_0 + \sum_{k=0}^{+\infty} A_k \triangle B_k, \]  \hspace{1cm} (3a)

\[ \sum_{k=-\infty}^{+\infty} B_k \nabla A_k = [AB]_+ - [AB]_- + \sum_{k=-\infty}^{+\infty} A_k \triangle B_k. \]  \hspace{1cm} (3b)

Both formulae just displayed hold for terminating series and nonterminating series, provided, in the latter case, that one of both series is convergent and there exist the limits \( [AB]_{\pm} := \lim_{n \to \pm \infty} A_n B_{n+1} \).

**Proof.** Let \( m \) and \( n \) be two integers. According to the definition of the backward difference, we have

\[ \sum_{k=m}^{n} B_k \nabla A_k = \sum_{k=m}^{n} B_k (A_k - A_{k-1}) = \sum_{k=m}^{n} A_k B_k - \sum_{k=m}^{n} A_{k-1} B_k. \]

Replacing \( k \) by \( k + 1 \) for the last sum, we get the following expression:

\[ \sum_{k=m}^{n} B_k \nabla A_k = A_n B_{n+1} - A_{m-1} B_m + \sum_{k=m}^{n} A_k (B_k - B_{k+1}) \]

\[ = A_n B_{n+1} - A_{m-1} B_m + \sum_{k=m}^{n} A_k \triangle B_k \]

which is essentially the same as Abel’s original transformation (1) on summation by parts. For \( n \to +\infty \), the two cases \( m = 0 \) and \( m \to -\infty \) of the last equation result in the modified Abel’s lemma respectively for unilateral and bilateral series. \( \square \)
Recently, Abel’s lemma on summation by parts has successfully been used to give new proofs of Bailey’s bilateral $\psi_6$-series identity by Chu [12] and terminating well-poised $q$-series identities by Chu–Jia [13]. The objective of the present work is to explore systematically the applications of Abel’s lemma on summation by parts to basic hypergeometric series identities. Several $q$-series identities will be exemplified in a unified manner by means of Abel’s lemma on summation by parts such as the $q$-binomial theorem, Ramanujan’s bilateral $1\psi_1$-series identity, $q$-Gauss summation theorem, $q$-Pfaff–Saalschütz summation formula, $q$-Dougall–Dixon $6\phi_5$-series identity and Jackson’s very well-poised $8\phi_7$-series identity. In addition to providing new proofs for the identities just mentioned, we shall also establish few new summation and transformation formulae (even though this is not the main concern of the present paper). They will show that as a classical analytic weapon, Abel’s lemma on summation by parts is indeed a very natural and powerful method in dealing with basic hypergeometric series identities.

In order to facilitate the readability of the paper, we reproduce the notations of $q$-shifted factorial and basic hypergeometric series.

For two complex $x$ and $q$, the shifted-factorial of order $n$ with base $q$ is defined by

$$(x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1}) = (x; q)_{\infty}/(xq^n; q)_{\infty}$$

where $n$ is assumed in the last equation, respectively, to be a natural number for the polynomial in the middle and integral for the fraction at the end. Its fractional form is abbreviated compactly to

$$\left[ \begin{array}{c} \alpha, \beta, \ldots, \gamma \\ A, B, \ldots, C \end{array} \right| q)_n = (\alpha; q)_n(\beta; q)_n \cdots (\gamma; q)_n/(A; q)_n(B; q)_n \cdots (C; q)_n.$$

Following Bailey [6] and Slater [24], the basic hypergeometric series and the corresponding bilateral series are defined, respectively, by

$$1+m\psi_m \left[ \begin{array}{c} a_0, a_1, \ldots, a_m \\ b_1, \ldots, b_m \end{array} \right| q; z \right] = \sum_{n=0}^{+\infty} z^n \left[ \begin{array}{c} a_0, a_1, \ldots, a_m \\ q, b_1, \ldots, b_m \end{array} \right| q)_n,$$

$$m\psi_m \left[ \begin{array}{c} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_m \end{array} \right| q; z \right] = \sum_{n=-\infty}^{+\infty} z^n \left[ \begin{array}{c} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_m \end{array} \right| q)_n,$$

where the base $q$ will be restricted to $|q| < 1$ for nonterminating $q$-series.

1. The terminating $q$-binomial theorem

Define the function $f_m(x)$ by the terminating $q$-binomial sum

$$f_m(x) := \sum_{k=0}^{m} (-1)^k \left[ \begin{array}{c} m \\ k \end{array} \right] q^k x^k.$$

For the two sequences given by

$$A_k := (-1)^k \left[ \begin{array}{c} m - 1 \\ k \end{array} \right] q^{\frac{k(k+1)}{2}} \quad \text{and} \quad B_k := x^k$$
it is trivial to check \( A_{-1} = 0 \) and the differences

\[
\nabla A_k = (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} \quad \text{and} \quad \triangle B_k = (1 - x)x^k.
\]

Then by means of the modified Abel’s lemma on summation by parts, we can manipulate \( f_m(x) \) as follows:

\[
f_m(x) = \sum_{k=0}^{m} B_k \nabla A_k = \sum_{k=0}^{m} A_k \triangle B_k
\]

\[
= (1 - x) \sum_{k=0}^{m-1} (-1)^k \begin{bmatrix} m - 1 \\ k \end{bmatrix} q^{\binom{k}{2}} (qx)^k
\]

which reads as the following relation

\[
f_m(x) = (1 - x) f_{m-1}(qx).
\]

Iterating this equation \( m \)-times, we find the recurrence relation

\[
f_m(x) = (x; q)_m \times f_0(q^m x).
\]

On account of the fact that \( f_0(x) = 1 \), we have the \( q \)-finite difference formula.

**Proposition 1 (The terminating \( q \)-binomial theorem: [17, II-4]).**

\[
(x; q)_m = \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} x^k.
\]

Now replacing \( m \) by \( m + n \) and \( x \) by \( x q^{-m} \) respectively, and then noting the relation

\[
(q^{-m} x; q)_{m+n} = (q^{-m} x; q)_m (x; q)_n = (-1)^m q^{-\left(m + \frac{1}{2}\right)} x^m (q/x; q)_m (x; q)_n
\]

we can reformulate the \( q \)-binomial theorem as

\[
(x; q)_n (q/x; q)_m = \sum_{k=0}^{m+n} (-1)^{k-m} \begin{bmatrix} m + n \\ k \end{bmatrix} q^{\binom{k}{2}} x^{k-m}
\]

which becomes, under summation index substitution \( k \to m + k \), the following finite form of the Jacobi triple product identity

\[
(x; q)_n (q/x; q)_m = \sum_{k=-m}^{n} (-1)^k \begin{bmatrix} m + n \\ m + k \end{bmatrix} q^{\binom{k}{2}} x^k.
\]
Letting \( m, n \to \infty \) in (5) and recalling the limiting relation
\[
\begin{bmatrix} m+n \\ m+k \end{bmatrix} = \frac{(q; q)_{m+n}}{(q; q)_{m+k}} \left( \frac{m, n \to \infty}{(q; q)_n} \right) \frac{1}{(q; q)_k} \quad \text{where } |q| < 1
\]
we recover the famous Jacobi triple product identity (cf. [4, §10.4] and [17, II-28])
\[
[q, x, q/x; q]_\infty = \sum_{k=-\infty}^{+\infty} (-1)^k q \left( \frac{q}{x} \right)^k x^k \quad \text{for } |q| < 1. \tag{6}
\]

2. Euler’s first \( q \)-exponential formula

Define the function \( g(x) \) by
\[
g(x) := \sum_{k=0}^{\infty} \frac{x^k}{(q; q)_k} \quad \text{where } |x| < 1.
\]

For the two sequences given by
\[
A_k := \frac{1}{(q; q)_k} \quad \text{and} \quad B_k := x^k
\]
it is easy to compute two extreme values
\[
A_{-1} B_0 = [AB]_+ = 0
\]
and the finite differences
\[
\nabla A_k = \frac{q^k}{(q; q)_k} \quad \text{and} \quad \triangle B_k = (1 - x) x^k.
\]

In view of the modified Abel’s lemma on summation by parts, we can manipulate \( g(x) \) as follows:
\[
g(x) = \frac{1}{1 - x} \sum_{k=0}^{\infty} A_k \triangle B_k = \frac{1}{1 - x} \sum_{k=0}^{\infty} B_k \nabla A_k
\]
\[
= \frac{1}{1 - x} \sum_{k=0}^{\infty} \frac{(qx)^k}{(q; q)_k} = \frac{g(qx)}{1 - x}.
\]

Iterating this process \( m \)-times, we find the following recurrence relation
\[
g(x) = \frac{g(x^m)}{(x; q)_m}.
\]

Letting \( m \to \infty \) and noting that \( g(0) = 1 \) in the last relation, we derive the first \( q \)-exponential expansion formula.
Proposition 2 (The first q-exponential function: [17, II-1]).

\[
\frac{1}{(x; q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q; q)_k} \quad \text{where } |x| < 1.
\]

3. Euler’s second q-exponential formula

Define the function \( h(x) \) by

\[
h(x) := \sum_{k=0}^{\infty} \frac{(-1)^k q^{(\frac{k}{2})} x^k}{(q; q)_k}.
\]

For the two sequences given by

\[
A_k := \frac{(-1)^k q^{(\frac{k+1}{2})}}{(q; q)_k} \quad \text{and} \quad B_k := x^k
\]

it is not hard to verify two extreme values

\[
A_{-1} B_0 = [A B]_+ = 0
\]

and the finite differences

\[
\nabla A_k = \frac{(-1)^k q^{(\frac{k}{2})}}{(q; q)_k} \quad \text{and} \quad \Delta B_k = (1 - x) x^k.
\]

According to the modified Abel’s lemma on summation by parts, we can manipulate \( h(x) \) as follows:

\[
h(x) = \sum_{k=0}^{\infty} B_k \nabla A_k = \sum_{k=0}^{\infty} A_k \Delta B_k
\]

\[
= (1 - x) \sum_{k=0}^{\infty} \frac{(-1)^k q^{(\frac{k}{2})} (qx)^k}{(q; q)_k} = (1 - x) h(qx).
\]

Iterating this process \( m \)-times, we find the following recurrence relation

\[
h(x) = (x; q)_m \times h(q^m x).
\]

Letting \( m \to \infty \) and noting that \( h(0) = 1 \) in the last relation, we derive the second q-exponential function expansion.

Proposition 3 (The second q-exponential function: [17, II-2]).

\[
(x; q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{(\frac{k}{2})} x^k}{(q; q)_k}.
\]
4. The nonterminating $q$-binomial theorem

Define the two sequences by

$$A_k := \frac{(qaz; q)_k}{(q; q)_k} \quad \text{and} \quad B_k := \frac{(a; q)_k}{(az; q)_k} z^k.$$  

It is not difficult to calculate two extreme values

$$A_{-1} B_0 = [AB]_+ = 0 \quad \text{with} \quad |z| < 1$$

and the finite differences

$$\nabla A_k = \frac{(az; q)_k}{(q; q)_k} q^k \quad \text{and} \quad \triangle B_k = \frac{1 - z}{1 - az (aqz; q)_k} z^k.$$  

Applying the modified Abel’s lemma on summation by parts, we can manipulate the $_1\phi_0$-series as follows:

$$\begin{align*}
_1\phi_0 \left[ \frac{a}{-} \mid q; z \right] &= \frac{1 - az}{1 - z} \sum_{k=0}^{\infty} A_k \triangle B_k \\
&= \frac{1 - az}{1 - z} \sum_{k=0}^{\infty} B_k \nabla A_k \\
&= \frac{1 - az}{1 - z} \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} (qz)^k
\end{align*}$$

which results in the following relation

$$\begin{align*}
_1\phi_0 \left[ \frac{a}{-} \mid q; z \right] &= \frac{1 - az}{1 - z} _1\phi_0 \left[ \frac{a}{-} \mid q; qz \right].
\end{align*}$$

The reader can find an alternative derivation of this relation in [17, §1.3.1].

Iterating the last equation $m$-times, we find the recurrence relation

$$\begin{align*}
_1\phi_0 \left[ \frac{a}{-} \mid q; z \right] &= \frac{(az; q)_m}{(z; q)_m} \times _1\phi_0 \left[ \frac{a}{-} \mid q; q^m z \right].
\end{align*}$$

Letting $m \to \infty$ in the last relation, we derive the $q$-binomial expansion formula.

**Proposition 4 (The nonterminating $q$-binomial theorem: [17, II-3]).**

$$\begin{align*}
_1\phi_0 \left[ \frac{a}{-} \mid q; z \right] &= \frac{(az; q)_\infty}{(z; q)_\infty} \quad \text{where} \quad |z| < 1.
\end{align*}$$
5. Ramanujan’s bilateral $1\psi_1$-series identity

Define the two sequences by

$$A_k := \frac{(a; q)_k}{(c; q)_k}$$
and
$$B_k := z^k.$$

We have easily two extreme values

$$[AB]_+ = [AB]_- = 0 \quad \text{with} \quad |c/a| < |z| < 1$$

and the finite differences

$$\nabla A_k = \frac{1 - c/a}{1 - q/a} \frac{(a/q; q)_k}{(c; q)_k} q^k$$
and
$$\triangle B_k = (1 - z) z^k.$$

By means of the modified Abel’s lemma on summation by parts for nonterminating bilateral series, we can manipulate the bilateral $1\psi_1$-series as follows:

$$1\psi_1 \left[ \frac{a}{c} \mid q; z \right] = \sum_{k=-\infty}^{+\infty} A_k \triangle B_k \frac{1}{1 - z} = \sum_{k=-\infty}^{+\infty} B_k \nabla A_k \frac{1}{1 - z}$$

$$= \frac{1 - c/a}{(1 - z)(1 - q/a)} \sum_{k=-\infty}^{+\infty} \frac{(a/q; q)_k}{(c; q)_k} (q z)^k$$

$$= \frac{1 - c/a}{(1 - z)(1 - q/a)} 1\psi_1 \left[ \frac{a}{c} \mid q; qz \right].$$

Andrews and Askey [3] (see also [17, §5.2]) have established this recurrence relation in a different manner. But the derivation presented here is more direct and simple.

Iterating this process $n$-times, we find the following recurrence relation

$$1\psi_1 \left[ \frac{a}{c} \mid q; z \right] = \frac{(c/a; q)_n}{(z; q)_n(q/a; q)_n} \times 1\psi_1 \left[ \frac{a/q^n}{c} \mid q; q^n z \right]. \quad (7)$$

Alternatively, for the two sequences defined by

$$C_k := (az/c)^k$$
and
$$D_k := \frac{(a; q)_k}{(c; q)_k} \frac{c^k}{a^k}$$
we can compute two extreme values

$$[CD]_+ = [CD]_- = 0 \quad \text{with} \quad |z| < 1$$

and the finite differences

$$\nabla C_k = (1 - c/az)(az/c)^k$$
and
$$\triangle D_k = \frac{1 - c/a}{1 - c} \frac{(a; q)_k}{(q_c; q)_k} \frac{c^k}{a^k}.$$
Applying the modified Abel’s lemma on summation by parts for nonterminating bilateral series, we can similarly reformulate the $1\psi_1$-series as follows:

$$\left[ a \right| c \left| q; z \right]_{\infty} = \sum_{k=-\infty}^{+\infty} D_k \nabla C_k \frac{1}{1-c/az} = \sum_{k=-\infty}^{+\infty} C_k \triangle D_k \frac{1}{1-c/az}$$

$$= \frac{1-c/a}{(1-c)(1-c/az)} \sum_{k=-\infty}^{+\infty} (a;q)_k z^k$$

$$= \frac{1-c/a}{(1-c)(1-c/az)} 1\psi_1 \left[ a \right| cq \left| q; z \right].$$

Iterating this process $m$-times, we derive another recurrence relation

$$1\psi_1 \left[ a \right| c \left| q; z \right] = (c/a;q)_m (c;q)_m (c/az;q)_m \times 1\psi_1 \left[ a \right| q^m c \left| q; z \right]. \quad (8)$$

The combination of (7) with (8) yields

$$1\psi_1 \left[ a \right| c \left| q; z \right] = \frac{(c/a;q)_m (c;q)_m (c/az;q)_m}{(c;q)_m (c/az;q)_m} \times 1\psi_1 \left[ a \right| q^n c \left| q^n z \right]. \quad (9)$$

Letting $m, n \to \infty$ and recalling Jacobi’s triple product identity (6)

$$1\psi_1 \left[ a \right| q^n c \left| q^n z \right] \implies \sum_{k=-\infty}^{+\infty} (-1)^k q^{(\frac{k}{2})} (az)^k = [q, az, q/az; q]_{\infty}$$

we derive the bilateral $1\psi_1$-series identity.

**Proposition 5 (Ramanujan’s bilateral $1\psi_1$-series identity: [14, §18] and [17, II-29]).**

$$1\psi_1 \left[ a \right| c \left| q; z \right] = \left[ q, az, q/az, c/a \left| c, z, c/az, q/a \right| q \right]_{\infty} \text{ where } |c/a| < |z| < 1.$$

The last formula can also be derived from (8) directly as follows. Letting $m \to +\infty$ in (8), we have the relation

$$1\psi_1 \left[ a \right| c \left| q; z \right] = (c/a;q)_{\infty} (c/az;q)_{\infty} \times 1\psi_1 \left[ a \right| 0 \left| q; z \right]. \quad (10)$$

The $1\psi_1$-series on the right-hand side can be evaluated by first putting $c = q$ and then invoking the $q$-binomial theorem as

$$1\psi_1 \left[ a \right| 0 \left| q; z \right] = (c;q)_{\infty} (c/az;q)_{\infty} (q/a;q)_{\infty} \times 1\psi_1 \left[ a \right| c \left| q; z \right]$$

$$= (q;q)_{\infty} (q/az;q)_{\infty} (q/a;q)_{\infty} \times \phi_0 \left[ a \right| q; z]$$
Substituting this into (10) results immediately in Ramanujan’s identity on \( \psi_1 \)-series displayed in the last proposition. For other proofs of this important formula, see [2,3], [16, §2] and [18–21].

6. \( q \)-Gauss summation theorem

Define the two sequences by

\[
A_k := \frac{(qb; q)_k}{(q; q)_k} \frac{1}{b^k} \quad \text{and} \quad B_k := \frac{(a; q)_k}{(c; q)_k} \frac{c^k}{a^k}.
\]

It is not hard to have two extreme values

\[
A_{-1} B_0 = [AB]_+ = 0 \quad \text{with} \ |c/ab| < 1
\]

and the finite differences

\[
\nabla A_k = \frac{(b; q)_k}{(q; q)_k} \frac{1}{b^k} \quad \text{and} \quad \triangle B_k = \frac{1 - c/a}{1 - c} \frac{(a; q)_k}{(qc; q)_k} \frac{c^k}{a^k}.
\]

According to the modified Abel lemma on summation by parts for nonterminating series, we can reformulate the following \( 2\phi_1 \)-series:

\[
2\phi_1\left[ a, b \mid q; \frac{c}{ab} \right] = \sum_{k=0}^{\infty} B_k \nabla A_k = \sum_{k=0}^{\infty} A_k \triangle B_k
\]

\[
= \frac{1 - c/a}{1 - c} \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} (qb; q)_k \left( \frac{c}{ab} \right)^k
\]= \frac{1 - c/a}{1 - c} 2\phi_1\left[ a, qb \mid qc; \frac{c}{ab} \right].
\]

Iterating this process \( m \)-times, we find the following recurrence relation

\[
2\phi_1\left[ a, b \mid q; \frac{c}{ab} \right] = \frac{(c/a; q)_m}{(c; q)_m} 2\phi_1\left[ a, q^m b \mid q; \frac{c}{ab} \right].
\]

Letting \( m \to \infty \) in the last relation and then applying the \( q \)-binomial expansion formula, we derive the \( q \)-Gauss summation theorem.

Proposition 6 (The \( q \)-Gauss summation formula: [17, II-8]).

\[
2\phi_1\left[ a, b \mid q; \frac{c}{ab} \right] = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty} \quad \text{where} \ |c/ab| < 1.
\]
The formula just displayed can alternatively be proved as follows. For the two sequences given by

\[ C_k := \frac{(qab/c; q)_k}{(q; q)_k} \left( \frac{c}{ab} \right)^k \quad \text{and} \quad D_k := \left[ \begin{array}{c} a, b \\ c, ab/c | q \end{array} \right]_k. \]

It is not difficult to check two extreme values

\[ C_{-1}D_0 = [CD]_+ = 0 \quad \text{with} \quad |c/ab| < 1 \]

and the finite differences

\[ \nabla C_k = \frac{(ab/c; q)_k}{(q; q)_k} \left( \frac{c}{ab} \right)^k \quad \text{and} \quad \Delta D_k = q^k \left[ \begin{array}{c} a, b \\ qc, ab/c | q \end{array} \right]_k \frac{(1-c/a)(1-c/b)}{(1-c)(1-c/ab)} . \]

Then we can similarly reformulate the \(2\phi_1\)-series as follows:

\[ 2\phi_1 \left[ \begin{array}{c} a, b \\ c | q; \frac{c}{ab} \end{array} \right] = \sum_{k=0}^{\infty} D_k \nabla C_k = \sum_{k=0}^{\infty} C_k \Delta D_k = \frac{(1-c/a)(1-c/b)}{(1-c)(1-c/ab)} \sum_{k=0}^{\infty} \frac{(a; q)_k(b; q)_k}{(q; q)_k(qc; q)_k} \left( \frac{qc}{ab} \right)^k \]

Iterating this process \(m\)-times, we find the following recurrence relation

\[ 2\phi_1 \left[ \begin{array}{c} a, b \\ c | q; \frac{c}{ab} \end{array} \right] = \left[ \begin{array}{c} c/a, c/b \\ c, c/ab | q \end{array} \right]_m \times 2\phi_1 \left[ \begin{array}{c} a, b \\ qc, qc/ab | q \right]_. \]

Letting \(m \to \infty\) in the last relation, we recover again the \(q\)-Gauss summation theorem displayed in Proposition 6.

7. \(q\)-Pfaff–Saalschütz summation formula

For the shifted factorial fractions with two pairs of balanced parameters, it is not hard to check the following differences.

**Lemma 7 (Balanced differences).**

\[ \nabla \left[ \frac{q\lambda}{ab}, \frac{d/\lambda}{q} \left| \begin{array}{c} b/\lambda, d/\lambda \\ bq, dq \end{array} \right] \right] = q^k \left[ \begin{array}{c} b/\lambda, d/\lambda \\ bq, dq \end{array} \right]_k \frac{(1-\lambda)(1-\lambda b/d)}{(1-\lambda b)(1-\lambda/d)}. \]

\[ \Delta \left[ \frac{b/\lambda}{b}, \frac{d/\lambda}{d} \left| \begin{array}{c} b/\lambda, d/\lambda \\ bq, dq \end{array} \right] \right] = q^k \left[ \begin{array}{c} b/\lambda, d/\lambda \\ bq, dq \end{array} \right]_k \frac{(1-\lambda)(1-\lambda b/d)}{(1-b)(1-d)} \left( \frac{d}{\lambda} \right). \]
In particular, with the two sequences given by

\[ A_k := \left[ \begin{array}{c|c} qx, qa/x \\ q, qa \end{array} \right]_k \quad \text{and} \quad B_k := \left[ \begin{array}{c|c} qa, qa/bc \\ qa/b, qa/c \end{array} \right]_k \]

we have the boundary condition

\[ A_{-1}B_0 = 0 \quad \text{and} \quad [AB]_+ = \left[ \begin{array}{c|c} qx, qa/x, qa/bc \\ q, qa/b, qa/c \end{array} \right]_\infty \]

as well as the finite differences

\[ \nabla A_k = q^k \left[ \begin{array}{c|c} x, a/x \\ q, qa \end{array} \right]_k \quad \text{and} \quad \triangle B_k = q^k \left[ \begin{array}{c|c} qa, qa/bc \\ q^2a/b, q^2a/c \end{array} \right]_k \frac{(1-b)(1-c)(qa/bc)}{(1-qa/b)(1-qa/c)}. \]

Then the modified Abel’s lemma on summation by parts enables us to manipulate the following nonterminating balanced series:

\[ 3\phi_2 \left[ \begin{array}{c|c} x, a/x, qa/bc \\ qa/b, qa/c \end{array} \right]_k = \sum_{k=0}^{\infty} B_k \nabla A_k = [AB]_+ + \sum_{k=0}^{\infty} A_k \triangle B_k = \left[ \begin{array}{c|c} qx, qa/x, qa/bc \\ q, qa/b, qa/c \end{array} \right]_\infty \]

The last balanced \(3\phi_2\)-series can be derived from the \(3\phi_2\)-series on the left under parameter replacements \(a \rightarrow q^2a\), \(b \rightarrow qb\), \(c \rightarrow qc\) and \(x \rightarrow qx\). Iterating this process \(m\)-times, we can proceed the following:

\[ 3\phi_2 \left[ \begin{array}{c|c} x, a/x, qa/bc \\ qa/b, qa/c \end{array} \right]_k = \left[ \begin{array}{c|c} b, c \\ qa/b, qa/c \end{array} \right]_m \left( \frac{qa}{bc} \right)_m \left[ \begin{array}{c|c} q^m x, q^ma/x, qa/bc \\ q^{1+m}a/b, q^{1+m}a/c \end{array} \right]_\infty \]

Rewriting the last factorial fraction

\[ \left[ \begin{array}{c|c} q^{1+k}x, q^{1+k}a/x, qa/bc \\ q, q^{1+k}a/b, q^{1+k}a/c \end{array} \right]_\infty = \left[ \begin{array}{c|c} qa/b, qa/c \end{array} \right]_k \times \left[ \begin{array}{c|c} qx, qa/x, qa/bc \end{array} \right]_\infty \]

we establish the following balanced transformation.
Proposition 8 (Balanced series transformation).

\[
\begin{align*}
\phi_2^{[3]}[x, a/x, qa/bc, qa/b, qa/c | q; q] &= \phi_2^{[3]}[b, c | qa/bc] (q \frac{a}{bc})^m \\
&+ \phi_2^{[3]}[qx, qa/x, qa/bc, qa/b, qa/c | q] \sum_{k=0}^{m-1} \phi_2^{[3]}[b, c | qx, qa/x | q] \left( q \frac{a}{bc} \right)^k.
\end{align*}
\]

Letting \( x = q^{-m} \), we get the \( q \)-Pfaff–Saalschütz summation formula.

Corollary 9 (\( q \)-Pfaff–Saalschütz summation formula [9, Eq. 2.2a]).

\[
\phi_2^{[3]}[q^{-m}, q^n a, qa/bc, qa/b, qa/c | q; q] = \phi_2^{[3]}[b, c | qa/bc] (q \frac{a}{bc})^m.
\]

It is equivalent to the following original form (cf. [17, II-12]):

\[
\phi_2^{[3]}[q^{-n}, a, b | c, q^{1-n} ab/c | q; q] = \phi_2^{[3]}[c/a, c/b | c/ab | q]_n.
\]

When \( m \to \infty \), we derive the following transformation formula (cf. [17, III-10]).

Corollary 10 (Balanced series transformation: \( |qa/bc| < 1 \)).

\[
\phi_2^{[3]}[x, a/x, qa/bc, qa/b, qa/c | q; q] = \phi_2^{[3]}[qx, qa/x, qa/bc, qa/b, qa/c | q] \sum_{k=0}^{\infty} \phi_2^{[3]}[b, c | qx, qa/x | q] \left( q \frac{a}{bc} \right)^k.
\]

8. Very well-poised \( \phi_5 \)-series identity

For the shifted factorial fractions with two pairs of well-poised parameters, it is not difficult to verify the following differences.

Lemma 11 (Well-poised differences).

\[
\nabla\left[ \phi_5^{[6]}[qb, qd, qA/b, A/A/d | q] \left( \frac{A}{bd} \right)^k \right] = \frac{1 - Aq^{2k}}{1 - A} \phi_5^{[6]}[b, d, qA/b, qA/d | q] \left( \frac{A}{bd} \right)^k \\
\times \frac{(1 - A)(1 - bd/A)}{(1 - b)(1 - d)};
\]

\[
\triangle\left[ \phi_5^{[6]}[b, d, A/b, A/d | q] \left( \frac{A}{bd} \right)^k \right] = \frac{1 - Aq^{2k}}{1 - A} \phi_5^{[6]}[b, d, qA/b, qA/d | q] \left( \frac{A}{bd} \right)^k \\
\times \frac{(1 - A)(1 - A/bd)}{(1 - A/b)(1 - A/d)}.
\]
In particular, with the two sequences given by

\[ A_k := \left[ \frac{qa}{q}, \frac{qc}{qa/c} \right]_k \left( \frac{1}{c} \right)^k \quad \text{and} \quad B_k := \left[ \frac{b}{qa/b}, \frac{d}{qa/d} \right]_k \left( \frac{qa}{bd} \right)^k \]

we have the boundary condition

\[ A_{-1}B_0 = [AB]_+ = 0 \quad \text{with} \quad |qa/bcd| < 1 \]
as well as the finite differences

\[
\nabla A_k = \frac{1-aq^{2k}}{1-a} \left[ \frac{a, c}{q, qa/c} \right]_k \left( \frac{1}{c} \right)^k ,
\]

\[
\triangle B_k = \frac{1-aq^{2k+1}}{1-qa} \left[ \frac{b, d}{q^2a/b, q^2a/d} \right]_k \left( \frac{qa}{bd} \right)^k \left( 1-qa \right) \left( 1-qa/bd \right) \left( 1-qa/b \right) \left( 1-qa/d \right) .
\]

Then by means of the modified Abel’s lemma on summation by parts, we can reformulate the following nonterminating very well-poised series:

\[
_{6} \phi_{5} \left[ \frac{a, \sqrt{a}, -q\sqrt{a}, b, c, d}{\sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d} \right; q; \frac{qa}{bcd} \right] = \sum_{k=0}^{\infty} B_k \nabla A_k = \sum_{k=0}^{\infty} A_k \triangle B_k = \frac{(1-qa)(1-qa/bd)}{(1-qa/b)(1-qa/d)} \times _{6} \phi_{5} \left[ \frac{qa}{\sqrt{qa}}, -\sqrt{qa}, b, \sqrt{qa} \right; \frac{qa}{bcd} \right] .
\]

Observe that the last \(_{6} \phi_{5}\)-series can also be obtained from the \(_{6} \phi_{5}\)-series displayed in the first line with \(a\) and \(c\) being replaced by \(qa\) and \(qc\) respectively. Iterating this process \(m\)-times, we establish the following relation.

**Proposition 12** (Very well-poised transformation).

\[
_{6} \phi_{5} \left[ \frac{a, \sqrt{a}, -q\sqrt{a}, b, c, d}{\sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d} \right; q; \frac{qa}{bcd} \right] = _{6} \phi_{5} \left[ \frac{q^m a, \sqrt{q^m a}, -q\sqrt{q^m a}, b, q^m c, d}{\sqrt{q^m a}, -\sqrt{q^m a}, q^1 a/b, qa/c, q^1 a/d} \right; q; \frac{qa}{bcd} \right] \times \left[ \frac{qa, qa/bd}{qa/b, qa/d} \right]_m
\]

provided \(|qa/bcd| < 1\).

When \(c = q^{-m}\), we recover from this proposition the terminating very well-poised \(_{6} \phi_{5}\)-series identity.
Corollary 13 (Terminating very well-poised $6\phi_5$-series identity: [17, II-21]).

\[
6\phi_5 \left[ \frac{a, q\sqrt{a}, -q\sqrt{a}, b, d, q^{-m}}{\sqrt{a}, -\sqrt{a}, \frac{qa}{b}, \frac{qa}{d}, q^{1+m}a} \right| q; \frac{q^{1+m}a}{bd} \right] = \left[ \frac{qa, qa/bd}{\frac{qa}{b}, \frac{qa}{d}} \right]_m.
\]

Letting $m \to \infty$ in Proposition 12, we derive the following transformation

\[
6\phi_5 \left[ \frac{a, q\sqrt{a}, -q\sqrt{a}, b, c, d}{\sqrt{a}, -\sqrt{a}, \frac{qa}{b}, \frac{qa}{c}, \frac{qa}{d}} \right| q; \frac{qa}{bcd} \right] = \left[ \frac{qa, qa/bc, qa/bd, qa/cd}{\frac{qa}{b}, \frac{qa}{c}, \frac{qa}{d}, \frac{bcd}{qa}} \right]_\infty.
\]

Evaluating the last $2\phi_1$-series by means of the $q$-Gauss summation theorem (cf. [17, II-8]), we get the following nonterminating very well-poised $6\phi_5$-series identity:

Corollary 14 (Nonterminating very well-poised $6\phi_5$-series identity: [10, §2] and [17, II-20]).

\[
6\phi_5 \left[ \frac{a, q\sqrt{a}, -q\sqrt{a}, b, c, d}{\sqrt{a}, -\sqrt{a}, \frac{qa}{b}, \frac{qa}{c}, \frac{qa}{d}} \right| q; \frac{qa}{bcd} \right] = \left[ \frac{b, d}{\frac{qa/c}{q}} \right]_\infty \text{ provided } |qa/bcd| < 1.
\]

Alternatively, define the two sequences by

\[
C_k := \left[ \frac{qa, bcd/a}{q, \frac{qa}{bcd}} \right] (\frac{qa}{bcd})^k \quad \text{and} \quad D_k := \left[ \frac{b, c, d}{\frac{qa/c}{q}, \frac{qa}{d}, \frac{bcd}{qa}} \right]_k.
\]

We can verify the boundary condition

\[
C_{-1}D_0 = [CD]_+ = 0 \quad \text{with } |qa/bcd| < 1
\]

as well as the finite differences

\[
\nabla C_k = \frac{1 - qa^{2k}}{1 - a} \left[ \frac{a, bcd/q a}{q, \frac{qa}{bcd}} \right] (\frac{qa}{bcd})^k,
\]

\[
\Delta D_k = \frac{1 - a q^{2k+1}}{1 - qa} \left[ \frac{b, c, d, q^2a^2/bcd}{q^2a/b, q^2a/c, q^2a/d, bcd/qa} \right]_k (1 - qa)(1 - qa/bc)(1 - qa/bd)(1 - qa/cd) \times \frac{1 - qa/b}{1 - qa/c}(1 - qa/d)(1 - qa/bcd).
\]

Applying the modified Abel’s lemma on summation by parts, we can manipulate the $6\phi_5$-series as follows:
\[ 6\phi_5 \left[ a, q\sqrt{a}, -q\sqrt{a}, \frac{b}{\sqrt{a}}, \sqrt{a}, \frac{qa}{b}, qa/c, qa/d \middle| q; \frac{qa}{bcd} \right] = \sum_{k=0}^{\infty} D_k \nabla C_k = \sum_{k=0}^{\infty} C_k \triangle D_k \]

\[ = \frac{(1 - qa)(1 - qa/bc)(1 - qa/bd)(1 - qa/cd)}{(1 - qa/b)(1 - qa/c)(1 - qa/d)(1 - qa/bcd)} \times 6\phi_5 \left[ qa, q\sqrt{qa}, \frac{b}{\sqrt{qa}}, -q\sqrt{qa}, \frac{qa}{b}, qa/c, qa/d \middle| q; \frac{q^2a}{bcd} \right]. \]

Observe that the last $6\phi_5$-series can also be obtained from the $6\phi_5$-series displayed in the first line with $a$ being replaced by $qa$. Iterating this process $m$-times, we establish the following relation.

**Proposition 15** (Very well-poised transformation).

\[ 6\phi_5 \left[ a, q\sqrt{a}, -q\sqrt{a}, \frac{b}{\sqrt{a}}, \sqrt{a}, \frac{qa}{b}, qa/c, qa/d \middle| q; \frac{qa}{bcd} \right] = 6\phi_5 \left[ q^m a, q\sqrt{q^m a}, -q\sqrt{q^m a}, \frac{b}{\sqrt{q^m a}}, \sqrt{q^m a}, \frac{q^m+1 a}{b}, q^m+1 a/c, q^m+1 a/d \middle| q; \frac{q^m+1 a}{bcd} \right] \]

\[ \times \left[ qa, qa/bc, qa/bd, qa/cd \middle| q \right]_m \text{ provided } |qa/bcd| < 1. \]

Letting $m \to \infty$ in the last proposition recovers directly the nonterminating well-poised $6\phi_5$-series identity displayed in Corollary 14. From the proofs of the $q$-Gauss summation theorem and the nonterminating well-poised $6\phi_5$-series identity, we see that there is much flexibility to choose two sequences when utilizing Abel’s lemma on summation by parts to evaluate infinite series.

9. **Jackson’s terminating very well-poised $8\phi_7$-series identity**

For the shifted factorial fractions with four pairs of well-poised parameters, we can show the following differences.

**Lemma 16** (Well-poised differences: $bcde = A^2$).

\[ \nabla \left[ qb, qc, qd, qe \middle| q \right]_{kA/b, qA/c, qA/d, qA/e} = \frac{1 - Aq^{2k}}{1 - A} \left[ b, c, d, e \middle| q \right]_{kA/b, qA/c, qA/d, qA/e} q^k \]

\[ \times \frac{(1 - A)(1 - A/bc)(1 - A/bd)(1 - A/cd)}{(1 - b)(1 - c)(1 - d)(1 - e)} \left( \frac{bcd}{A} \right); \]

\[ \triangle \left[ b, c, d, e \middle| q \right]_{kA/b, qA/c, qA/d, qA/e} = \frac{1 - Aq^{2k}}{1 - A} \left[ b, c, d, e \middle| q \right]_{kA/b, qA/c, qA/d, qA/e} q^k \]

Proposition 17 (Well-poised nonterminating series transformation: \( qa^2 = bcdef \)).

\[
8\phi_7 \left[ a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \mid q; q \right] = 8\phi_7 \left[ q^n a, q\sqrt{q^n a}, -q\sqrt{q^n a}, b, c, d, q^m e, q^m f \mid q; q \right] \\
\times \left[ qa, qa/bc, qa/bd, qa/cd \mid q \right]_m - \left[ qa, b, c, d, qe, qf \mid q, qa/b, qa/c, qa/d, qa/e, qa/f \mid q \right]_{\infty} \\
\times \frac{q a/bd}{1 - qa/bc} \sum_{k=0}^{m-1} \left[ qa/bc, qa/bd, qa/cd \mid q \right]_k q^k.
\]

Proof. For \( qa^2 = bcdef \), define the two sequences by

\[
A_k := \left[ qa, qe, qf, qa/ef \mid q \right]_k \text{ and } B_k := \left[ b, c, d, qef \mid q \right]_k.
\]

According to Lemma 16, we have

\[
A_{-1}B_0 = 0 \text{ and } [AB]_+ = \frac{1}{1 - a/ef} \left[ qa, b, c, d, qe, qf \mid q, qa/b, qa/c, qa/d, qa/e, qa/f \mid q \right]_{\infty}
\]

as well as the finite differences

\[
\nabla A_k = \frac{1 - a q^{2k}}{1 - a} \left[ a, e, f, a/ef \mid q \right]_k q^k, \\
\times \frac{1 - a q^{2k+1}}{1 - qa} \left[ b, c, d, qef \mid q^2a/b, q^2a/c, q^2a/d, qa/ef \mid q \right]_k q^k.
\]

Then applying the modified Abel’s lemma on summation by parts, we can manipulate the series as follows:

\[
\sum_{k=0}^{\infty} B_k \nabla A_k = [AB]_+ + \sum_{k=0}^{\infty} A_k \Delta B_k = \frac{1}{1 - a/ef} \left[ qa, b, c, d, qe, qf \mid q, qa/b, qa/c, qa/d, qa/e, qa/f \mid q \right]_{\infty} \\
\]
\[ \times \varphi_7 \left[ qa, q\sqrt{qa}, -q\sqrt{qa}, b, c, d, e, f \middle| q; q \right]. \]

Observe that the last \( \varphi_7 \)-series is essentially the same as original \( \varphi_7 \)-series under parameter replacements \( a \rightarrow qa, e \rightarrow qe \) and \( f \rightarrow qf \). Iterating this process \( m \)-times, we have the equation

\[ \varphi_7 \left[ a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \middle| q; q \right] = \varphi_7 \left[ q^m a, q\sqrt{q^m a}, -q\sqrt{q^m a}, b, c, d, e, f \middle| q; q \right] \]

\[ \times \left[ qa, qa/bc, qa/bd, qa/cd \middle| q \right]_m + \sum_{k=0}^{m-1} \left[ qa, qa/bc, qa/bd, qa/cd \middle| q \right]_k \]

\[ \times \frac{1}{1-q^{-ka/ef}} \left[ q^{1+k} a, b, c, d, q^{1+k} e, q^{1+k} f \middle| q \right]_\infty. \]

Rewriting the last factorial fraction, we establish the transformation formula stated in the proposition. \( \square \)

Letting \( f = q^{-m} \) in Proposition 17, we find Jackson’s very well-poised \( \varphi_7 \)-summation formula.

**Corollary 18** (Very well-poised \( \varphi_7 \)-identity: cf. [6, §8.3], [11, Eq. 1.5], [17, II-22] and [24, IV-8]).

\[ \varphi_7 \left[ a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-m} \middle| q; q \right] = \left[ qa, qa/bc, qa/bd, qa/cd \middle| q \right]_m \text{ provided } q^{1+m} a^2 = bcde. \]

Letting \( m \rightarrow \infty \) in Proposition 17, we find the following transformation from a very well-poised \( \varphi_7 \)-series to a balanced \( 3\varphi_2 \)-series (cf. [17, III-36]).

**Corollary 19** (Transformation from very well-poised \( \varphi_7 \)-series to balanced \( 3\varphi_2 \)-series).

\[ \varphi_7 \left[ a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \middle| q; q \right] = \left[ qa, qa/bc, qa/bd, qa/cd \middle| q \right] \times 3\varphi_2 \left[ b, c, d \middle| qa/e, qa/f \middle| q \right]_\infty \]

\[ - \frac{qa/bcd}{1-qa/bcd} \left[ qa, b, c, d, qe, qf \middle| q \right] \times \sum_{k=0}^{+\infty} \left[ qa/bc, qa/bd, qa/cd \middle| q \right]_k q^k \text{ provided } qa^2 = bcdef. \]
The informed reader may notice the difference of the last relation from Watson’s $q$-Whipple transformation formula (cf. [17, III-18]):

$$
\phi_7 \left[ \begin{array}{cccccc}
 a, q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, \\
 \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, \\
 q^{n+1} & a^2 & \end{array} \right]_{\frac{q^2+n}{bcde}} = \phi_3 \left[ \begin{array}{cccccc}
 qa, & qa/bc, & qa/b, & qa/c, & qa/d, \\
 qa/bc, & qa/b, & qa/c, & qa/d, \\
 q & q^{n} & \end{array} \right]_{\frac{q}{q}}.
$$

In Corollary 19, replacing $e$ and $f$ respectively by $q^{-m}e$ and $q^{1+m}a^2/bcde$ and then letting $m \to \infty$, we recover the nonterminating very well-poised $6\phi_5$-series identity stated in Corollary 14:

$$
\phi_5 \left[ \begin{array}{cccccc}
 a, q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, \\
 \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, \\
 \end{array} \right]_{\frac{q}{qbcd}} \left[ \begin{array}{cccc}
 qa, & qa/bc, & qa/bd, & qa/cd, \\
 qa/bc, & qa/bd, & qa/cd, \\
 q & q & \end{array} \right]_{\frac{q}{q}} = \phi_7 \left[ \begin{array}{cccccc}
 a, q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, \\
 \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, \\
 q^{n+1} & a^2 & \end{array} \right]_{\frac{q^{n+1}}{bcde}}.
$$

provided that the parameters are subject to $|qa/bcd| < 1$ for convergence.

10. $10\phi_9$-Generalization of Jackson’s terminating very well-poised series identity

According to the linear combination

$$
(1 - q^k\lambda)(1 - q^k\frac{a}{\lambda}) = \frac{(\lambda - d)(1 - a/d\lambda)}{(b - d)(1 - a/bd)}(1 - q^k\frac{a}{b})(1 - q^k\frac{a}{d}) + \frac{(b - \lambda)(1 - a/b\lambda)}{(b - d)(1 - a/bd)}(1 - q^k\frac{d}{b})(1 - q^k\frac{d}{d})
$$

we derive the following expression

$$
\phi_9 \left[ \begin{array}{cccccc}
 a, q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, \\
 \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, \\
 q^{n+1} & a^2 & \end{array} \right]_{\frac{q}{q}} = \phi_7 \left[ \begin{array}{cccccc}
 a, q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, \\
 \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, \\
 q^{n+1} & a^2 & \end{array} \right]_{\frac{q}{q}}
$$

$$
\times \phi_7 \left[ \begin{array}{cccccc}
 a, q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, \\
 \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, \\
 q^{n+1} & a^2 & \end{array} \right]_{\frac{q}{q}}
$$

When $q^n a^2 = bcde$, two $8\phi_7$-series just displayed are both well poised and balanced and can therefore be evaluated by means of Jackson’s terminating very well-poised summation formula as
Substituting these evaluations into $10\phi_9$-series expression and then simplifying the result, we obtain the following very well-poised series identity.

**Theorem 20** (Very well-poised and balanced $10\phi_9$-series identity: $q^n a^2 = bcde$).
**Proposition 21** (Very well-poised and 3-balanced \(8\phi_7\)-series identity: Lakin [23, Eq. 7], see also [5] and [8, Eq. 3.3b]).

\[
\begin{align*}
\phi_7 & \left[ a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n}\right| q; q^2 \\
& = \left\{ q^n + \frac{(1 - q^n)(1 - a/bcd)(1 - q^n a^2/abcd)}{(1 - a/bc)(1 - a/bd)(1 - a/cd)} \right\} \\
& \times \left[ q a, a/bc, a/bd, a/cd \left| q \right|_n \right],
\end{align*}
\]

where \(q^n a^2 = bcde\).

11. Bibasic transformation and symmetric formula

Define two sequences by factorial fractions

\[
A_k := \left[ pb, pc, pd, pe \left| p \right|_k \right],
\]
\[
B_k := \left[ B, C, D, E \left| q \right|_k \right].
\]

When \(a^2 = bcde\) and \(A^2 = BCDE\), we can compute, by means of Lemma 16, the following differences

\[
\nabla A_k = \frac{1 - ap^{2k}}{1 - a} \left[ b, c, d, e \left| p \right|_k \right] p^k \\
\times \frac{bcd (1 - a)(1 - a/bc)(1 - a/bd)(1 - a/cd)}{a (1 - b)(1 - c)(1 - d)(1 - e)};
\]
\[
\Delta B_k = \frac{1 - Aq^{2k}}{1 - A} \left[ B, C, D, E \left| q \right|_k \right] q^k \\
\]

We can also evaluate the following two limits

\[
[AB]_+ = \left[ B, C, D, E \left| q \right|_\infty \right] \left[ pb, pc, pd, pe \left| p \right|_\infty \right],
\]
\[
[AB]_- = \left[ qB/A, qC/A, qD/A, qE/A \left| q \right|_\infty \right] \left[ b/a, c/a, d/a, e/a \left| 1/b, 1/c, 1/d, 1/e \right|_\infty \right].
\]

According to the modified Abel’s lemma on summation by parts for bilateral series, we establish the following bilateral bibasic series transformation.

**Theorem 22** (Bilateral bibasic series transformation). For the indeterminate \(p\) and \(q\) with \(0 < |p| < 1\) and with \(0 < |q| < 1\), there holds transformation
\[
\sum_{n=-\infty}^{+\infty} \frac{1 - ap^{2n}}{1 - a} \left[ \begin{array}{c}
 b, c, d, e \\
pa/b, pa/c, pa/d, pa/e \end{array} \right] p^n \left[ \begin{array}{c}
 B, C, D, E \\
A/B, A/C, A/D, A/E \end{array} \right] q_n
\]

\[
= \frac{(a/bcd)(1-b)(1-c)(1-d)(1-e)}{(1-a)(1-a/bc)(1-a/bd)(1-a/cd)} \times \left[ \begin{array}{c}
 B, C, D, E \\
A/B, A/C, A/D, A/E \end{array} \right] p^n \left[ \begin{array}{c}
 b/a, c/a, d/a, e/a \\
pa/b, pa/c, pa/d, pa/e \end{array} \right] q_n
\]

\[
- \left[ \begin{array}{c}
 qB/A, qC/A, qD/A, qE/A \\
q/B, q/C, q/D, q/E \end{array} \right] q_n
\]

\[
+ \sum_{k=-\infty}^{+\infty} \frac{1 - Aq^{2k}}{1 - A} \left[ \begin{array}{c}
 B, C, D, E \\
qA/B, qA/C, qA/D, qA/E \end{array} \right] q_k
\]

\[
\times \left[ \begin{array}{c}
pb, pc, pd, pe \\
pa/b, pa/c, pa/d, pa/e \end{array} \right] q_n
\]

\[
\]

provided that \( a^2 = bcde \) and \( A^2 = BCDE \), which imply that the first series is 2-balanced with respect to \( p \) and the second to \( q \).

**Proof.** We need only to verify the convergence. For the first series, the truncated part \( \sum_{n\geq 0} \) along the positive direction is convergent for \(|p| < 1\) and \(|q| < 1\), which make the series to be comparable with the geometric series \( \sum_{n\geq 0} p^n \). By means of transformation

\[
\frac{(x; q)_n}{(y; q)_n} = \left( \frac{y}{x} \right)^n \times \frac{(q/y; q)_n}{(q/x; q)_n}
\]

the truncated part \( \sum_{n<0} \) of the first series along the negative direction can be expressed under replacement \( n \rightarrow -n \) as

\[
\sum_{n=1}^{\infty} \frac{1 - p^{2n}/a}{1 - 1/a} \left[ \begin{array}{c}
 b/a, c/a, d/a, e/a \\
p/b, p/c, p/d, p/e \end{array} \right] p^n \left[ \begin{array}{c}
 qB/A, qC/A, qD/A, qE/A \\
q/B, q/C, q/D, q/E \end{array} \right] q_n
\]

This is again a convergent series for \(|p| < 1\) and \(|q| < 1\). Therefore the first bilateral series with respect to \( n \) is convergent. Symmetrically, the second bilateral series with respect to \( k \) is convergent either. \( \square \)

In particular, letting \( e = a \), we derive the following transformation between bibasic unilateral series.

**Proposition 23** (Bibasic series transformation). For the indeterminate \( p \) and \( q \) with \( 0 < |p| < 1 \) and with \( 0 < |q| < 1 \), there holds transformation
\[
\sum_{n=0}^{\infty} \frac{1 - ap^{2n}}{1 - a} \left[ \begin{array}{cccc} a, & b, & c, & d \\ p, & pa/b, & pa/c, & pa/d \end{array} \right| p \right] p^n \left[ \begin{array}{cccc} B, & C, & D, & E \\ A/B, & A/C, & A/D, & A/E \end{array} \right| q \right]_n
\]

\[
= \left[ \begin{array}{cccc} B, & C, & D, & E \\ A/B, & A/C, & A/D, & A/E \end{array} \right| q \right] \sum_{n=0}^{\infty} \frac{1}{1 - A} \left[ \begin{array}{cccc} p, & pb, & pc, & pd \\ p, & pa/b, & pa/c, & pa/d \end{array} \right| p \right]_n
\]

\[
\frac{(1 - A)(1 - A/BC)(1 - A/BD)(1 - A/CD)}{(1 - A/B)(1 - A/C)(1 - A/D)(1 - A/BCD)} \sum_{k=0}^{\infty} \frac{1 - Aq^{2k}}{1 - A} \left[ \begin{array}{cccc} B, & C, & D, & E \\ qA/B, & qA/C, & qA/D, & qA/E \end{array} \right| q \right] q^k \left[ \begin{array}{cccc} p, & pb, & pc, & pd \\ p, & pa/b, & pa/c, & pa/d \end{array} \right| p \right]_k
\]

provided that \(a = bcd\) and \(A^2 = BCDE\), which imply that the first series is 2-balanced with respect to \(p\) and the second to \(q\).

Performing parameter replacements in the last transformation

\[
A \rightarrow q^{-2m} / A,
\]

\[
B \rightarrow q^{-m} B / A,
\]

\[
C \rightarrow q^{-m} C / A,
\]

\[
D \rightarrow q^{-m} D / A,
\]

\[
E \rightarrow q^{-m}
\]

and then reversing the finite series on the right-hand side by \(k \rightarrow m - k\), we get the following symmetric transformation.

**Corollary 24** (Bibasic symmetric transformation). For complex parameters \(\{a, b, c, d\}\) and \(\{A, B, C, D\}\) satisfying \(a = bcd\) and \(A = BCD\) respectively, there holds bibasic symmetric transformation

\[
\sum_{n=0}^{m} \frac{1 - ap^{2n}}{1 - a} \left[ \begin{array}{cccc} a, & b, & c, & d \\ p, & pa/b, & pa/c, & pa/d \end{array} \right| p \right] p^n \left[ \begin{array}{cccc} q^{-m}, & q^{-m} B / A, & q^{-m} C / A, & q^{-m} D / A \\ q^{-m}, & q^{-m} B / A, & q^{-m} C / A, & q^{-m} D / A \end{array} \right| q \right]_n
\]

\[
= \sum_{k=0}^{m} \frac{1 - Aq^{2k}}{1 - A} \left[ \begin{array}{cccc} A, & B, & C, & D \\ q, & qA/B, & qA/C, & qA/D \end{array} \right| q \right] q^k
\]

\[
\times \left[ \begin{array}{cccc} p^{-m}, & p^{-m} b / a, & p^{-m} c / a, & p^{-m} d / a \\ p^{-m}, & p^{-m} b / a, & p^{-m} c / a, & p^{-m} d / a \end{array} \right| p \right]_k
\]

\[
\times \left[ \begin{array}{cccc} p, & pb, & pc, & pd \\ p, & pa/b, & pa/c, & pa/d \end{array} \right]_m \left[ \begin{array}{cccc} q, & qA/B, & qA/C, & qA/D \\ q, & qA/B, & qA/C, & qA/D \end{array} \right]_m.
In fact, if letting
\[ \Xi[p : a, b, c, d \mid q : A, B, C, D] := [p, pa/b, pa/c, pa/d \mid p]_m \]
\[ \times \sum_{n=0}^{m} \frac{1 - ap^{2n}}{1 - a} \left[ p, pa/b, pa/c, pa/d \mid p \right]_n p^n \]
\[ \times \left[ q^{-m} B/A, q^{-m} C/A, q^{-m} D/A, q^{-m} \mid q \right]_n \]
then we can reformulate the transformation in Corollary 24 as the following symmetric expression:
\[ \Xi[p : a, b, c, d \mid q : A, B, C, D] = \Xi[q : A, B, C, D \mid p : a, b, c, d] . \]

When \( p = q \), the last symmetric relation reduces to the “split-poised” transformation due to Gasper [15, Eq. 2.11]:
\[ ^{10}\phi_9[a, q \sqrt{a}, -q \sqrt{a}, b, c, d, q^{-m} B/A, q^{-m} C/A, q^{-m} D/A \mid q; q] \]
\[ = ^{10}\phi_9[A, q \sqrt{A}, -q \sqrt{A}, B, C, D, q^{-m} B/A, q^{-m} C/A, q^{-m} D/A \mid q; q] \]
\[ \times \left[ qA, qB, qC, qD, qA/B, qA/C, qA/D, qA, qB, qC, qD, qa/b, qa/c, qa/d \mid q \right]_m \]
where \( a = bcd \) and \( A = BCD \) are assumed as before.

References

[1] N.H. Abel, Untersuchungen über die Reihe \( 1 + \frac{m}{2}x + \frac{m(m-1)}{2!} x^2 + \cdots \), J. Reine Angew. Math. 1 (1826) 311–339.