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# Twisted modules over vertex algebras on algebraic curves ☆

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**Abstract**

We extend the geometric approach to vertex algebras developed by the first author to twisted modules, allowing us to treat orbifold models in conformal field theory. Let $V$ be a vertex algebra, $H$ a finite group of automorphisms of $V$, and $C$ an algebraic curve such that $H \subset \mathrm{Aut}(C)$. We show that a suitable collection of twisted $V$-modules gives rise to a section of a certain sheaf on the quotient $X = C/H$. We introduce the notion of conformal blocks for twisted modules, and analyze them in the case of the Heisenberg and affine Kac–Moody vertex algebras. We also give a chiral algebra interpretation of twisted modules.



## 1. Introduction

Conformal field theory (CFT) in two dimensions provides a rich setting in which several areas of mathematics such as representation theory and algebraic geometry interact in a natural way. In recent years, much effort has been spent on setting up a precise mathematical framework for CFT. The algebraic aspect of the theory has been formalized in the language of vertex algebras (see [B,FLM,K,FB]). In order to understand the rich geometry behind CFT, this algebraic approach must be combined with a geometric formalism.

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In [FB], an algebro-geometric approach to vertex algebras is introduced (see [H,BD] for other approaches). Starting with a conformal vertex algebra $V$ and an algebraic curve $X$, one can construct a vector bundle $\mathscr{V}_X$ on $X$ such that vertex operators become (endomorphism-valued) sections of $\mathscr{V}_X^*$. This gives a coordinate-free description of vertex operators and allows one to make contact with the fascinating geometry pertaining to $X$ and related moduli spaces.

If a vertex algebra has a group of automorphisms, then its representation theory may be enhanced by the inclusion of twisted modules. The systematic study of twisted modules was initiated in [FLM] where twisted vertex operators were used in the construction of the Moonshine Module vertex algebra (see Chapter 9 of [FLM] and the works [Le1,Le2]). The notion of the twisted module was formulated in [FFR,D] following [FLM]. Twisted modules (or twisted sectors as they are known in the physics literature) appear as important ingredients of the so-called orbifold models of conformal field theory (see [DHVW,DVVV]). They have been extensively studied in recent years (see, e.g., [Li,DLM,BKT]).

In this paper we extend the geometric formalism developed in [FB] to twisted modules over vertex algebras. Let $C$ be a smooth projective curve, and $H \subset \mathrm{Aut}(C)$ a finite group of automorphisms of $C$ such that the stabilizer of the action of $H$ on at a generic point of $C$ consists of the identity element of $H$. Suppose furthermore that $V$ is a conformal vertex algebra, and that $H$ acts on $V$ by conformal automorphisms. We show that with these data, the vector bundle $\mathscr{V}_C$ acquires a $H$-equivariant structure, lifting the action of $H$ on $C$. Let $X = C/H$ be the quotient curve, and $v : C \to X$ the quotient map, ramified at the fixed points of $H$. Denote by $\overset{\circ}{C} \subset C$ the locus of points in $C$ whose stabilizer in $H$ is the identity element. Let $\overset{\circ}{X} \subset X$ be the image of $\overset{\circ}{C}$ in $X$ and $\overset{\circ}{v} : \overset{\circ}{C} \to \overset{\circ}{X}$ the restriction of $v$ to $\overset{\circ}{C}$. Thus, $\overset{\circ}{C}$ is a principal $H$-bundle over $\overset{\circ}{X}$. The vector bundle $\mathscr{V}_{\overset{\circ}{C}}$ over $\overset{\circ}{C}$ carries a $H$-equivariant structure and hence descends to a vector bundle on $\overset{\circ}{X}$ which we denote by $\mathscr{V}^H_{\overset{\circ}{X}}$.

Let $x \in X$. Then $x$ corresponds to a $H$-orbit $\mathbf{O}_x$ in $C$. For each point $p \in v^{-1}(x)$, the stabilizer $H_p$ is a cyclic group, which has a canonical generator $h_p$, the monodromy around $p$ (generically, $H_p = \{e\}$ and $h_p = e$). We call a collection $\{M_p^{h_p}\}$ of $h_p$-twisted modules satisfying certain compatibilities, a $V$-module along $v^{-1}(x)$. For example, if $h_p = e$, then each $M_p^{h_p}$ is an ordinary $V$-module and the requirement is that if $p' = g(p) \in v^{-1}(p)$, then $M_{p'}^{h_{p'}}$ is obtained from $M_p^{h_p}$ by twisting the $V$-action by the automorphism of $V$ corresponding to $g$. If, on the other hand, $H = \mathbb{Z}/N\mathbb{Z}$ and $h_p$ is a generator of $H$, then $M_p^{h_p}$ can be an arbitrary $h_p$-twisted $V$-module.

We attach to a $V$-module $\mathscr{M}_x$ along $v^{-1}(x)$ a section $\mathscr{Y}^{\mathscr{M}_x}$ of $\mathscr{V}^{H,*}_{\overset{\circ}{X}}$ on $\mathscr{D}_x^\times$, the punctured disc at $x$. Using this structure we define the spaces of conformal blocks in the twisted setting. The space of conformal blocks is associated to a pair $(C, H)$ as above and a collection of $V$-modules along $v^{-1}(x)$ attached to a set of points of $X \backslash \overset{\circ}{X}$, and a (possibly empty) collection of $V$-modules along $v^{-1}(x), x \in \overset{\circ}{X}$. We give two equivalent definitions of the space of conformal blocks: using the action of a certain Lie algebra obtained from Fourier coefficients of vertex operators, and using analytic continuation (as in [FB]). In the case of the Heisenberg and affine Kac–Moody vertex algebras this definition may be simplified using twisted versions of the Heisenberg and affine Lie algebras, respectively.

Finally, we explain the connection with the chiral algebra formalism. The right $\mathscr{D}_X$-module $\mathscr{A} = \mathscr{V}_X \otimes \Omega_X$ is a chiral algebra on $X$ in the sense of Beilinson and Drinfeld [BD] (see [FB, Chapter 18]). The action of $H$ on $V$ induces an action of $H$ by automorphisms of $\mathscr{A}$. Then the twist $\mathscr{A}^{\overset{\circ}{C}}$ of $\mathscr{A}|_{\overset{\circ}{X}}$ by the $H$-torsor $\overset{\circ}{C}$, $\mathscr{A}^{\overset{\circ}{C}} = \mathscr{A}|_{\overset{\circ}{X}} \underset{H}{\times} \overset{\circ}{C}$, is also a chiral algebra. Twisted $V$-modules correspond to $\mathscr{A}^{\overset{\circ}{C}}$-modules supported at the points $x \in X \backslash \overset{\circ}{X}$, and the above space of conformal blocks may be defined in terms of these $\mathscr{A}^{\overset{\circ}{C}}$-modules.

## 2. Vertex algebras and modules

In this paper we will use the language of vertex algebras, their modules, and twisted modules. For an introduction to vertex algebras and their modules see [FLM,K,FB], and for background on twisted modules, see [FFR,D,DLM].

We recall that a conformal vertex algebra is a $\mathbb{Z}_+$-graded vector space

$$V = \bigoplus_{n=0}^{\infty} V_n,$$

together with a vacuum vector $|0\rangle \in V_0$, a translation operator $T$ of degree 1, a conformal vector $\omega \in V_2$, and a vertex operation

$$Y : V \to \operatorname{End} V[[z^{\pm 1}]],$$

$$A \mapsto Y(A,z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}.$$

These data must satisfy certain axioms (see [FLM,K,FB]). In what follows we will denote the collection of such data simply by $V$.

A vector space $M$ is called a $V$-module if it is equipped with an operation

$$Y^M : V \to \operatorname{End} M[[z^{\pm 1}]],$$

$$A \mapsto Y^M(A,z) = \sum_{n \in \mathbb{Z}} A^M_{(n)} z^{-n-1}$$

such that for any $v \in M$ we have $A^M_{(n)} v = 0$ for large enough $n$. This operation must satisfy the following axioms:

- $Y^M(|0\rangle, z) = \mathrm{Id}_M$;
- For any $v \in M$ there exists an element

$$f_v \in M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

such that the formal power series

$$Y^M(A,z)Y^M(B,w)v \quad \text{and} \quad Y_M(Y(A,z-w)B,w)v$$

are expansions of $f_v$ in $M((z))((w))$ and $M((w))((z-w))$, respectively.

The power series $Y^M(A,z)$ are called vertex operators. We write the vertex operator corresponding to $\omega$ as

$$Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L^M_n z^{-n-2},$$

where $L^M_n$ are linear operators on $V$ generating the Virasoro algebra. Following [D], we call $M$ admissible if $L^M_0$ acts semi-simply with integral eigenvalues.

Now let $\sigma_V$ be a conformal automorphism of $V$, i.e., an automorphism of the underlying vector space preserving all of the above structures (in particular $\sigma_V(\omega) = \omega$). We will assume that $\sigma_V$ has finite order $N > 1$. A vector space $M^\sigma$ is called a *$\sigma_V$-twisted* $V$-module (or simply twisted module) if it is equipped with an operation

$$Y^{M^\sigma} : V \to \mathrm{End}\, M^\sigma[[z^{\pm\frac{1}{N}}]],$$

$$A \mapsto Y^{M^\sigma}(A, z^{\frac{1}{N}}) = \sum_{n \in \frac{1}{N}\mathbb{Z}} A^{M^\sigma}_{(n)} z^{-n-1}$$

such that for any $v \in M^\sigma$ we have $A^{M^\sigma}_{(n)} v = 0$ for large enough $n$. Please note that we use the notation $Y^{M^\sigma}(A, z^{\frac{1}{N}})$ rather than $Y^{M^\sigma}(A,z)$ in the twisted setting. This operation must satisfy the following axioms (see [FFR,D,DLM,Li]):

- $Y^{M^\sigma}(|0\rangle, z^{\frac{1}{N}}) = \mathrm{Id}_{M^\sigma}$.
- For any $v \in M^\sigma$, there exists an element

$$f_v \in M^\sigma[[z^{\frac{1}{N}}, w^{\frac{1}{N}}]][z^{-\frac{1}{N}}, w^{-\frac{1}{N}}, (z-w)^{-1}]$$

such that the formal power series

$$Y^{M^\sigma}(A, z^{\frac{1}{N}}) Y^{M^\sigma}(B, w^{\frac{1}{N}})v \quad \text{and} \quad Y^{M^\sigma}(Y(A, z-w)B, w^{\frac{1}{N}})v$$

are expansions of $f_v$ in $M^\sigma((z^{\frac{1}{N}}))((w^{\frac{1}{N}}))$ and $M^\sigma((w^{\frac{1}{N}}))((z-w))$, respectively.

- If $A \in V$ is such that $\sigma_V(A) = e^{\frac{2\pi i m}{N}} A$, then $A^{M^\sigma}_{(n)} = 0$ unless $n \in \frac{m}{N} + \mathbb{Z}$.

The series $Y^{M^\sigma}(A, z)$ are called twisted vertex operators. In particular, the Fourier coefficients of the twisted vertex operator

$$Y^{M^\sigma}(\omega, z^{\frac{1}{N}}) = \sum_{n \in \mathbb{Z}} L_n^{M^\sigma} z^{-n-2},$$

generate an action of the Virasoro algebra on $M^\sigma$. The $\sigma_V$-twisted module $M^\sigma$ is called admissible if $L_0^{M^\sigma}$ acts semi-simply with eigenvalues in $\frac{1}{N}\mathbb{Z}$.

One shows in the same way as in [FB, Section 4.1], that the axioms imply the following commutation relations between the coefficients of twisted vertex operators:

$$[A^{M^\sigma}_{(m)}, B^{M^\sigma}_{(k)}] = \sum_{n \geqslant 0} \binom{m}{n} (A_{(n)} \cdot B)^{M^\sigma}_{(m+k-n)}, \tag{2.1}$$

where by definition

$$\binom{m}{n} = \frac{m(m-1)\ldots(m-n+1)}{n!}, \quad n \in \mathbb{Z}_{>0}; \quad \binom{m}{0} = 1.$$

We also have the following analogue of Proposition 4.1 of [FB]:

**Lemma 2.1.** *For any $A \in V$, $Y^{M^\sigma}(TA, z^{\frac{1}{N}}) = \partial_z Y^{M^\sigma}(A, z^{\frac{1}{N}})$.*

**Proof.** We apply axiom (2) in the situation where $B = |0\rangle$. Then

$$Y^{M^\sigma}(Y(A, z-w)|0\rangle, w^{\frac{1}{N}})v = \sum_{n \geqslant 0} Y^{M^\sigma}(A_{(-n-1)}|0\rangle, w^{\frac{1}{N}})v(z-w)^n.$$

But $A_{(-2)}|0\rangle = TA$, therefore $Y^{M^\sigma}(TA, w^{\frac{1}{N}})v$ appears as the coefficient in front of $(z-w)$ in this series. Hence it should coincide with the coefficient in front of $(z-w)$ in the expansion of $Y^{M^\sigma}(A, z^{\frac{1}{N}})v$ in a power series in $w^{\frac{1}{N}}$ and $(z-w)$. But the latter is equal to $\partial_w Y^{M^\sigma}(A, w^{\frac{1}{N}})$.

Applying formula (2.1) in the case when $A = \omega$ and $m = 1$ (so that $A_{(m)} = L_0$), we obtain that

$$[L_0^{M^\sigma}, B^{M^\sigma}_{(k)}] = (L_0 \cdot B)^{M^\sigma}_{(k)} + (L_{-1} \cdot B)^{M^\sigma}_{(k+1)}.$$

But in a conformal vertex algebra $L_{-1} \cdot B = TB$ and $(TB)^{M^\sigma}_{(k+1)} = (-k-1)B_{(k)}$ by Lemma 2.1. Therefore if $B$ is homogeneous of degree $\Delta$, then

$$[L_0^{M^\sigma}, B^{M^\sigma}_{(k)}] = (\Delta - k - 1)B^{M^\sigma}_{(k)}. \tag{2.2}$$

Suppose that $M^\sigma$ is an admissible module. Then we define a linear operator $S_\sigma$ on $M^\sigma$ as follows. It acts on the eigenvectors of $L_0^{M^\sigma}$ with eigenvalue $\frac{m}{N}$ by multiplication by $e^{\frac{2\pi i m}{N}}$. Hence we obtain an action of the cyclic group of order $N$ generated by $\sigma$ on $M^\sigma$, $\sigma \mapsto S_\sigma$. According to the axioms of twisted module and formula (2.2) we have the following identity:

$$S_\sigma^{-1} Y^{M^\sigma}(\sigma \cdot A, z^{\frac{1}{N}}) S_\sigma = Y^{M^\sigma}(A, z^{\frac{1}{N}}). \tag{2.3}$$

Finally, we remark that there is an analogue of the Reconstruction Theorem for twisted modules. Namely, suppose that $V$ is generated by vectors $a^\alpha \in V, \alpha \in S$, in the sense of the usual Reconstruction Theorem (see Theorem 4.5 of [K] or Theorem 3.6.1 of [FB]). Then if $M^\sigma$ is a $\sigma$-twisted $V$-module, the twisted vertex operators $Y^{M^\sigma}(A, z^{\frac{1}{N}})$ for all $A \in V$ may be reconstructed from the series $Y^{M^\sigma}(a^\alpha, z^{\frac{1}{N}}), \alpha \in S$. This follows from H. Li's formula for $Y^{M^\sigma}(A_{(n)}B, z^{\frac{1}{N}})$ in terms of $Y^{M^\sigma}(A, z^{\frac{1}{N}})$ and $Y^{M^\sigma}(B, z^{\frac{1}{N}})$ [Li]. But this formula is more complicated than its untwisted analogue, so the resulting formula for a general twisted vertex operator usually looks rather cumbersome (see for example formula (7.1) below).

## 3. Torsors and twists

Let $M^\sigma$ be an admissible conformal $\sigma_V$-twisted $V$-module where $\mathrm{ord}(\sigma_V) = N$. In this section we define a group $\mathrm{Aut}_N \mathcal{O}$ which naturally acts on $M^\sigma$, as well as natural torsors for $\mathrm{Aut}_N \mathcal{O}$. This will allow us to twist $M^\sigma$ by a certain torsor of formal coordinates.

### *3.1. The group* $\mathrm{Aut}_N \mathcal{O}$

Let $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$ denote the group of continuous algebra automorphisms of $\mathbb{C}[[z^{\frac{1}{N}}]]$. Since $\mathbb{C}[[z^{\frac{1}{N}}]]$ is topologically generated by $z^{\frac{1}{N}}$, an automorphism $\rho$ of $\mathbb{C}[[z^{\frac{1}{N}}]]$ is completely determined by the image of $z^{\frac{1}{N}}$, which is a series of the form

$$\rho(z^{\frac{1}{N}}) = \sum_{n \in \frac{1}{N}\mathbb{Z}, n > 0} c_n z^n, \tag{3.1}$$

where $c_{\frac{1}{N}} \neq 0$. Hence we identify $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$ with the space of power series in $z^{\frac{1}{N}}$ having non-zero linear term. For more on the structure of the group $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$, see Section 5.1 of [FB]. Recall that we denote $\mathbb{C}[[z]]$ by $\mathcal{O}$.

**Definition 3.1.** $\mathrm{Aut}_N\mathcal{O}$ is the subgroup of $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$ preserving the subalgebra $\mathbb{C}[[z]] \subset \mathbb{C}[[z^{\frac{1}{N}}]]$.

Thus, $\mathrm{Aut}_N\mathcal{O}$ consists of power series of the form

$$\rho(z^{\frac{1}{N}}) = \sum_{n \in \frac{1}{N}+\mathbb{Z}, n>0} c_n z^n, \quad c_{\frac{1}{N}} \neq 0. \tag{3.2}$$

There is a homomorphism $\mu$: $\mathrm{Aut}_N\mathcal{O} \to \mathrm{Aut}\,\mathcal{O}$ which takes $\rho \in \mathbb{C}[[z^{\frac{1}{N}}]]$ to the automorphism of $\mathbb{C}[[z]]$ that it induces. At the level of power series, this is just the map $\mu : \rho(z) \mapsto \rho(z)^N$. The kernel consists of the automorphisms of the form $z^{\frac{1}{N}} \mapsto \varepsilon z^{\frac{1}{N}}$, where $\varepsilon$ is an $N$th root of unity, so we have the following exact sequence:

$$1 \to \mathbb{Z}/N\mathbb{Z} \to \mathrm{Aut}_N\mathcal{O} \to \mathrm{Aut}\,\mathcal{O} \to 1$$

making $\mathrm{Aut}_N\mathcal{O}$ a central extension of $\mathrm{Aut}\,\mathcal{O}$ by the cyclic group $\mathbb{Z}/N\mathbb{Z}$.

The Lie algebra of $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$ is

$$\mathrm{Der}^{(0)}\mathbb{C}[[z^{\frac{1}{N}}]] = z^{\frac{1}{N}}\mathbb{C}[[z^{\frac{1}{N}}]]\partial_{z^{\frac{1}{N}}},$$

and the Lie algebra of $\mathrm{Aut}_N\mathcal{O}$ is its Lie subalgebra $\mathrm{Der}^{(0)}_N\mathcal{O} = z^{\frac{1}{N}}\mathbb{C}[[z]]\partial_{z^{\frac{1}{N}}}$. The homomorphism $\mu$ induces an isomorphism of the corresponding Lie algebras sending

$$z^{k+\frac{1}{N}}\partial_{z^{\frac{1}{N}}} \mapsto Nz^{k+1}\partial_z, \quad k \in \mathbb{Z}, k \geqslant 0.$$

### 3.2. *The* $\mathrm{Aut}_N\mathcal{O}$*-torsor of special coordinates*

Let $(\mathcal{D}, \sigma_{\mathcal{D}})$ be a pair consisting of a formal disc $\mathcal{D} = \mathrm{Spec}\, R$, where $R \cong \mathbb{C}[[z]]$ and an automorphism $\sigma_{\mathcal{D}}$ of $\mathcal{D}$ (equivalently, of $R$) of order $N$. We denote by $\bar{\mathcal{D}}$ the quotient of $\mathcal{D}$ by $\langle \sigma_{\mathcal{D}} \rangle$, i.e., the disc $\mathrm{Spec}\, R^{\sigma_{\mathcal{D}}}$, where $R^{\sigma_{\mathcal{D}}}$ is the subalgebra of $\sigma_{\mathcal{D}}$-invariant elements.

A formal coordinate $t$ is called a *special coordinate* with respect to $\sigma_{\mathcal{D}}$ if $\sigma_{\mathcal{D}}(t) = \varepsilon t$, where $\varepsilon$ is an $N$th root of unity, or equivalently, if $t^N$ is a formal coordinate on $\bar{\mathcal{D}}$. We denote by $\mathscr{A}ut(\mathscr{D})$ the set of all formal coordinates on $\mathscr{D}$ and by $\mathscr{A}ut_N(\mathscr{D})$ the subset of $\mathscr{A}ut(\mathscr{D})$ consisting of special formal coordinates. The set $\mathscr{A}ut_N(\mathscr{D})$ carries a simply transitive right action of the group $\mathrm{Aut}_N\mathcal{O}$ given by $t \mapsto \rho(t)$, where $\rho$ is the power series given in (3.2), i.e., $\mathscr{A}ut_N(\mathscr{D})$ is an $\mathrm{Aut}_N\mathcal{O}$-torsor.

### 3.3. Twisting modules by $\mathscr{A}ut_N(\mathscr{D})$

Let $M^\sigma$ be an admissible $\sigma_V$-twisted module over a conformal vertex algebra $V$. Define a representation $r^{M^\sigma}$ of the Lie algebra $\mathrm{Der}_N^{(0)}\mathcal{O}$ on $M^\sigma$ by the formula

$$z^{k+\frac{1}{N}}\partial_{z^{\frac{1}{N}}} \to -N \cdot L_k^{M^\sigma}.$$

It follows from the definition of a twisted module that the operators $L_k^{M^\sigma}, k>0$, act locally nilpotently on $M^\sigma$ and that the eigenvalues of $L_0^{M^\sigma}$ lie in $\frac{1}{N}\mathbb{Z}$, so that the operator $N \cdot L_0^{M^\sigma}$ has integer eigenvalues. This implies that the Lie algebra representation $r^{M^\sigma}$ may be exponentiated to a representation $R^{M^\sigma}$ of the group $\mathrm{Aut}_N\mathcal{O}$.

In particular, the subgroup $\mathbb{Z}/N\mathbb{Z}$ of $\mathrm{Aut}_N\mathcal{O}$ acts on $M^\sigma$ by the formula $i \mapsto S_\sigma^i$, where $S_\sigma$ is the operator defined in Section 2.

We now twist the module $M^\sigma$ by the action of $\mathrm{Aut}_N\mathcal{O}$ and define the vector space

$$\mathscr{M}^\sigma(\mathscr{D}) \overset{\mathrm{def}}{=} \mathscr{A}ut_N(\mathscr{D}) \underset{\mathrm{Aut}_N\mathcal{O}}{\times} M^\sigma. \tag{3.3}$$

Thus, vectors in $\mathscr{M}^\sigma(\mathscr{D})$ are pairs $(t, v)$, up to the equivalence relation

$$(\rho(t), v) \sim (t, R^{M^\sigma}(v)), \quad t \in \mathscr{A}ut_N(\mathscr{D}), v \in M^\sigma.$$

When $\mathscr{D} = \mathscr{D}_x$, the formal neighborhood of a point $x$ on an algebraic curve $X$, we will use the notation $\mathscr{M}_x^\sigma$.

## 4. Twisted vertex operators as sections

Our goal is to give a coordinate-independent description of the operation $Y^{M^\sigma}$. In order to do this we need to find how the operation $Y^{M^\sigma}$ transforms under changes of special coordinates. This is the subject of this section.

### 4.1. The transformation formula for twisted vertex operators

Let $\mathcal{O} = \mathbb{C}[[t]]$. Denote by $R^V$ the representation of the group $\mathrm{Aut}\mathcal{O}$ on $V$ obtained by exponentiating the representation $r^V$ of the Lie algebra $\mathrm{Der}^{(0)}\mathcal{O}$ sending $t^{n+1}\partial_t$ to $-L_n, n \geqslant 0$ (see Section 5.2 of [FB]). Recall that for any $\rho(t^{\frac{1}{N}}) \in \mathrm{Aut}_N\mathcal{O}$, we have

$\rho(t^{\frac{1}{N}})^N \in \mathrm{Aut}\mathcal{O}$. For any $\tau(t) \in \mathrm{Aut}\mathcal{O}$ we denote by $\tau_z$ the element of $\mathrm{Aut}(\mathbb{C}[[z]]\widehat{\otimes}\mathcal{O})$ obtained by expanding $\tau(z+t) - \tau(z)$ in powers of $t$ (see Section 5.4.5 of [FB]). Then we have the following analogue of Lemma 5.4.6 from [FB] (that lemma is originally due to Y.-Z. Huang [H]).

**Lemma 4.1.** *For any* $A \in V$, $\rho \in \mathrm{Aut}_N\mathcal{O}$

$$R^{M^\sigma}(\rho)Y^{M^\sigma}(R^V((\rho^N)_z)^{-1}A, \rho(z^{\frac{1}{N}}))R^{M^\sigma}(\rho)^{-1} = Y^{M^\sigma}(A, z^{\frac{1}{N}}). \tag{4.1}$$

**Proof.** The exponential map $\mathrm{Der}_N^{(0)} \to \mathrm{Aut}_N\mathcal{O}$ is surjective, so it suffices to consider the infinitesimal version of (4.1). Write

$$\rho = \exp(\varepsilon v(z^{\frac{1}{N}})\partial_{z^{\frac{1}{N}}}) \cdot z^{\frac{1}{N}},$$

where

$$v(z^{\frac{1}{N}}) = -\sum_{k\in\mathbb{Z}, k\geqslant 0} v_k z^{k+\frac{1}{N}}.$$

We have

$$\rho^N = \exp(\varepsilon u(z)\partial_z) \cdot z,$$

where

$$u(z) = -N \sum_{k\in\mathbb{Z}, k\geqslant 0} v_k z^{k+1}.$$

To check that formula (4.1) holds, it suffices to check that the ε-linear term in it vanishes. Denote $r^{M^\sigma}(v(z^{\frac{1}{N}})\partial_{z^{\frac{1}{N}}})$ by $r_v^{M^\sigma}$ and the ε-linear term in $R^V((\rho^N)_z)$ by $r_{u,z}^V$. The ε-linear term in (4.1) reads

$$(\mathrm{Id} + \varepsilon r_v^{M^\sigma})Y^{M^\sigma}((\mathrm{Id} - \varepsilon r_{u,z}^V)A, z^{\frac{1}{N}} + \varepsilon v(z^{\frac{1}{N}}))(\mathrm{Id} - \varepsilon r_v^{M^\sigma}) - Y^{M^\sigma}(A, z^{\frac{1}{N}})$$

$$= \varepsilon[r_v^{M^\sigma}, Y^{M^\sigma}(a, z^{\frac{1}{N}})] - \varepsilon Y^{M^\sigma}(r_{u,z}^V \cdot A, z^{\frac{1}{N}}) + \varepsilon v(z^{\frac{1}{N}})\partial_{z^{\frac{1}{N}}} Y^{M^\sigma}(A, z^{\frac{1}{N}}). \tag{4.2}$$

We find that

$$r_{u,z}^V \cdot A = -\sum_{m\geqslant 0} \frac{1}{(m+1)!}(\partial_z^{m+1}u(z))L_m \cdot A$$

and

$$r_v^{M^\sigma} = N \sum_{m \in \mathbb{Z}, m \geqslant 0} v_m L_m^{M^\sigma},$$

so that vanishing of (4.2) is equivalent to the identity

$$[r_v^{M^\sigma}, Y^{M^\sigma}(A, z^{\frac{1}{N}})] = -\sum_{m \geqslant -1} \frac{1}{(m+1)!} (\partial_z^{m+1} u(z)) Y^{M^\sigma}(L_m \cdot A, z^{\frac{1}{N}}).$$

Since

$$v(z^{\frac{1}{N}}) \partial_{z^{\frac{1}{N}}} = u(z) \partial_z,$$

this identity follows from the OPE between a twisted vertex operator and the Virasoro field in the same way as in Section 5.2.3 of [FB]. □

### 4.2. Example: primary fields

Recall that a vector $A \in V$ is called a primary vector of conformal dimension $\Delta$ if it satisfies

$$L_n A = 0, \quad n > 0; \quad L_0 A = \Delta A.$$

As shown in Lemma 5.3.4 of [FB], the corresponding vertex operator $Y(A, z)$ transforms under coordinate changes as an endomorphism-valued $\Delta$-differential on the punctured disc. Now formula (4.1) implies an analogous transformation formula for the corresponding twisted vertex operator $Y^{M^\sigma}(A, z^{\frac{1}{N}})$.

**Corollary 4.1.** *Let $A \in V$ be a primary vector of conformal dimension $\Delta$, and $\rho \in \mathrm{Aut}_N \mathcal{O}$. Then*

$$R^{M^\sigma}(\rho) Y^{M^\sigma}(A, \rho(z^{\frac{1}{N}})) R^{M^\sigma}(\rho)^{-1} (\partial_z(\rho^N(z^{\frac{1}{N}})))^\Delta = Y^{M^\sigma}(A, z^{\frac{1}{N}}). \tag{4.3}$$

## 5. Coordinate-independent interpretation of twisted vertex operators

### 5.1. Recollections from [FB]

Let $X$ be a smooth curve and $\mathscr{A}ut_X$ the principal $\mathrm{Aut}\mathcal{O}$-bundle of formal coordinates on $X$. The fiber of $\mathscr{A}ut_X$ at $x \in X$ is the $\mathrm{Aut}\mathcal{O}$-torsor $\mathscr{A}ut_x$ of formal coordinates at $x$ (see Section 5.4 of [FB] for details). Given a conformal vertex algebra $V$, set

$$\mathscr{V} = \mathscr{V}_X = \mathscr{A}ut_X \underset{\mathrm{Aut}\mathcal{O}}{\times} V. \tag{5.1}$$

This is a vector bundle whose fiber at $x \in X$ is the $\mathscr{A}ut_x$-twist of $V$,

$$\mathscr{V}_x = \mathscr{A}ut_x \underset{\mathrm{Aut}\mathcal{O}}{\times} V.$$

i.e., the set of pairs $(z, A)$ where $z$ is a formal coordinate at $x$ and $A \in V$, modulo the equivalence condition $(\rho(z), A) \sim (z, R^V(\rho) \cdot A)$ for $\rho \in \mathrm{Aut}\mathcal{O}$.

As explained in Chapter 5 of [FB], for any $x \in X$ the vertex operation $Y$ gives rise to a canonical section $\mathscr{Y}_x$ of the dual bundle $\mathscr{V}^*$ on the punctured disc $\mathscr{D}_x^\times$ with values in $\mathrm{End}\mathscr{V}_x$. Equivalently, we have a canonical linear map

$$\mathscr{Y}_x^\vee : \Gamma(\mathscr{D}_x^\times, \mathscr{V} \otimes \Omega_X) \to \mathrm{End}\mathscr{V}_x, \quad s \mapsto \mathrm{Res}_x \langle \mathscr{Y}_x, s \rangle.$$

If we choose a formal coordinate $z$ at $x$ and use it to trivialize $\mathscr{V}|_{\mathscr{D}_x}$, then $\mathscr{Y}_x^\vee$ is given by the formula

$$\mathscr{Y}_x^\vee (A \otimes z^n dz) = A_{(n)}.$$

Furthermore, the map $\mathscr{Y}_x^\vee$ factors through the quotient

$$U(\mathscr{V}_x) = \Gamma(\mathscr{D}_x^\times, \mathscr{V} \otimes \Omega_X)/\mathrm{Im}\, \nabla.$$

The latter is a Lie algebra and the resulting map $U(\mathscr{V}_x) \to \mathrm{End}\mathscr{V}_x$ is a Lie algebra homomorphism (see Section 8.2 of [FB]).

More generally, let $M$ be an admissible $V$-module. We attach to it a vector bundle $\mathscr{M}$ on $X$ in the same way as above. The module operation $Y^M$ then gives rise to a canonical section $\mathscr{Y}_x^M$ of $\mathscr{V}^*|_{\mathscr{D}_x^\times}$ with values in $\mathrm{End}\mathscr{M}_x$. Equivalently, we have a canonical linear map

$$\mathscr{Y}_x^{M,\vee} : \Gamma(\mathscr{D}_x^\times, \mathscr{V} \otimes \Omega_X) \to \mathrm{End}\mathscr{M}_x,$$

which factors through $U(\mathscr{V}_x)$ (see Section 6.3.6 of [FB]).

In this section we obtain analogous results for twisted modules over vertex algebras.

### 5.2. The vector bundle $\mathscr{V}_X^H$

Let $C$ be a smooth projective curve, and $H \subset \mathrm{Aut}(C)$ a finite group of automorphisms of $C$. Suppose furthermore that $V$ is a conformal vertex algebra, and that $H$ acts on $V$ by conformal automorphisms. The vector bundle $\mathscr{V}_C$ carries a $H$-equivariant structure lifting the action of $H$ on $C$. It is given by

$$h \cdot (p, (A, z)) \stackrel{\mathrm{def}}{=} (h(p), (h(A), z \circ h^{-1})), \tag{5.2}$$

where $z \circ h^{-1}$ is the coordinate induced at $h(p)$ from $z$. Let $X = C/H$ be the quotient curve, and $v : C \to X$ the quotient map, ramified at the points where $H$ has non-trivial stabilizers. Denote by $\overset{\circ}{C} \subset C$ (resp. $\overset{\circ}{X} \subset X$) the complement of the ramification points (resp. branch points) of $v$, and by $\overset{\circ}{v} : \overset{\circ}{C} \to \overset{\circ}{X}$ the restriction of $v$. Thus, $\overset{\circ}{C}$ is a $H$-principal bundle over $\overset{\circ}{X}$. The action of $H$ on $\overset{\circ}{C}$ is free, and $\mathscr{V}_{\overset{\circ}{C}}$ descends to a vector bundle $\mathscr{V}^H_{\overset{\circ}{X}}$ on $\overset{\circ}{X}$. More explicitly,

$$\mathscr{V}^H_{\overset{\circ}{X}} = \mathscr{A}ut_{\overset{\circ}{C}} \underset{\mathrm{Aut}\mathcal{O} \times H}{\times} V \tag{5.3}$$

Here, $H$ acts on $\mathscr{A}ut_{\overset{\circ}{C}}$ by $h(p,z) = (h(p), z \circ h^{-1})$, and this action commutes with the action of $\mathrm{Aut}\mathcal{O}$. The actions of $H$ and $\mathrm{Aut}\mathcal{O}$ on $V$ commute because $H$ is a conformal automorphism of $V$, and thus commutes with the Virasoro action.

The vector bundle $\mathscr{V}^H_{\overset{\circ}{X}}$ possesses a flat connection $\nabla^H$. If $z$ is a local coordinate $x \in \overset{\circ}{X}$, $\nabla^H$ is given by the expression $d + L^V_{-1} \otimes dz$.

### 5.3. Modules along $H$-orbits

Let $x \in X$. Then every point $p \in v^{-1}(x)$ has a cyclic stabilizer of order $N$, which we denote $H_p$. Each $H_p$ has a canonical generator $h_p$, which corresponds to the monodromy of a small loop around $x$. For a generic point $p$, $H_p = \{e\}$ and we set $h_p = e$. Suppose that we are given the following data:

(1) A collection of admissible $V$-modules $\{M_p^{h_p}\}_{p \in v^{-1}(x)}$, one for each point in the fiber, such that $M_p^{h_p}$ is $h_p$-twisted.

(2) A collection of maps $S_{g,p,g(p)} : M_p^{h_p} \mapsto M_{g(p)}^{h_{g(p)}}$, $g \in H$, $p \in v^{-1}(x)$, commuting with the action of $\mathrm{Aut}_N\mathcal{O}$ and satisfying

$$S_{gk,p,gk(p)} = S_{g,k(p),gk(p)} \circ S_{k,p,k(p)},$$

$$S^{-1}_{g,p,g(p)} = S_{g^{-1},g(p),p},$$

and

$$S^{-1}_{g,p,g(p)} Y^{M_{g(p)}^{h_{g(p)}}}(g \cdot A, z) S_{g,p,g(p)} = Y^{M_p^{h_p}}(A,z).$$

(3) If $g \in H_p$, then $S_{g,p,p} = S_g$, where $S_g$ is the operator defined in Section 2.

Given a collection $\{M_p^{h_p}\}_{p \in v^{-1}(x)}$, we can form the collection $\{\mathscr{M}_p^{h_p}(\mathscr{D}_p)\}_{p \in v^{-1}(x)}$, where $\mathscr{M}_p^{h_p}(\mathscr{D}_p)$ is the $\mathrm{Aut}_N\mathcal{O}$-twist of $M_p^{h_p}$ by the torsor of special coordinates at $p$.

Let

$$\overline{\mathcal{M}_x} = \bigoplus_{p \in v^{-1}(x)} \mathcal{M}_p^{h_p}(\mathcal{D}_p)$$

This is a representation of $H$, where $H$ acts as follows. If $A \in M_p^{h_p}$, $z_p^{\frac{1}{N}}$ is a special coordinate at $p$, and $g \in H$, then

$$g \cdot (A, z_p^{\frac{1}{N}}) = (S_{g,p,g(p)} \cdot A, z_p^{\frac{1}{N}} \circ g^{-1})$$

Note that this action is well-defined since the $S$-operators commute with the action of $\mathrm{Aut}_N \mathcal{O}$. Now, let $\mathcal{M}_x = (\overline{\mathcal{M}_x})^H$, the space of $H$-invariants of $\overline{\mathcal{M}_x}$. The composition of the inclusion $\mathcal{M}_x \to \overline{\mathcal{M}_x}$ and the projection $\overline{\mathcal{M}_x} \to \mathcal{M}_p^{h_p}(\mathcal{D}_p)$ is an isomorphism for all $p \in v^{-1}(x)$. For $v_p \in \mathcal{M}_p^{h_p}(\mathcal{D}_p)$, denote by $[v_p]$ the corresponding vector in $\mathcal{M}_x$. Note that for each $(A, z_p^{\frac{1}{N}})_p \in \mathcal{M}_p^{h_p}(\mathcal{D}_p)$, and $g \in H$, $[(A, z_p^{\frac{1}{N}})_p] = [(S_{g,p,g(p)} \cdot A, z_p^{\frac{1}{N}} \circ g^{-1})_{g(p)}]$ in $\mathcal{M}_x$.

**Definition 5.1.** We call $\mathcal{M}_x$ a *$V$-module along $v^{-1}(x)$*.

Henceforth, we will suppress the square brackets for elements of $\mathcal{M}_x$ and refer to $[(A, z_p^{\frac{1}{N}})_p]$ simply as $(A, z_p^{\frac{1}{N}})$.

### 5.4. *Construction of Modules along H-orbits*

In this section we wish to give a construction of a module along $v^{-1}(x)$ starting with a point $p \in v^{-1}(x)$ and a $h_p$-twisted module $M_p^{h_p}$. Note that when $H_p$ is trivial, this is just an ordinary $V$-module $M$.

Thus, suppose we are given $p \in v^{-1}(x)$, and a $h_p$-twisted module $M_p^{h_p}$. Observe that the monodromy generator at the point $g(p)$ is $h_{g(p)} = g h_p g^{-1}$, i.e. the monodromies are conjugate.

(1) For $g \in H$, define the module $M_{g(p)}^{g h_p g^{-1}}$ to be $M_p^{h_p}$ as a vector space, with the $V$-module structure given by the vertex operator

$$Y^{M_{g(p)}^{g h_p g^{-1}}}(A, z) = Y^{M_p^{h_p}}(g^{-1} \cdot A, z) \tag{5.4}$$

It is easily checked that this equips $M_{g(p)}^{g h_p g^{-1}}$ with the structure of a $g h_p g^{-1}$-twisted module. Furthermore, if $g \in H_p$, this construction results in a $h_p$-twisted module isomorphic to $M_p^{h_p}$.

(2) Recall that for $q \in \nu^{-1}(x)$, $M_q^{h_q}$ is canonically isomorphic to $M_p^{h_p}$ as a vector space by the previous item. Thus, if $g(q) \neq q$, define $S_{g,q,g(q)}$ to be the identity map.
(3) If $g \in H_q$, then $g$ is conjugate to an element $g' \in H_p$. Define $S_{g,q,q} = S_{g',p,p}$ also using the canonical identification.

It is easy to check that this construction is well-defined, and satisfies the requirements of Definition 5.1.

**Remark 1.** If $H_p$ is trivial, and $M = V$, then for any $g \in H$, the new module structure (5.4) is isomorphic to the old one, and so the resulting module $\mathcal{M}_x$ along $\nu^{-1}(x)$ is isomorphic to $\mathcal{V}_x^H$, the fiber of the sheaf $\mathcal{V}^H$ at $x$.

**Remark 2.** If $H = H_p$, then $p$ is unique, and so any $h_p$-twisted module results in a module along $\nu^{-1}(x)$.

## 5.5. *Twisted vertex operators as sections of* $\mathcal{V}_{\mathring{X}}^{H,*}$

We begin with the observation that a section of $\mathcal{V}_{\mathring{X}}^H$ over $U \subset \mathring{X}$ is the same as a $H$-invariant section of $\mathcal{V}_{\mathring{C}}$ over $\nu^{-1}(U)$, and likewise for $\mathcal{V}_{\mathring{X}}^{H,*}$. Thus, defining a section of $\mathcal{V}_{\mathring{X}}^{H,*}$ on $\mathcal{D}_x^\times$ is equivalent to defining an $H$-invariant section of $\mathcal{V}_{\mathring{C}}^*$ on $\coprod_{p \in \nu^{-1}(x)} \mathcal{D}_p^\times$.

Let $p \in \nu^{-1}(x)$, and let $z_p^{\frac{1}{N}}$ be an $h_p$-special formal coordinate at $p$. This coordinate gives us a trivialization $\imath_{z_p}$ of $\mathcal{V}_{\mathring{C}}|_{\mathcal{D}_p^\times}$. We will denote by $\imath_{z_p}(A)$ the section of $\mathcal{V}_{\mathring{C}}|_{\mathcal{D}_p^\times}$ corresponding to $A \in V$ with respect to this trivialization. The coordinate $z_p^{\frac{1}{N}}$ also gives us an identification of $\mathcal{M}_p^{h_p}(\mathcal{D}_p)$ with $M_p^{h_p}$.

**Theorem 5.1.** *Let $x \in X$, and $\mathcal{M}_x$ a $V$-module along $\nu^{-1}(x)$. For each $p \in \nu^{-1}(x)$, choose a $h_p$-special coordinate $z_p^{\frac{1}{N}}$ at $p$. Define an* $\mathrm{End}(\mathcal{M}_x)$*-valued section $\mathcal{Y}^{\mathcal{M}_x}$ of $\mathcal{V}_{\mathring{C}}^*$ on $\coprod_{p \in \nu^{-1}(x)} \mathcal{D}_p^\times$ by the formula*

$$\langle (z_p^{\frac{1}{N}}, \phi), \mathcal{Y}^{\mathcal{M}_x}(\imath_{z_p}(A)) \cdot (z_p^{\frac{1}{N}}, v) \rangle = \langle \phi, Y^{M_p^{h_p}}(A, z_p^{\frac{1}{N}}) \cdot v \rangle. \tag{5.5}$$

*Then this section $\mathcal{Y}^{\mathcal{M}_x}$ is independent of the choice of special coordinate $z_p^{\frac{1}{N}}$ on each $\mathcal{D}_p^\times$. Furthermore, it is $H$-invariant.*

**Proof.** We begin by checking coordinate-independence. Choose a $p \in \nu^{-1}(x)$. Let $w^{\frac{1}{N}}$ be another special coordinate at $p$. Then there exists a unique $\rho \in \mathrm{Aut}_N \mathcal{O}$ such that

$w^{\frac{1}{N}} = \rho(z_p^{\frac{1}{N}})$ (thus, $w = \rho^N(z_p^{\frac{1}{N}})$). We attach to $w^{\frac{1}{N}}$ a section $\tilde{\mathscr{Y}}_p^{M^\sigma}$ of $\mathscr{V}^*_{\overset{\circ}{C}}|_{\mathscr{D}_p^\times}$ with values in $\mathrm{End}(\mathscr{M}_x)$ by the formula

$$\langle (w^{\frac{1}{N}}, \tilde{\phi}), \tilde{\mathscr{Y}}^{\mathscr{M}_x}(\imath_w(\tilde{A})) \cdot (w^{\frac{1}{N}}, \tilde{v}) \rangle = \langle \tilde{\phi}, Y^{M^{h_p}}(\tilde{A}, w^{\frac{1}{N}}) \cdot \tilde{v} \rangle.$$

We must show that $\tilde{\mathscr{Y}}^{\mathscr{M}_x} = \mathscr{Y}^{\mathscr{M}_x}$. We have

$$(z_p^{\frac{1}{N}}, \phi) = (w^{\frac{1}{N}}, \phi \cdot R^{M^{h_p}}(\rho))$$

$$(z_p^{\frac{1}{N}}, v) = (w^{\frac{1}{N}}, R^{M^{h_p}}(\rho)^{-1} \cdot v)$$

As explained in [FB], the section $\imath_{z_p}(A)$ of $\mathscr{V}_{\overset{\circ}{C}}$ appears in the coordinate $w = \rho^N(z_p^{\frac{1}{N}})$ as $\imath_w(R^V((\rho^N)_z)^{-1} \cdot A)$. It follows that

$$\imath_{z_p}(A) = \imath_w(R^V((\rho_z^N)^{-1}) \cdot A).$$

Therefore

$$\begin{aligned}&\langle (z_p^{\frac{1}{N}}, \phi), \tilde{\mathscr{Y}}^{\mathscr{M}_x}(\imath_{z_p}(A) \cdot (z_p^{\frac{1}{N}}, v) \rangle \\ &\quad = \langle \phi, R^{M^{h_p}}(\rho) Y^{M^\sigma}(R^V((\rho^N)_z)^{-1} \cdot A, \rho(z_p^{\frac{1}{N}})) R^{M^{h_p}}(\rho)^{-1} v \rangle.\end{aligned}$$

By (4.1), $\tilde{\mathscr{Y}}^{\mathscr{M}_x} = \mathscr{Y}^{\mathscr{M}_x}$, and we obtain that our section is coordinate-independent.

We now proceed to show that $\mathscr{Y}^{\mathscr{M}_x}$ is $H$-invariant. This amounts to checking, for $g \in H$

$$\langle (z_p^{\frac{1}{N}}, \phi), \mathscr{Y}^{\mathscr{M}_x}(g \cdot \imath_{z_p}(A)) \cdot (z_p^{\frac{1}{N}}, v) \rangle = g \cdot \langle (z_p^{\frac{1}{N}}, \phi), \mathscr{Y}^{\mathscr{M}_x}(\imath_{z_p}(A)) \cdot (z_p^{\frac{1}{N}}, v) \rangle$$

where the action on the right is by pullback of functions. The right-hand side is $\langle \phi, Y^{M_p^{h_p}}(A, z_p^{\frac{1}{N}} \circ g^{-1}) \cdot v \rangle$. On the left, we have

$$g \cdot \imath_{z_p}(A) = \imath_{z_p \circ g^{-1}}(g \cdot A)$$

$$(z_p^{\frac{1}{N}}, \phi) \cong (z_p^{\frac{1}{N}} \circ g^{-1}, \phi \circ S^{-1}_{g,p,g(p)}) \in \mathscr{M}_x^*$$

$$(z_p^{\frac{1}{N}}, v) \cong (z_p^{\frac{1}{N}} \circ g^{-1}, S_{g,p,g(p)} \cdot v) \in \mathscr{M}_x$$

Thus we get

$$\langle (z_p^{\frac{1}{N}}, \phi), \mathcal{Y}^{\mathcal{M}_x}(g \cdot \imath_{z_p}(A) \cdot (z_p^{\frac{1}{N}}, v) \rangle$$

$$= \langle (z_p^{\frac{1}{N}} \circ g^{-1}, \phi \circ S^{-1}_{g,p,g(p)}), \mathcal{Y}^{\mathcal{M}_x}(\imath_{z_p \circ g^{-1}}(g \cdot A)), (z_p^{\frac{1}{N}} \circ g^{-1}, S_{g,p,g(p)} \cdot v) \rangle$$

$$= \langle \phi, S^{-1}_{g,p,g(p)} Y^{M^{h_{g(p)}}_{g(p)}}(g \cdot A, z_p^{\frac{1}{N}} \circ g^{-1}) S_{g,p,g(p)} \cdot v \rangle$$

$$= \langle \phi, Y^{M_p^{h_p}}(A, z_p^{\frac{1}{N}} \circ g^{-1}) \cdot v \rangle$$

$$= g \cdot \langle (z_p^{\frac{1}{N}}, \phi), \mathcal{Y}^{\mathcal{M}_x}(\imath_{z_p}(A)) \cdot (z_p^{\frac{1}{N}}, v) \rangle \qquad \square$$

**Remark 3.** In view of the comments at the beginning of Section 5.5, $\mathcal{Y}^{\mathcal{M}_x}$ is the pullback under $v$ of a unique section of $\mathcal{V}^{H,*}_{\overset{\circ}{X}}$ on $\mathcal{D}_x^{\times}$. We will abuse notation by denoting the latter by $\mathcal{Y}^{\mathcal{M}_x}$ as well.

In the case of twisted primary fields, (4.3) implies the following analogue of Proposition 5.3.8 of [FB]:

**Proposition 5.1.** *Let $x \in X$, and $\mathcal{M}_x$ a $V$-module along $v^{-1}(x)$. For each $p \in v^{-1}(x)$, choose a $h_p$-special coordinate $z_p^{\frac{1}{N}}$ at $p$. Define an* $\mathrm{End}(\mathcal{M}_x)$*-valued $\Delta$-differential $\varpi$ on* $\coprod_{p \in v^{-1}(x)} \mathcal{D}_p^{\times}$ by the formula

$$\langle (z_p^{\frac{1}{N}}, \phi), \varpi \cdot (z_p^{\frac{1}{N}}, v) \rangle = \langle \phi, Y^{M_p^{h_p}}(A, z_p^{\frac{1}{N}}) \cdot v \rangle (dz_p)^{\Delta}$$

$$= N^{\Delta} z_p^{\Delta \frac{(N-1)}{N}} \langle \phi, Y^{M_p^{h_p}}(A, z_p^{\frac{1}{N}}) \cdot v \rangle (dz_p^{\frac{1}{N}})^{\Delta}$$

*Then $\varpi$ is independent of the choice of $z_p^{\frac{1}{N}}$*'s.

Recall from Section 5.4.9 of [FB] that a primary vector $A \in V$ determines a line subbundle $j_A : \Omega_C^{-\Delta} \hookrightarrow \mathcal{V}_C$, and by dualizing a surjection $j_A^* : \mathcal{V}_C^* \twoheadrightarrow \Omega_C^{\Delta}$. The section $\varpi$ of $\Omega^{\Delta}_{\overset{\circ}{C}}|_{\coprod_{p \in v^{-1}(x)} \mathcal{D}_p^{\times}}$ appearing in Proposition 5.1 is just the image of the section $\mathcal{Y}^{\mathcal{M}_x}$ under $j_A^*$.

*5.6. Dual version*

Let $x \in X$, and $\mathcal{M}_x$ a $V$-module along $v^{-1}(x)$. As in the case of ordinary vertex operators (see Section 5.4.8 of [FB]), dualizing the construction we obtain a linear map

$$\mathscr{Y}^{\mathscr{M}_x,\vee} : \Gamma(\mathscr{D}_x^\times, \mathscr{V}_{\overset{\circ}{X}}^H \otimes \Omega_{\overset{\circ}{X}}) \to \mathrm{End}\,\mathscr{M}_x.$$

Given by

$$s \to \mathrm{Res}_x \langle \mathscr{Y}^{\mathscr{M}_x}, s \rangle \tag{5.6}$$

Moreover, this map factors through the quotient

$$U(\mathscr{V}_x^H) \stackrel{\mathrm{def}}{=} \Gamma(\mathscr{D}_x^\times, \mathscr{V}_{\overset{\circ}{X}}^H \otimes \Omega_{\overset{\circ}{X}}) / \mathrm{Im}\, \nabla^H,$$

which has a natural Lie algebra structure. The corresponding map $U(\mathscr{V}_x^H) \to \mathrm{End}\,\mathscr{M}_x$ is a homomorphism of Lie algebras. Note that $x$ does not have to lie in $\overset{\circ}{X}$, but can be *any* point of $X$.

### 5.7. A sheaf of Lie algebras

Following Section 8.2.5 of [FB], let us consider the following complex of sheaves (in Zariski topology) on $\overset{\circ}{X}$:

$$0 \to \mathscr{V}_{\overset{\circ}{X}}^H \xrightarrow{\nabla} \mathscr{V}_{\overset{\circ}{X}}^H \otimes \Omega_{\overset{\circ}{X}} \to 0,$$

where $\mathscr{V}_{\overset{\circ}{X}}^H \otimes \Omega_{\overset{\circ}{X}}$ is placed in cohomological degree 0 and $\mathscr{V}_{\overset{\circ}{X}}^H$ is placed in cohomological degree $-1$ (shifted de Rham complex). Let $h(\mathscr{V}_{\overset{\circ}{X}}^H)$ denote the sheaf of the 0th cohomology, assigning to every Zariski open subset $\Sigma \subset \overset{\circ}{X}$ the vector space

$$U_\Sigma(\mathscr{V}_{\overset{\circ}{X}}^H) \stackrel{\mathrm{def}}{=} \Gamma(\Sigma, \mathscr{V}_{\overset{\circ}{X}}^H \otimes \Omega_{\overset{\circ}{X}}) / \mathrm{Im}\, \nabla^H$$

One can show as in Chapter 18 of [FB] that this is a sheaf of Lie algebras.

According to formula (5.6), for any $x \in \Sigma'$, where $\Sigma' \subset X$ is such that $\Sigma' \cap \overset{\circ}{X} = \Sigma$, restriction induces a Lie algebra homomorphism $U_\Sigma(\mathscr{V}_{\overset{\circ}{X}}^H) \to U(\mathscr{V}_x^H)$. We denote the image by $U_\Sigma(\mathscr{V}_x^H)$.

*5.8. Interpretation in terms of chiral algebras*

A. Beilinson and V. Drinfeld have introduced in [BD] the notion of chiral algebra (see also [G]). A chiral algebra on a smooth curve $X$ is a right $\mathscr{D}$-module $\mathscr{A}$ on $X$ together with homomorphisms of $\mathscr{D}$-modules $\Omega \to \mathscr{A}$ and

$$j_* j^* (\mathscr{A} \boxtimes \mathscr{A}) \to \Delta_! (\mathscr{A}),$$

where $\Delta : X \to X^2$ is the diagonal embedding, and $j : (X^2 \backslash \Delta) \to X^2$ is the complement of the diagonal. These homomorphisms must satisfy certain axioms.

As shown in Chapter 18 of [FB], for any conformal vertex algebra $V$ and any smooth curve $X$, the right $\mathscr{D}$-module $\mathscr{V}_X \otimes \Omega_X$ is naturally a chiral algebra.

Recall that a module over a chiral algebra $\mathscr{A}$ on $X$ is a right $\mathscr{D}$-module $\mathscr{R}$ on $X$ together with a homomorphism of $\mathscr{D}$-modules

$$a : j_* j^* (\mathscr{A} \boxtimes \mathscr{R}) \to \Delta_! (\mathscr{R}).$$

This homomorphism should satisfy the axioms of [BD].

Suppose that $\mathscr{R}$ is supported at a point $x \in X$ and denote its fiber at $x$ by $\mathscr{R}_x$. Then $\mathscr{R} = i_{x!}(\mathscr{R}_x)$, where $i_x$ is the embedding $x \to X$. Applying the de Rham functor along the second factor to the map $a$ we obtain a map

$$a_x : \; j_{x*} j_x^* (\mathscr{A}) \otimes \mathscr{R}_x \to \mathscr{R},$$

where $j_x : \; (X \backslash x) \to X$. The chiral module axioms may be reformulated in terms of this map. Note that it is not necessary for $\mathscr{A}$ to be defined at $x$ in order for this definition to make sense. If $\mathscr{A}$ is defined on $X \backslash x$, we simply replace $j_{x*} j_x^* (\mathscr{A})$ by $j_{x*}(\mathscr{A})$.

If $M$ is a module over a conformal vertex algebra $V$, we associate to it the space $\mathscr{M}_x$ as in Section 5.1. Then the right $\mathscr{D}$-module $i_{x!}(\mathscr{M}_x)$ is a module over the chiral algebra $\mathscr{V} \otimes \Omega_X$ supported at $x$ and the corresponding map $a_x^M$ is defined as follows. Choose a formal coordinate $z$ at $x$ and use it to trivialize $\mathscr{V}|_{\mathscr{D}_x}$ and $\mathscr{M}_x$ and to identify

$$i_{x!}(\mathscr{M}_x) = M((z))dz / M[[z]]dz.$$

Then

$$a_x^M (\iota_z(A) \otimes f(z)dz, B) = Y^M(A, z) B \otimes f(z) dz \mod M[[z]]dz.$$

The independence of $a_x^M$ on $z$ is proved in the same way as the independence of $\mathscr{Y}_x^M$. Note that applying to $a_x^M$ the de Rham cohomology functor we obtain the map $\mathscr{Y}_x^{M,\vee}$ (see Section 5.1).

Now let $\mathscr{V}^H_{\overset{\circ}{X}}$ be the vector bundle on $\overset{\circ}{X}$ with connection defined by formula (5.3). Then $\mathscr{V}^H_{\overset{\circ}{X}} \otimes \Omega_X$ is a right $\mathscr{D}$-module on $\overset{\circ}{X}$. Suppose that $H = \mathbb{Z}/N\mathbb{Z} = \langle \sigma \rangle$. Given a full ramification point $p \in C$ and a twisted $V$-module $M^{\sigma}$, we define a map

$$a_p^{M^{\sigma}}: j_{x*}(\mathscr{V}^H_{\overset{\circ}{X}} \otimes \Omega_X) \otimes \mathscr{M}_p^{\sigma} \to i_{p!}(\mathscr{M}_p^{\sigma}),$$

where $x = v(p) \in X \backslash \overset{\circ}{X}$, as follows. Observe that sections of $\mathscr{V}^H_{\overset{\circ}{X}} \otimes \Omega_X|_{\mathscr{D}_x^{\times}}$ are the same as $\sigma$-invariant sections of $v^*(\mathscr{V}_X) \otimes \Omega_C|_{\mathscr{D}_p^{\times}}$. Choose a special coordinate $z^{\frac{1}{N}}$ at $p$ and use it to trivialize $v^*(\mathscr{V}) \otimes \Omega_C|_{\mathscr{D}_p^{\times}}$ and $\mathscr{M}_p$ and to identify

$$i_{p!}(\mathscr{M}_p^{\sigma}) = M((z^{\frac{1}{N}}))dz^{\frac{1}{N}} / M[[z^{\frac{1}{N}}]]dz^{\frac{1}{N}}.$$

Then

$$a_p^{M^{\sigma}}(\imath_z(A) \otimes f(z^{\frac{1}{N}})dz^{\frac{1}{N}}, B) \stackrel{\text{def}}{=} Y^{M^{\sigma}}(A, z^{\frac{1}{N}})B \otimes f(z^{\frac{1}{N}})dz^{\frac{1}{N}} \bmod M[[z^{\frac{1}{N}}]]dz^{\frac{1}{N}}.$$

The independence of $a_p^{M^{\sigma}}$ on $z^{\frac{1}{N}}$ may be proved in the same way as the independence of $\mathscr{Y}_p^{M^{\sigma}}$ was proved. Applying to $a_p^{M^{\sigma}}$ the de Rham cohomology functor we obtain the map $\mathscr{Y}_p^{M^{\sigma},\vee}$ from Section 5.6.

Now let $H$ be an arbitrary finite group acting (generically with trivial stabilizers) on a smooth curve $C$. Then as before we have an $H$-torsor $\overset{\circ}{C}$ over $\overset{\circ}{X} \subset X = C/H$. Let $\mathscr{A}$ be a chiral algebra on $\overset{\circ}{X}$ equipped with an action of $H$ by automorphisms. Then the $\overset{\circ}{C}$-twist of $\mathscr{A}$,

$$\mathscr{A}^{\overset{\circ}{C}} = \overset{\circ}{C} \underset{H}{\times} \mathscr{A},$$

inherits the chiral algebra structure from $\mathscr{A}$. So we can consider $\mathscr{A}^{\overset{\circ}{C}}$-modules supported at arbitrary points $x \in X$. If $\mathscr{A} = \mathscr{V}_X \otimes \Omega_X$, where $V$ is a conformal vertex algebra on which $H$ acts by automorphisms, then such modules may be constructed from twisted $V$-modules. Namely, to each $V$-module along $v^{-1}(x)$ (see Definition 5.1) we can attach to it in the same way as above a $\mathscr{A}^{\overset{\circ}{C}}$-module supported at $x$.

## 6. Conformal blocks

We use the notation of Section 5.2. Let $\{x_i\}_{i=1\cdots m}$ be a collection of distinct points of $X$, which contains all of the branch points of $v$. Let $\{\mathscr{M}_{x_i}\}_{i=1\cdots m}$ be a collection of $V$-modules, such that $\mathscr{M}_{x_i}$ is a module along $v^{-1}(x_i)$. Let

$$\mathscr{F} = \bigotimes_{i=1}^{m} \mathscr{M}_{x_i}.$$

Given $\phi \in \mathscr{F}^*$, $A_i \in \mathscr{M}_{x_i}$,

$$\langle \phi, A_1 \otimes \cdots \otimes \mathscr{Y}^{\mathscr{M}_{x_i}} \cdot A_i \otimes \cdots \otimes A_m \rangle \tag{6.1}$$

is a section of $\mathscr{V}^{H,*}_{\overset{\circ}{X}}$ on $\mathscr{D}^{\times}_{x_i}$.

We can now define the generalized space of conformal blocks, extending Definition 9.1.1 of [FB]:

**Definition 6.1.** The space of conformal blocks

$$\mathscr{C}_V(X, \{x_i\}, \mathscr{M}_{x_i})_{i=1\cdots m}$$

is by definition the vector space of all linear functionals $\phi \in \mathscr{F}^*$ such that for any $A_i \in \mathscr{M}_{x_i}$, sections (6.1) can be extended to the *same* section of $\mathscr{V}^{H,*}_{\overset{\circ}{X}}$, regular over $X \backslash \{x_i\}$.

**Remark.** Observe that in this definition *all* branch points of $v$ are required to carry module insertions, so that $X \backslash \{x_i\} \subset \overset{\circ}{X}$. $\square$

We can pull back sections (6.1) by $v$ to sections of $\mathscr{V}^{*}_{\overset{\circ}{C}}$ on $\coprod_{p \in v^{-1}(x_i)} \mathscr{D}^{\times}_{p}$. Equivalently, Definition 6.1 can be rephrased as follows:

**Definition 6.2.** The space of conformal blocks

$$\mathscr{C}_V(X, \{x_i\}, \mathscr{M}_{x_i})_{i=1\cdots m}$$

is by definition the vector space of all linear functionals $\phi \in \mathscr{F}^*$ such that for any $A_i \in \mathscr{M}_{x_i}$, the pullbacks of sections (6.1) can be extended to the *same* $H$-invariant section of $\mathscr{V}^{*}_{\overset{\circ}{C}}$, regular over $C \backslash \{v^{-1}(x_i)\}$.

### *6.1. Alternative definition*

Composing map (5.6) with the map

$$U_{X \backslash \{x_i\}}(\mathscr{V}^{H}_{\overset{\circ}{X}}) \to \bigoplus_{i=1}^{m} U(\mathscr{V}^{H}_{x_i})$$

we obtain an action of the Lie algebra $U_{X \backslash \{x_i\}}(\mathscr{V}^{H})$ on $\mathscr{F}$. We will employ the Strong Residue Theorem (see [T]):

**Theorem 6.1.** *Let $\mathscr{E}$ be a vector bundle on a smooth projective curve $Z$. Let $t_1, \ldots, t_n \in Z$ be a set of distinct points. Then a section*

$$\tau \in \bigoplus_{i=1}^{n} \Gamma\left(\mathscr{D}_{t_i}^{\times}, \mathscr{E}^*\right)$$

*has the property that*

$$\sum_{i=1}^{n} \mathrm{Res}_{t_i} \langle \mu, \tau \rangle = 0, \quad \forall \mu \in \Gamma(Z - \{t_1, \cdots, t_n\}, \mathscr{E} \otimes \Omega_Z)$$

*if and only if $\tau$ can be extended to a regular section of $\mathscr{E}$ over $Z - \{t_1, \ldots, t_n\}$ (i.e., $\tau \in \Gamma(Z - \{t_1, \ldots, t_n\}, \mathscr{E}^*)$)*

Applying Theorem 6.1 to Definition 6.1, we obtain that $\phi \in \mathscr{F}^*$ is a conformal block if and only if it vanishes on all elements of the form $s \cdot v, s \in U_{X \backslash \{x_i\}}(\mathscr{V}^{H}_{\overset{\circ}{X}}), v \in \mathscr{F}$. This leads to the following equivalent definition, extending Definition 9.1.2 of [FB]:

**Definition 6.3.** The space of *coinvariants* is the vector space

$$\mathscr{H}_V(X, \{x_i\}, \mathscr{M}_{x_i})_{i=1 \cdots =} \mathscr{F} / U_{X \backslash \{x_i\}}(\mathscr{V}^{H}_{\overset{\circ}{X}}) \cdot \mathscr{F}.$$

The space of *conformal blocks* is its dual: the vector space of $U_{X \backslash \{x_i\}}(\mathscr{V}^{H}_{\overset{\circ}{X}})$-invariant functionals on $\mathscr{F}$

$$\mathscr{C}_V(X, \{x_i\}, \mathscr{M}_{x_i})_{i=1 \cdots m} = \mathrm{Hom}_{U_{X \backslash \{x_i\}}(\mathscr{V}^{H}_{\overset{\circ}{X}})}(\mathscr{F}, \mathbb{C}).$$

## 7. Example: Heisenberg vertex algebra

The definition of conformal blocks given in Section 6 is quite abstract, and involves a priori all of the fields of the vertex algebra $V$. When vertex algebras are generated by a finite number of fields the definition of conformal block can be simplified to involve only those generating fields. In this section we illustrate this in the case of the Heisenberg vertex algebra with an order 2 automorphism.

### *7.1. The vertex algebra $\pi$ and its $\mathbb{Z}/2\mathbb{Z}$-twisted sector*

Let $\mathscr{H}$ (resp. $\mathscr{H}^{\sigma}$) denote the Lie algebra with generators $\{\tilde{b}_n, \tilde{K}\}_{n \in \mathbb{Z}}$ (resp. $\{b_n, K\}_{n \in \frac{1}{2} + \mathbb{Z}}$), and commutation relations

$$[\tilde{b}_n, \tilde{b}_m] = n\delta_{n,-m}\tilde{K}$$

(resp. same but with $b_n$'s) where $\tilde{K}$ (resp. $K$) is central. Let $\mathscr{H}_+$ (resp. $\mathscr{H}^\sigma_+$) be the subalgebras generated by $\{\tilde{b}_n\}_{n \geqslant 0}$ (resp. $\{b_n\}_{n>0}$). For $\lambda \in \mathbb{C}$, let $\widetilde{\mathbb{C}}^\lambda$ denote the 1-dimensional representation of $\mathscr{H}_+ \oplus \mathbb{C} \cdot \tilde{K}$ on which $\tilde{b}_n, n>0$ acts by 0, $\tilde{b}_0$ acts by $\lambda$, and $\tilde{K}$ acts by the identity. Let

$$\pi^\lambda = \operatorname{Ind}^{\mathscr{H}}_{\mathscr{H}_+ \oplus \mathbb{C} \cdot \tilde{K}} \widetilde{\mathbb{C}^\lambda}.$$

It is well known that $\pi = \pi^0$ has the structure of a vertex algebra (see for instance Chapter 2 of [FB]), generated by the field assignment

$$Y^\pi(\tilde{b}_{-1}|0\rangle, z) = \tilde{b}(z) = \sum_{n \in \mathbb{Z}} \tilde{b}_n z^{-n-1}.$$

Let us take $\omega = \frac{1}{2}\tilde{b}^2_{-1}$ to be the conformal vector of $\pi$. With this conformal structure, $\pi$ has a conformal automorphism $\sigma$ of order 2, induced from the automorphism of $\mathscr{H}$ which acts by $\tilde{b}_n \to -\tilde{b}_n$. All $\pi^\lambda$ have the structure of conformal $\pi$-modules. If $\lambda = \sqrt{M}$, $M$ even, then $\pi^\lambda$ is admissible. We write

$$Y^{\pi^\lambda}(\tilde{b}_{-1}|0\rangle, z) = \tilde{b}^\lambda(z) = \sum_{n \in \mathbb{Z}} \tilde{b}_n z^{-n-1}.$$

Now let $\mathbb{C}$ denote the 1-dimensional representation of $\mathscr{H}^\sigma_+ \oplus \mathbb{C} \cdot K$ on which $K$ acts by the identity. The $\mathscr{H}^\sigma$-module

$$\pi^\sigma = \operatorname{Ind}^{\mathscr{H}^\sigma}_{\mathscr{H}^\sigma_+ \oplus \mathbb{C} \cdot K} \mathbb{C}$$

has the structure of an admissible conformal $\sigma$-twisted $\pi$-module, generated (in the sense of [Li]) by the field assignment

$$Y^{\pi^\sigma}(\tilde{b}_{-1}|0\rangle, z) = b(z^{\frac{1}{2}}) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} b_n z^{-n-1}.$$

The twisted vertex operator assigned to an arbitrary vector $v \in \pi$ is given as follows (see [FLM,KP,D]). Let

$$W^{\pi^\sigma}(\tilde{b}_{n_1} \dots \tilde{b}_{n_k}|0\rangle, z) = \frac{1}{(-n_1-1)!} \cdots \frac{1}{(-n_k-1)!} : \partial_z^{-n_1-1} b(z) \dots \partial_z^{-n_k-1} b(z) :$$

and set

$$\Delta_z = \sum_{m,n \geqslant 0} c_{mn} \tilde{b}_m \tilde{b}_n z^{-m-n}$$

where the constants $c_{mn}$ are determined by the formula

$$\sum_{m,n\geqslant 0} c_{mn}x^m y^n = -\log\left(\frac{(1+x)^{1/2}+(1+y)^{1/2}}{2}\right)$$

Then for any $v\in\pi$ we have

$$Y^{\pi^\sigma}(v,z) = W^{\pi^\sigma}(\exp\Delta_z\cdot v, z). \tag{7.1}$$

### 7.2. The Lie algebra $\mathscr{H}_{\mathrm{out}}$

Using the notation of earlier sections, we now restrict to the case where $V=\pi$, $H\cong\langle\sigma_C\rangle$, where $\sigma_C$ has order 2, and $v: C\to X$ has degree 2. This is for example the case when $C$ is hyperelliptic, $\sigma_C$ is the hyperelliptic involution, and $X=\mathbb{CP}^1$. The vector bundles $\mathscr{V}_{\overset{\circ}{C}}, \mathscr{V}^H_{\overset{\circ}{X}}$ will be denoted $\Pi_{\overset{\circ}{C}}, \Pi^\sigma_{\overset{\circ}{X}}$ respectively.

For $x\in X$ we can construct $\pi$-modules along $v^{-1}(x)$ by applying the construction in Section 5.4. If $\pi^{-1}(x)$ consists of one point, we obtain a module $\pi^\sigma_x$ along $v^{-1}(x)$ starting with $\pi^\sigma$. If $\pi^{-1}(x)$ consists of two points, we obtain a module $\pi^{\lambda,p}_x$ along $v^{-1}(x)$ starting with a point $p$ in the fiber and a $\pi^\lambda, \lambda\in\sqrt{2}\mathbb{Z}$.

Let $\{x_i\}_{i=1\cdots m}$ be a collection of points of $X$ containing all of the branch points of $v$, and $\{\pi_{x_i}\}$ a collection of $\pi$-modules along $v^{-1}(x_i)$, where $\pi_{x_i}\cong\pi^\sigma_{x_i}$ or $\pi_{x_i}\cong\pi^{\lambda_i,p_i}_{x_i}$ depending on whether $v^{-1}(x_i)$ consists of one or two points. The vector $\tilde{b}_{-1}|0\rangle\in\pi$ is primary and has conformal weight 1. Applying Proposition 5.1, for each $x_i$, we obtain an $\mathrm{End}(\pi_{x_i})$-valued 1-form $\varpi_i$ on $\coprod_{p\in v^{-1}(x_i)}\mathscr{D}^\times_p$. Let

$$\mathscr{F} = \bigotimes_{i=1}^m \pi_{x_i}$$

and let $\mathscr{H}_{\mathrm{out}}(C_{\mathrm{aff}})$ be the abelian Lie algebra $\mathbb{C}[C\backslash\{v^{-1}(x_i)\}]$ of regular functions on $C\backslash\{v^{-1}(x_i)\}$. If $A_i\in\pi_{x_i}$, then $f\in\mathscr{H}_{\mathrm{out}}(C_{\mathrm{aff}})$ acts on $\mathscr{F}$ by

$$f\cdot(A_1\otimes\cdots\otimes A_m) = \sum_{i=1}^m\sum_{p\in v^{-1}(x_i)} A_1\otimes\cdots\otimes(\mathrm{Res}_p f\varpi_i)A_i\otimes\cdots\otimes A_m. \tag{7.2}$$

We will say that a meromorphic function $f$ on $C$ is *even* if $\sigma^*_C(f)=f$ and *odd* if $\sigma^*_C(f)=-f$. If $f$ is even, then $\sum_{p\in v^{-1}(x_i)}\mathrm{Res}_p f\varpi_i = 0$, so only odd functions act non-trivially. Let $\mathscr{H}_{\mathrm{out}}(C_{\mathrm{aff}})^o$ denote the space of odd elements in $\mathbb{C}[C\backslash\{v^{-1}(x_i)\}]$.

### 7.3. Coinvariants and conformal blocks

Now we give a simpler, alternative definition of the spaces of coinvariants and conformal blocks for (twisted) $\pi$-modules, extending Definition 8.1.7 in [FB].

**Definition 7.1.** The space of *coinvariants* associated to $(X, \{x_i\}, \{\pi_{x_i}\})$ is the vector space

$$\widetilde{\mathscr{H}}_{\pi}(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m} = \mathscr{F} / \mathscr{H}_{\text{out}}(C_{\text{aff}})^{\text{o}} \cdot \mathscr{F}$$

The space of *conformal blocks* associated to $(X, \{x_i\}, \{\pi_{x_i}\})$ is the vector space

$$\widetilde{\mathscr{C}}_{\pi}(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m} = \text{Hom}_{\mathscr{H}_{\text{out}}}(C_{\text{aff}})^{\text{o}}(\mathscr{F}, \mathbb{C})$$

of $\mathscr{H}_{\text{out}}(C_{\text{aff}})^{\text{o}}$-invariant functionals on $\mathscr{F}$.

**Remark.** The discussion at the end of Section 7.2 implies that these definitions remain the same if we replace $\mathscr{H}_{\text{out}}(C_{\text{aff}})^{\text{o}}$ by $\mathscr{H}_{\text{out}}(C_{\text{aff}})$.

We can ask how the space of conformal blocks changes under the addition of points. Suppose then that to our collection of points $\{x_i\}_{i=1\cdots m}$ we add $x_{m+1}$. Since $\{x_i\}_{i=1\cdots m}$ contains all the branch points of $v$, $v^{-1}(x_{m+1})$ consists of two points. Set $\pi_{x_{m+1}} = \pi^{0,p}_{x_{m+1}}$, $p \in v^{-1}(x_{m+1})$. Observe that in the case of the vacuum representation, $\pi^{0,p}_{x_{m+1}} = \pi^{0,\sigma_C(p)}_{x_{m+1}}$, so there is no choice of point in the fiber. There exists a natural map:

$$\widetilde{\mathscr{C}}_{\pi}(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m+1} \to \widetilde{\mathscr{C}}_{\pi}(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m} \tag{7.3}$$

given by

$$\phi \to \phi|_{\otimes_{i=1}^{m} \pi_{x_i} \otimes |0\rangle}, \tag{7.4}$$

i.e., we restrict the functional $\phi$ to the vacuum vector in $\pi_{x_{m+1}}$. The following lemma, analogous to Proposition 8.3.2 in [FB], will be proved in Section 7.5.

**Lemma 7.1.** *Map* (7.4) *is an isomorphism.*

### 7.4. Equivalence of Definitions 6.1 and 7.1

We now have two seemingly different definitions of the space of conformal blocks: the general Definitions 6.1 and 7.1, which is specific to the case $V = \pi$. We will show that these two definitions agree. Applying the Strong Residue Theorem 6.1 to our collection of End($\mathscr{F}$)-valued one-forms, we obtain

**Corollary 7.1.** *A functional $\phi$ is a conformal block if and only if $\forall A_i \in \pi_{x_i}$, the one-forms*

$$\langle \phi, A_1 \otimes \cdots \otimes (\varpi_i \cdot A_i) \otimes \cdots \otimes A_m \rangle \in \Gamma\left( \coprod_{p \in v^{-1}(x_i)} \mathscr{D}_p^{\times}, \Omega_{\mathring{C}} \right) \tag{7.5}$$

*can be extended to a single one-form $\varpi_\phi$ on $C \backslash \{v^{-1}(x_i)\}$.*

**Remark.** $\sigma_C$ acts on the space of holomorphic one-forms. It is clear that $\varpi_\phi$ is odd under this action.

We are now ready to prove the equivalence of the two definitions of conformal blocks. The proof is a generalization of the proof of Theorem 8.3.3 of [FB].

**Theorem 7.1.** *Let $\phi$ be a linear functional on $\mathscr{F}$ such that $\forall A_i \in \pi_{x_i}$, the one-forms* (7.5) *can be extended to a single, odd, regular one-form $\varpi_\phi$ on $C \backslash \{v^{-1}(x_i)\}$. Then the sections*

$$\langle \phi, A_1 \otimes \cdots \otimes (\mathscr{Y}^{\pi_{x_i}} \cdot A_i) \otimes \cdots \otimes A_m \rangle \in \Gamma\left( \coprod_{p \in v^{-1}(x_i)} \mathscr{D}_p^\times, \Pi_C^* \right) \tag{7.6}$$

*can be extended to a single, invariant, regular section of $\Pi_C^*$ on $C \backslash \{v^{-1}(x_i)\}$, and vice versa.*

**Proof.** From the discussion following Theorem 5.1, there exists a map

$$\Pi^*_{\overset{\circ}{C}} \to \Omega_{\overset{\circ}{C}}$$

such that the one-forms (7.5) are the projections of sections (6.1) on $\coprod_{p \in v^{-1}(x_i)} \mathscr{D}_p^\times$. Thus if (6.1) extend to $C \backslash \{v^{-1}(x_i)\}$, so will (7.5). Denote $C \backslash \{v^{-1}(x_i)\}$ by $C_{\text{aff}}$.

Let $\hat{C}^2$ denote $C^2_{\text{aff}} \backslash \Xi$, where $\Xi$ is the divisor in $C^2_{\text{aff}}$ consisting of pairs $(x, y) \in C^2_{\text{aff}}$ satisfying $\sigma_C^k(x) = \sigma_C^l(y)$ for some integers $k, l$. Let $r_1 : \hat{C}^2 \to C_{\text{aff}}$ denote the projection on the first factor, whose fiber over $q \in C_{\text{aff}}$ is

$$C_{\text{aff}} \backslash \mathbf{O}_q$$

where $\mathbf{O}_q$ denotes the $H$-orbit of $q$. Let

$$\tilde{\mathscr{O}} = (r_1)_* \mathscr{O}_{\hat{C}^2}.$$

This is a quasi-coherent sheaf on $C_{\text{aff}}$ whose fiber at $q \in C_{\text{aff}}$ is $\mathbb{C}[C_{\text{aff}} \backslash \mathbf{O}_q]$. Let $\mathscr{G}$ denote the sheaf $\mathscr{F} \otimes \Pi_C|_{C_{\text{aff}}}$ on $C_{\text{aff}}$, where we treat $\mathscr{F}$ as a constant sheaf. For $q \in C_{\text{aff}}$, the fiber $\mathscr{G}_q$ is $\mathscr{F} \otimes (\Pi_C)_q \cong \mathscr{F} \otimes \pi_{v(q)}$, where by $\pi_{v(q)}$ we mean the module along $v^{-1}(v(q))$ constructed out of the vacuum $\pi$. The sheaf $\tilde{\mathscr{O}}$ acts on $\mathscr{G}$ in such a way that fiberwise we obtain the action (7.2) of $\mathbb{C}[C_{\text{aff}} \backslash \mathbf{O}_q]$ on

$$\mathscr{F} \otimes \Pi_{v(q)} = \bigotimes_{i=1}^m \pi_{x_i} \otimes \pi_{v(q)}$$

Introduce the sheaf of homomorphisms

$$\widetilde{\mathscr{C}} = \mathrm{Hom}_{\tilde{\mathscr{O}}}(\mathscr{G}, \mathscr{O}_{C_{\text{aff}}})$$

whose fiber at $q \in C_{\text{aff}}$ is the space of conformal blocks

$$\widetilde{\mathscr{C}}_\pi(X, \{x_i, v(q)\}, \pi_{x_i}, \pi_{v(q)})_{i=1\cdots m}.$$

Lemma 7.1 identifies the fibers of $\widetilde{\mathscr{C}}$ with $\widetilde{\mathscr{C}}_\pi(X, \{x_i\}, \pi_{x_i})_{i=1\cdots r}$, thus providing a canonical trivialization of $\widetilde{\mathscr{C}}$. Thus, given

$$\phi \in \widetilde{\mathscr{C}}_\pi(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m}$$

we obtain a section $\tilde{\phi}$ of $\widetilde{\mathscr{C}}$ on $C_{\text{aff}}$. Let $\tilde{\phi}_q$ denote the value of this section at $q \in C$. Let $A_i \in \pi_{x_i}$ and

$$v = \bigotimes_{i=1}^{m} A_i \in \mathscr{F}$$

Contracting $\tilde{\phi}$ with $v \in \mathscr{F}$, we obtain a section $\tilde{\phi}^v$ of $\Pi_C^*$. By construction, the value of $\tilde{\phi}^v$ on $B \in (\Pi_C)_q = \pi_{v(q)}$ is $\tilde{\phi}_q(v \otimes B)$. We wish to show that the section $\tilde{\phi}^v$ is the analytic continuation of sections (6.1). We begin with the following lemma. $\square$

**Lemma 7.2.** *There exists a regular one-form on $C_{\text{aff}} \backslash \mathbf{O}_q$ whose restriction to the union $\coprod_{p \in v^{-1}(x_i)} \mathscr{D}_p^\times$ is equal to*

$$\langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes \varpi_i \cdot A_i \otimes \cdots \otimes A_m \otimes B \rangle,$$

*and its restriction to $\coprod_{p \in v^{-1}(v(q))} \mathscr{D}_p^\times$ is*

$$\langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \varpi_q \cdot B \rangle$$

**Proof.** Since $\tilde{\phi}_q$ is a conformal block, we obtain, using the definition of the action of $\mathbb{C}[C_{\text{aff}} \backslash \mathbf{O}_q]$ on $\mathscr{F} \otimes \pi_{v(q)}$,

$$\begin{aligned} 0 = \langle \tilde{\phi}_q, f \cdot (v \otimes B) \rangle = & \sum_{i=1\cdots m} \sum_{p \in v^{-1}(x_i)} \text{Res}_p \langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes (f \varpi_i) \cdot A_i \otimes \cdots \otimes B \rangle \\ & + \sum_{p=(q,\sigma_C(q))} \text{Res}_p \langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \cdots \otimes (f \varpi_q) \cdot B \rangle. \end{aligned}$$

Thus, the strong residue theorem implies that there exists a regular one-form on $C_{\text{aff}} \backslash \mathbf{O}_q$ having the desired properties. $\square$

**Remark 4.** If $x \in X, p \in v^{-1}(x)$, $v^{-1}(x)$ consists of $\frac{2}{N}$ points, and $z^{\frac{1}{N}}$ is a formal coordinate at $p$, then the endomorphism-valued one-form $\varpi$ on $\mathscr{D}_p^\times$ has the expression

$$\varpi = N z^{\frac{N-1}{N}} b(z^{\frac{1}{N}}) dz^{\frac{1}{N}}$$

where

$$b(z^{\frac{1}{N}}) = \sum_{n \in \frac{1}{N}+\mathbb{Z}} b_n z^{-n-1}$$

Now we prove Theorem 7.1. Suppose that $q$ is near $p \in v^{-1}(x_i)$. Choose a small analytic neighborhood $U$ of $p$ with special coordinate $z_i^{\frac{1}{N}}$ centered on $p$, such that $q \in U_i$. If $v$ is unramified, $N = 1$, and $z_i^{\frac{1}{N}}$ is any coordinate centered at $p$, otherwise $N = 2$ and $z_i^{\frac{1}{2}}$ is a $\sigma_C$-special coordinate. Since $q \neq p$, $w = z_i - z_i(q)$ is a coordinate centered at $q$, in some neighborhood $W$ of $q$. Near $q$, we can trivialize

$$\pi \times W \cong \Pi_C|_W$$

via

$$(B, q) \to (B, z_i - z_i(q)).$$

It remains to prove the following:

**Lemma 7.3.** $\forall B \in \pi$,

$$\begin{aligned}\langle \tilde{\phi}_q, v \otimes B \rangle &= \langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(B, q) \cdot A_i \otimes \cdots \otimes A_m \otimes |0\rangle \rangle \\ &= \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(B, q) \cdot A_i \otimes \cdots \otimes A_m \rangle.\end{aligned}$$

**Proof.** The second equality follows from Lemma 7.1. The first equality is proved by induction. It obviously holds for $B = |0\rangle$. Now, denote by $\pi^{(r)}$ the subspace of $\pi$ spanned by all monomials of the form $\widetilde{b_{i_1}} \cdots \tilde{b}_{i_k}|0\rangle$, where $k \leqslant r$. Suppose that we have proved Lemma 7.3 for all $B \in \pi^{(r)}$. The inductive step is to prove the equality for elements of the form $B' = \widetilde{b_n} \cdot B$. By our inductive hypothesis, we know that if $B \in \pi^{(r)}$, then

$$\begin{aligned}&\langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes (N z_i^{\frac{N-1}{N}} b(z_i^{\frac{1}{N}})) \cdot A_i \otimes \cdots \otimes B \rangle dz_i^{\frac{1}{N}} \\ &\quad = \langle \phi, A_1 \otimes \cdots \otimes N Y^{\pi_{x_i}}(B, q) z_i^{\frac{N-1}{N}} b(z_i^{\frac{1}{N}}) \cdot A_i \otimes \cdots \otimes A_m \rangle dz_i^{\frac{1}{N}}.\end{aligned}$$

According to Lemma 7.2 above, we also have

$$\langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes (N z_i^{\frac{N-1}{N}} b(z_i^{\frac{1}{N}})) \cdot A_i \otimes \cdots \otimes A_m \otimes B \rangle dz_i^{\frac{1}{N}} = \langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \tilde{b}(w) \cdot B \rangle dw.$$

Using locality and associativity, we obtain

$$
\begin{aligned}
&\langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(C,q) b(z_i^{\frac{1}{N}}) \cdot A_i \otimes \cdots \otimes A_m \rangle N z_i^{\frac{N-1}{N}} dz_i^{\frac{1}{N}} \\
&\quad = \langle \phi, A_1 \otimes \cdots \otimes b(z_i^{\frac{1}{N}}) Y^{\pi_{x_i}}(C,q) \cdot A_i \otimes \cdots \otimes A_m \rangle N z_I^{\frac{N-1}{N}} dz_i^{\frac{1}{N}} \\
&\quad = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\tilde{b}(z_i - z_i(q)) \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m \rangle N z_i^{\frac{N-1}{N}} dz_i^{\frac{1}{N}} \\
&\quad = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\tilde{b}(w) \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m \rangle N z_i^{\frac{N-1}{N}} dz_i^{\frac{1}{N}} \\
&\quad = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\tilde{b}(w) \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m \rangle dw,
\end{aligned}
$$

where the last step holds because $N z_i^{\frac{N-1}{N}} dz_i^{\frac{1}{N}} = dw$. Combining these relations, we obtain

$$
\langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \tilde{b}(w) \cdot B \rangle dw = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\tilde{b}(w) \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m \rangle dw.
$$

Multiplying both sides by $w^n$ and taking residues, we find that

$$
\langle \tilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \tilde{b}_n \cdot B \rangle = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\tilde{b}_n \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m \rangle.
$$

Equivariance of sections (7.6) follows from the fact that they are invariant on all $\mathscr{D}_p^\times$ where $p$ is a fixed point of $\sigma_C$. This completes the proof of Theorem 7.1. $\square$

## *7.5. Proof of Lemma 7.1*

We start with the following fact.

**Lemma 7.4.** *Let $p \in v^{-1}(x_{m+1})$. For every principal part $f_-$ at $p$, there exists an odd function $f \in \mathbb{C}[C_{\mathrm{aff}} \backslash \mathbf{O}_p]$ whose principal part at $p$ is $f_-$.*

**Proof.** Let $D$ be an effective divisor symmetric under the action of $\sigma_C$ (i.e., if $D = \Sigma c_q \cdot q$, then $c_q = c_{\sigma_C(q)}$), supported on $\{v^{-1}(x_i)\}_{i=1\cdots m}$. Denote the canonical divisor of $C$ by $K_C$. For $\deg(D) > \deg(K_C)$, the Riemann–Roch theorem implies that

$$
\dim \mathscr{L}(D) = \deg(D) + 1 - g_C.
$$

It follows that

$$
Q_{n+1} = \mathscr{L}(D + (n+1) \cdot p + (n+1)\sigma_C(p)) / \mathscr{L}(D + n \cdot p + n \cdot \sigma_C(p))
$$

for $n \geqslant 0$ is two-dimensional. Furthermore, $Q_{n+1}$ carries an action of $\sigma_C$. Suppose now that $Q_{n+1}$ is spanned by the images of two even functions $f_1, f_2$. Since $f_i$ are even, they have poles of the same order at both $p$ and $\sigma_C(p)$, and so will any linear combination. But this contradicts the fact that

$$\mathscr{L}(D+(n+1)p+n\sigma_C(p))/\mathscr{L}(D+n\cdot p+n\cdot\sigma_C(p))$$

is one-dimensional. It follows that for each $n\geqslant 0$, $Q_{n+1}$ contains an odd function. $\square$

Now we prove the statement equivalent to Lemma 7.1 that the corresponding spaces of coinvariants

$$\tilde{\mathscr{H}}_\pi(X,\{x_i\},\pi_{x_i})_{i=1\cdots m} \quad \text{and} \quad \tilde{\mathscr{H}}_\pi(X,\{x_i\},\pi_{x_i})_{i=1\cdots m+1}$$

are isomorphic. Recall that $\pi_{x_{m+1}}$ here is the vacuum module $\pi^{0,p}_{x_{m+1}}$. Let $C'_{\text{aff}}=C_{\text{aff}}\backslash\mathbf{O}_p$, and $\mathscr{H}_{\text{out}}(C_{\text{aff}})p=\mathbb{C}[C'_{\text{aff}}]$. The space

$$\tilde{\mathscr{H}}_\pi(X,\{x_i\},\pi_{x_i})_{i=1\cdots m} \quad (\text{resp. } \tilde{\mathscr{H}}_\pi(X,\{x_i\},\pi_{x_i})_{i=1\cdots m+1}$$

is identified with the 0th homology of the Lie algebra $\mathscr{H}_{\text{out}}(C_{\text{aff}})^{\text{o}}$ (resp. $\mathscr{H}_{\text{out}}(C'_{\text{aff}})^{\text{o}}$) with coefficients in $\mathscr{F}$ (resp. $\mathscr{F}\otimes\pi_{x_{m+1}}$). Lemma 7.4 implies that the sequence

$$0\to\mathscr{H}_{\text{out}}(C_{\text{aff}})^{\text{o}}\to\mathscr{H}_{\text{out}}(C'_{\text{aff}})^{\text{o}}\xrightarrow{\mu} w_{s+1}^{-1}\mathbb{C}[w_{s+1}^{-1}]\to 0 \tag{7.7}$$

is exact, where $\mu$ is the map that attaches to a function its principal part at $p$. The homology of $\mathscr{H}_{\text{out}}(C'_{\text{aff}})^{\text{o}}$ with coefficients in $\mathscr{F}\otimes\pi_{x_{m+1}}$ is computed using the Chevalley complex

$$C^\bullet=\mathscr{F}\otimes\pi_{x_{m+1}}\otimes\bigwedge^\bullet(\mathscr{H}_{\text{out}}(C'_{\text{aff}})^{\text{o}})$$

with the differential $d: C^i\to C^{i-1}$ given by the formula

$$d=\sum_i f_i\otimes\psi_i^*$$

where $\{f_i\}$ is a basis in $\mathscr{H}_{\text{out}}(C'_{\text{aff}})^{\text{o}}$ and $\{\psi^*\}$ is the dual basis of $(\mathscr{H}_{\text{out}}(C'_{\text{aff}})^{\text{o}})^*$ acting on $\bigwedge^\bullet(\mathscr{H}_{\text{out}}(C'_{\text{aff}})^{\text{o}})$ by contraction.

Choose pull-backs $z_n, n<0$, of $w^n_{s+1}, n<0$, in $\mathscr{H}_{\text{out}}(C'_{\text{aff}})^{\text{o}}$ under $\mu$. Because of the exactness of sequence (7.7), we can choose a basis $\{f_i\}$ in $\mathscr{H}_{\text{out}}(C'_{\text{aff}})^{\text{o}}$ which is a union of $\{z_n\}_{n<0}$, and a basis of $\mathscr{H}_{\text{out}}(C_{\text{aff}})^{\text{o}}$. In this basis we may decompose

$$d=d_{C_{\text{aff}}}+\sum z_n\otimes\phi_n^*,$$

where $dC_{\text{aff}}$ is the differential for $\mathscr{H}_{\text{out}}(C_{\text{aff}})^{\text{o}}$, and $\phi_n^*$ denotes the element of the dual basis to $\{f_i\}$ corresponding to $z_n$.

We need to show that the homologies of this complex are isomorphic to the homologies of the complex $\mathscr{F}\otimes\bigwedge^\bullet(\mathscr{H}_{\text{out}}(C_{\text{aff}})^{\text{o}})$. Introduce an increasing filtration on $\pi_{x_{m+1}}$, letting $\pi_{m+1}^{(r)}$ be the span of all monomials of order less than or equal to $m$ in $\tilde{b}_n, n<0$. Now introduce a filtration $\{F_i\}$ on the Chevalley complex $C^\bullet$ by setting

$$F_i = \mathrm{span}\left\{ v \otimes B \otimes D | v \in \mathscr{F}, B \in \pi_{x_{m+1}}^{(m)}, D \in \bigwedge^{i-m}(\mathscr{H}_{\mathrm{out}}(C'_{\mathrm{aff}})^{\mathrm{o}}) \right\}.$$

Our differential preserves this filtration.

Consider now the spectral sequence associated to the filtered complex $C^\bullet$. The zeroth term $E^0$ is the associated graded space of the Chevalley complex, isomorphic to

$$\left(\pi_{s+1} \otimes \bigwedge^{\bullet}(\phi_n^*)_{n<0}\right) \otimes \left(\mathscr{F} \otimes \bigwedge^{\bullet}(\mathscr{H}_{\mathrm{out}}(C_{\mathrm{aff}})^{\mathrm{o}})\right).$$

The zeroth differential acts along the first factor of the above decomposition, and is given by the formula

$$d^0 = \sum_{n<0} b_n \otimes \phi_n^*,$$

because on the graded module the operator $z_n$ acts as $b_n, n<0$. But $\pi_{s+1}$ is isomorphic to the symmetric algebra with generators $b_n, n<0$, and our differential is simply the Koszul differential for this symmetric algebra. It is well-known that the zeroth homology of this complex is isomorphic to $\mathbb{C}$, and all other homologies vanish. Therefore all positive homologies of $d^0$ vanish, while the zeroth homology is $\mathscr{F} \otimes \bigwedge^\bullet(\mathscr{H}_{\mathrm{out}}(C_{\mathrm{aff}}))$. Hence, the $E^1$ term coincides as a vector space with the Chevalley complex of the homology of $\mathscr{H}_{\mathrm{out}}(C_{\mathrm{aff}})^{\mathrm{o}}$ with coefficients in $\mathscr{F}$. Also, the $E^1$ differential coincides with $d_{C_{\mathrm{aff}}}$, which is the corresponding Chevalley differential. We thus obtain the desired isomorphism

$$H_i(\mathscr{H}_{\mathrm{out}}(C'_{\mathrm{aff}})^{\mathrm{o}}, \mathscr{F} \otimes \pi_{m+1}) \cong H_i(\mathscr{H}_{\mathrm{out}}(C_{\mathrm{aff}})^{\mathrm{o}}, \mathscr{F}).$$

## 8. Affine vertex algebras

In Section 7.3 we have shown that in the case of the Heisenberg vertex algebra the space of conformal blocks had a simple realization as the dual of a certain space of twisted coinvariants. In this section we present a similar realization in the case of vertex algebras attached to affine Kac–Moody algebras.

### 8.1. The vacuum module $V_k(\mathfrak{g})$

Let $\mathfrak{g}$ denote a complex simple Lie algebra, $L\mathfrak{g} = \mathfrak{g} \otimes [t, t^{-1}]$ its loop algebra, and $\hat{\mathfrak{g}}$ the corresponding affine Kac–Moody Lie algebra. For $k \in \mathbb{C}$, let $\mathbb{C}_k$ denote the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C} \cdot K$ where $\mathfrak{g}[t]$ acts by 0, and $K$ acts by $k$. It is well known that the *vacuum module*

$$V_k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[t]+\mathbb{C}\cdot K}^{\hat{\mathfrak{g}}} \mathbb{C}_k$$

has the structure of a vertex algebra (see for instance Section 3.4.2 of [FB]).

Pick a basis $\{J^a\}_{a=1\cdots d}$ (where $d = \dim(\mathfrak{g})$) of $\mathfrak{g}$, and let $\{J_a\}_{a=1\cdots d}$ be its dual basis with respect to the normalized Killing form. Suppose that $k \neq -h^\vee$ (where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$) and set

$$S = \frac{1}{2(k+h^\vee)} \sum_{a=1}^{d} (J_a \otimes t^{-1})(J^a \otimes t^{-1})|0\rangle.$$

This is the *Sugawara vector* which determines a conformal structure on $V_k(\mathfrak{g})$ when $k \neq -h^\vee$. In what follows, we will always use this conformal structure on $V_k(\mathfrak{g})$.

Let $\sigma_{\mathfrak{g}}$ be an automorphism of $\mathfrak{g}$ of finite order $N$. Then $\sigma_g$ induces a conformal automorphism of $V_k(\mathfrak{g})$, which we will denote by $\sigma_{V_k(\mathfrak{g})}$.

In particular, consider the case when $\sigma_{\mathfrak{g}}$ is an outer automorphism (note that this is not necessary for the results below). Thus, $N = 2$ when $\mathfrak{g} = A_n, D_m, m \neq 4, E_6$, and $N = 3$ when $\mathfrak{g} = D_4$. The following result is proved in [Li].

**Lemma 8.1.** *The $\sigma_{V_k(\mathfrak{g})}$-twisted $V_k(\mathfrak{g})$-modules are precisely the $\hat{\mathfrak{g}}^\sigma$-modules from the category $\mathcal{O}$, where $\hat{\mathfrak{g}}^\sigma$ is the twisted affine Kac–Moody algebra associated to the automorphism $\sigma_{\mathfrak{g}}$.*

### *8.2. The Lie algebra $\mathfrak{g}^\sigma_{\mathrm{out}}(C_{\mathrm{aff}})$*

We keep the notation of Section 7. Let $C$ be an algebraic curve with an automorphism $\sigma_C$ of order $N$ (where $N = 2$ or 3 depending on $\mathfrak{g}$), and let $\{x_i\}_{i=1\cdots m}$ be a collection of points of $X$ containing the branch points of $v$. Denote $C \backslash \{v^{-1}(x_i)\}$ by $C_{\mathrm{aff}}$. Let us write $\mathfrak{g} = \oplus_{l=0}^{N-1} \mathfrak{g}_l$, where $\mathfrak{g}_l$ denotes the eigenspace of $\sigma_{\mathfrak{g}}$ corresponding to the eigenvalue $e^{\frac{2\pi i l}{N}}$. Then $\sigma_C$ acts on $\mathbb{C}[C_{\mathrm{aff}}]$-the ring of functions on $C_{\mathrm{aff}}$, and so we can write $\mathbb{C}[C_{\mathrm{aff}}] = \oplus \mathbb{C}[C_{\mathrm{aff}}]_l$, where $\mathbb{C}[C_{\mathrm{aff}}]_l$ consists of those functions $f$ such that $\sigma_C^*(f) = e^{\frac{2\pi i l}{N}} f$. Let

$$\mathfrak{g}^\sigma_{\mathrm{out}}(C_{\mathrm{aff}}) = \bigoplus_{l=1}^{N} (\mathfrak{g}_l \otimes \mathbb{C}[C_{\mathrm{aff}}]_l).$$

### *8.3. Coinvariants and conformal blocks*

For $x \in X$, $V$-modules along $v^{-1}(x)$ can be constructed from ordinary or twisted $V$-modules using the same technique that was used in the Heisenberg case in

Section 7.2. More precisely, if $x$ is a branch point of $v, p = v^{-1}(x)$, and $\sigma_{C,p}$ is the monodromy around $x$, then any $\sigma_{C,p}$-twisted $V_k(\mathfrak{g})$-module gives rise to a $V_k(\mathfrak{g})$-module along $v^{-1}(x)$. Similarly, if $v^{-1}(x)$ consists of $N$ points, then an ordinary $V_k(\mathfrak{g})$-module and a choice of point $p \in v^{-1}(x)$ gives rise to a $V_k(\mathfrak{g})$-modules along $v^{-1}(x)$.

Let $\{\mathscr{M}_{x_i}\}$ be a collection of $V_k(\mathfrak{g})$-modules along $\{x_i\}$ constructed in this manner. Thus for each $x_i$, we have a distinguished point $p_i \in v^{-1}(x_i)$. Pick special coordinates $z^{\frac{1}{N_i}}$ near $p_i$, where $N_i = 1$ if $v$ is unramified at $p_i$ and $N_i = N$ otherwise. Set

$$\mathscr{F} = \bigotimes_{i=m}^{r} \mathscr{M}_{x_i}.$$

Then $\mathfrak{g}^{\sigma}_{\text{out}}(C_{\text{aff}})$ acts on $\mathscr{F}$ as follows:

$$h \cdot (A_1 \otimes \cdots \otimes A_m) = \sum_i A_1 \otimes \cdots \otimes [h]_{p_i} \cdot A_i \otimes \cdots \otimes A_m,$$

where $[h]_p$ denotes the Laurent series expansion of $h$ around $p \in C$ in the special coordinate that was selected.

We are now ready to give an alternative, simplified definition of twisted coinvariants and conformal blocks for $V_k(\mathfrak{g})$, extending the definition of Section 8.2.1 in [FB]:

**Definition 8.1.** The space of *coinvariants* is the vector space

$$\tilde{\mathscr{H}}_{V_k(\mathfrak{g})}(X, \{x_i\}, \mathscr{M}_{x_i})_{i=1\cdots m} = \mathscr{F} / \mathfrak{g}^{\sigma}_{\text{out}}(C_{\text{aff}}) \cdot \mathscr{F}$$

The space of *conformal blocks* is its dual:

$$\widetilde{\mathscr{C}}_{V_k(\mathfrak{g})}(X, \{x_i\}, \mathscr{M}_{x_i})_{i=1\cdots m} = \text{Hom}_{\mathfrak{g}^{\sigma}_{\text{out}}(C_{\text{aff}})}(\mathscr{F}, \mathbb{C})$$

The following theorem is proved using the same methods as Theorem 7.1.

**Theorem 8.1.** *In the case of the vertex algebra* $V_k(\mathfrak{g})$, *Definition* 6.1 *is equivalent to Definition* 8.1.

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