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A new simultaneous method of fourth order for finding complex zeros in circular interval arithmetic

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Abstract

Starting from disjoint disks which contain polynomial complex zeros, the new iterative interval method for simultaneous finding of inclusive disks for complex zeros is formulated. The convergence theorem and the conditions for convergence are considered, and the convergence is shown to be fourth. Numerical examples are included. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Iterative method; Interval arithmetic; Zeros of polynomial; Lagrange interpolation; Order of convergence

1. Introduction

Let $\mathscr{P}$ be the vector space of the polynomials, let $\mathscr{P}_n$ be the vector space of the polynomials of degree less than or equal to n, and let $\mathscr{C}$ be the complex plane.

We consider a monic polynomial $P \in \mathscr{P}_n$, $n \geqslant 3$:

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{j=1}^{n} (z - \zeta_j) \tag{1}$$

with simple complex zeros $\zeta_1, \ldots, \zeta_n$ of P, and $a_i \in \mathscr{C}$.

Various authors developed the techniques for a posteriori error estimates for the approximations of polynomial zeros (see [2,3,5,6,10,11]). A quite different approach to error estimates is based on the use of interval arithmetic (see [4,7,9]). In this manner, not only very close zero approximations (given by the midpoints of intervals) but also upper error bounds for the zeros (given by the semi-widths

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of intervals) are obtained which means the automatic verification of results and a control of errors in each iteration. Recently, Zheng and Sun [12] presented a new approach for the simultaneous computation of all zeros of polynomials. Using some results derived in [12] we propose in this paper some new simultaneous methods for finding polynomial complex zeros in circular interval arithmetic.

In Section 3 we derive the new simultaneous method in circular interval arithmetic. The convergence analysis is given in Section 4. We prove that the convergence is of fourth order. Also, in this section, we give a more simple interval iterative method under some condition in remarks. Some numerical examples are included in Section 5.

2. Complex circular arithmetic

The circular arithmetic was introduced and for the first time systematically used by Gargantini and Henrici [7]. Here we give some basic properties of complex circular arithmetic necessary for this paper. For more details see the book [1].

If $\mathbb{Z}$ denotes the disk with center c and radius r,

$$\mathbb{Z} := \{z \in \mathscr{C}\colon\ |z - c| \leqslant r\},$$

we write $\mathbb{Z} = \{c; r\}$, $c = \text{mid}\ \mathbb{Z}$, $r = \text{rad}\ \mathbb{Z}$, for brevity.

We denote by $\mathscr{D}(\mathscr{C})$ the set of all the disks.

Definition 2.1. Let $\mathbb{Z}_1, \mathbb{Z}_2 \in \mathscr{D}(\mathscr{C})$, $\mathbb{Z}_1 = \{c_1; r_1\}$, $\mathbb{Z}_2 = \{c_2; r_2\}$. Then

$$\mathbb{Z}_1 \pm \mathbb{Z}_2 = \{c_1 \pm c_2; r_1 + r_2\},$$

$$\mathbb{Z}_1 \cdot \mathbb{Z}_2 = \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\},$$

$$\mathbb{Z}_1 : \mathbb{Z}_2 = \mathbb{Z}_1 \cdot 1/\mathbb{Z}_2 \quad \text{if } 0 \notin \mathbb{Z}_2,$$

$$1/\mathbb{Z}_2 = \{\bar{c}_2/(|c_2|^2 - r_2^2); r_2/(|c_2|^2 - r_2^2)\} \quad \text{if } 0 \notin \mathbb{Z}_2,$$

where $\bar{c}_2$ denotes the complex conjugate of c_2.

From Definition 2.1, we have

$$\sum_{k=1}^{m} \{c_k; r_k\} = \left\{ \sum_{k=1}^{m} c_k; \sum_{k=1}^{m} r_k \right\}$$

and

$$\prod_{k=1}^{m} \{c_k; r_k\} = \left\{ \prod_{k=1}^{m} c_k; \prod_{k=1}^{m} (|c_k| + r_k) - \prod_{k=1}^{m} |c_k| \right\}.$$

In particular,

$$w \pm \{c; r\} = \{w \pm c; r\},$$

$$w \cdot \{c; r\} = \{wc; |w| r\},$$

where $w \in \mathscr{C}$.

To enable the kth root of a disk to produce disks enclosing the mentioned regions, the following circular approximation to the set $\{z^{1/k};\ z \in \mathbb{Z}\}$ was introduced in [9]:

Let $c = |c|\exp(\mathrm{i}\theta)$ and $|c| > r$, i.e., the disk $\mathbb{Z} = \{c; r\}$ does not contain the origin. Then

$$\mathbb{Z}^{1/k} := \bigcup_{\lambda=0}^{k-1} \left\{ |c|^{1/k} \exp\left(\frac{\theta + 2\lambda\pi}{k}\right); |c|^{1/k} - (|c| - v)^{1/k} \right\},$$

especially,

$$\mathbb{Z}^{1/2} := \left\{ |c|^{1/2} \exp\left(\frac{\mathrm{i}\theta}{2}\right); |c|^{1/2} - (|c| - r)^{1/2} \right\} \cup \left\{ -|c|^{1/2} \exp\left(\frac{\mathrm{i}\theta}{2}\right); |c|^{1/2} - (|c| - r)^{1/2} \right\}. \tag{2}$$

3. A simultaneous method in circular interval arithmetic

Suppose that the disjoint disks $\mathbb{Z}_i = \{\mathrm{mid}(\mathbb{Z}_i); \mathrm{rad}(\mathbb{Z}_i)\} = \{z_i; r_i\}$ with center z_i and radius r_i contain the zeros ζ_i of the polynomial P, $i = 1, \ldots, n$.

We introduce the following notation:

$$u_i = \frac{P(z_i)}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)}, \tag{3}$$

$$s_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{z_i - z_j}, \tag{4}$$

$$t_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{(z_i - z_j)^2}, \tag{5}$$

$$t_i^* = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{(z_i - z_j)(\zeta_i - z_j)}. \tag{6}$$

Using Lagrangean interpolation of P with nodes $z_1, \ldots, z_n$, we have

$$\begin{aligned} P(z) &= \sum_{j=1}^{n} \frac{u_j}{z - z_j} \prod_{i=1}^{n} (z - z_i) + \frac{P^{(n)}(\zeta_2)}{n!} \prod_{i=1}^{n} (z - z_i) \\ &= \left(\sum_{j=1}^{n} \frac{u_j}{z - z_j} + 1 \right) \prod_{i=1}^{n} (z - z_i) \\ &= \left\{ \left(1 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_i}{z - z_j} \right) (z - z_i) + u_i \right\} \prod_{\substack{j=1\\ j\neq i}}^{n} (z - z_j). \end{aligned}$$

So

$$P'(z) = \left\{ 1 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{z-z_j} + u_i \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{z-z_j} \right.$$
$$\left. + (z-z_i)\left[\left(1 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{z-z_j}\right) \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{z-z_j} - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{(z-z_j)^2} \right] \right\} \prod_{\substack{j=1\\ j\neq i}}^{n} (z-z_j).$$

Thus,

$$\frac{P'(z_i)}{P(z_i)} - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{z_i-z_j} = \left(1 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{z_i-z_j}\right) \Big/ u_i = \frac{1+s_i}{u_i}. \tag{7}$$

Let

$$c_j = \frac{P(z_j)}{\prod_{\substack{v=0\\ v\neq j}}^{n} (z_j - z_v)}, \quad j = 0, 1, \ldots, n, \tag{8}$$

where z_0 be another point, $z_0 \in \mathscr{C}$.

Then

$$c_j = \frac{u_j}{z_j - z_0}, \quad j \neq 0. \tag{9}$$

Using Lagrangean interpolation of P with the nodes $z_0, z_1, \ldots, z_n$, we have

$$P(z) = \left(\sum_{j=0}^{n} \frac{c_j}{z-z_j} \right) \prod_{j=0}^{n} (z-z_j). \tag{10}$$

By (8), we have, for some i, $1 \leqslant i \leqslant n$,

$$\frac{c_0}{z-z_0} + \frac{c_i}{z-z_i} = \frac{P(z_0)}{(z-z_0)(z_0-z_i)\prod_{\substack{j=1\\ j\neq i}}^{n}(z_0-z_j)} + \frac{P(z_i)}{(z-z_i)(z_i-z_0)\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i-z_j)}$$
$$= \frac{1}{(z-z_0)\prod_{\substack{j=1\\ j\neq i}}^{n}(z_0-z_j)} \frac{P(z_0)-P(z_i)}{z_0-z_i}$$
$$- \frac{P(z_i)}{z_i-z_0} \left[\frac{1}{(z_i-z)\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i-z_j)} - \frac{1}{(z_0-z)\prod_{\substack{j=1\\ j\neq i}}^{n}(z_0-z_j)} \right].$$

Then, let $z_0 \to z_i$, we have

$$\frac{c_0}{z-z_0} + \frac{c_i}{z-z_i}$$
$$\to \frac{P'(z_i)}{(z-z_i)\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i-z_j)}$$

$$
\begin{aligned}
&-\frac{P(z_i)[(z_0-z)\prod_{j=1, j\neq i}^{n}(z_0-z_j)-(z_i-z)\prod_{j=1, j\neq i}^{n}(z_i-z_j)]}{\prod_{j=1, j\neq i}^{n}(z_i-z_j)(z_i-z)(z_0-z)(z_i-z_0)\prod_{j=1, j\neq i}^{n}(z_0-z_j)}\\
&\to \frac{1+\sum_{j=1, j\neq i}^{n}(u_i/(z_i-z_j))+u_i\sum_{j=1, j\neq i}^{n}1/(z_i-z_j)}{z-z_i}\\
&\quad+\frac{u_i}{(z_i-z)^2\prod_{j=1, j\neq i}^{n}(z_i-z_j)}\left[\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i-z_j)+(z_i-z)\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i-z_j)\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z_i-z_j}\right]\\
&=\frac{1+s_i}{z-z_i}+\frac{u_i}{(z-z_i)^2}.
\end{aligned}
$$

Thus, by (10), we obtain

$$
P(z)=\left\{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{u_j}{(z_j-z_i)(z-z_j)}+\frac{1+s_i}{z-z_i}+\frac{u_i}{(z-z_i)^2}\right\}(z-z_i)\prod_{j=1}^{n}(z-z_j)\quad (\text{for } z_0=z_i). \tag{11}
$$

Substituting $z=\zeta_i\notin\{z_1,\dots,z_n\}$ in (11), and $P(z)=P(\zeta_i)=0$, we have

$$
\frac{u_i}{(\zeta_i-z_i)^2}+\frac{1+s_i}{\zeta_i-z_i}-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{u_j}{(z_i-z_j)(\zeta_i-z_j)}=0. \tag{12}
$$

Therefore,

$$
\zeta_i=z_i-\frac{2u_i}{1+s_i+\sqrt{(1+s_i)^2+4u_i\sum_{j=1, j\neq i}^{n}u_j/(z_i-z_j)(\zeta_i-z_j)}}. \tag{13}
$$

Since $\zeta_i\in\mathbb{Z}_i$, $i=1,\dots,n$, from (13) it follows

$$
\zeta_i\in z_i-\frac{2u_i}{1+s_i+((1+s_i)^2+4u_i\sum_{j=1, j\neq i}^{n}u_j/(z_i-z_j)(\mathbb{Z}_i-z_j))_*^{1/2}}\triangleq\hat{\mathbb{Z}}_i, \tag{14}
$$

that is, from $\zeta_i\in\mathbb{Z}_i\Rightarrow\zeta_i\in\hat{\mathbb{Z}}_i$, where the symbol "$*$" denotes the value of the second root which is equal to $(1+s_i)^2+4u_i\sum_{j=1,j\neq i}^{n}u_j/(z_i-z_j)(\mathbb{Z}_i-z_j)$.

Let $\mathbb{Z}_j^{(0)}=\{z_j^{(0)};r_j^{(0)}\}$ be disjoint disks such that $\zeta_j\in\mathbb{Z}_j^{(0)}$, then Eq. (14) (or (13)) leads to the following interval method for the simultaneous inclusion of the distinct roots ζ_i, $i=1,\dots,n$:

$$
\mathbb{Z}_i^{(m+1)}=z_i^{(m)}-\frac{2u_i^{(m)}}{1+s_i^{(m)}+\left((1+s_i^{(m)})^2+4u_i^{(m)}\sum_{j=1, j\neq i}^{n}\frac{u_j^{(m)}}{(z_i^{(m)}-z_j^{(m)})(\mathbb{Z}_i^{(m)}-z_j^{(m)})}\right)_*^{1/2}} \tag{15}
$$

where

$$
u_i^{(m)}=\frac{P(z_i^{(m)})}{\prod_{j=1, j\neq i}^{n}(z_i^{(m)}-z_j^{(m)})},\quad s_i^{(m)}=\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{u_j^{(m)}}{(z_i^{(m)}-z_j^{(m)})}.
$$

4. Convergence

In this section, we carry out a convergence analysis of the interval iterative method (15). We assume that the distinct roots $\zeta_1,\dots,\zeta_n$ of the polynomial P are isolated in separated disks

$$\mathbb{Z}_i^{(0)} = \{z_i^{(0)}; r_i^{(0)}\}, \quad i=1,\dots,n$$

and let $\mathbb{Z}_i^{(m)} = \{z_i^{(m)}; r_i^{(m)}\}$, $i=1,\dots,n$, be the disks obtained in the mth iteration.

We introduce the following notation:

$$r^{(m)} = \max_{1\leqslant i\leqslant n} r_i^{(m)},$$

$$\varrho^{(m)} = \min_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \{|z| : z\in z_i^{(m)} - \mathbb{Z}_j^{(m)}\} = \min_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \{|z_i^{(m)} - z_j^{(m)}| - r_j^{(m)}\}, \quad m=0,1,2,\dots .$$

The value of $\varrho^{(m)}$ is a measure of the separation of the disks $\mathbb{Z}_i^{(m)}$, $i=1,\dots,n$, from each other.

For simplicity, we will omit sometimes the iteration index m and write,

$$z_i, \hat{z}_i, \mathbb{Z}_i, \hat{\mathbb{Z}}_i, u_i, \hat{u}_i, s_i, \hat{s}_i, r_i, \hat{r}_i, r, \hat{r}, \varrho, \hat{\varrho}$$

instead of

$$z_i^{(m)}, z_i^{(m+1)}, \mathbb{Z}_i^{(m)}, \mathbb{Z}_i^{(m+1)}, u_i^{(m)}, u_i^{(m+1)}, s_i^{(m)}, s_i^{(m+1)},$$

$$r_i^{(m)}, r_i^{(m+1)}, r^{(m)}, r^{(m+1)}, \varrho^{(m)}, \varrho^{(m+1)}, \text{ respectively.}$$

First we give the following assertion.

Lemma 1. *The definition u_i, r_i, r, ϱ are as above, then*

$$|u_i| < r\left(1+\frac{r}{\varrho}\right)^{n-1} \quad \textit{for any } i=1,\dots,n. \tag{16}$$

Proof.

$$|u_i| = \left|\frac{\prod_{j=1}^n (z_i-\zeta_j)}{\prod_{\substack{j=1\\ j\neq i}}^n (z_i - z_j)}\right| = \prod_{\substack{j=1\\ j\neq i}}^n \left|\frac{z_i-\zeta_i}{z_i-z_j}\right| |z_i-\zeta_i| \leqslant r_i \prod_{\substack{j=1\\ j\neq i}}^n \frac{|z_i-z_j|+r_j}{|z_i-z_j|}$$

$$< r\left(1+\frac{r}{\varrho}\right)^{n-1}, \quad i=1,\dots,n. \qquad \square$$

Lemma 2. *Let*

$$T_i = \sum_{\substack{j=1\\ j\neq i}}^n \frac{u_i}{(z_i-z_j)(\mathbb{Z}_i - z_j)},$$

then

$$T_i = \sum_{\substack{j=1\\ j\neq i}}^n \frac{u_j}{z_i-z_j}\frac{1}{|z_i-z_j|^2 - r_i^2}\{\bar{z}_i - \bar{z}_j; r_i\}. \tag{17}$$

Proof.

$$\mathbb{Z}_i - z_j = \{z_i - z_j; r_i\},$$

so

$$\frac{1}{\mathbb{Z}_i - z_j} = \frac{1}{|z_i - z_j|^2 - r_i^2}\{\bar{z}_i - \bar{z}_j; r_i\}.$$

Thus,

$$T_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{z_i - z_j} \cdot \frac{1}{|z_i - z_j|^2 - r_i^2}\{\bar{z}_i - \bar{z}_j; r_i\}. \qquad \square$$

From Lemma 2, we have

$$\begin{aligned}
(1+s_i)^2 + 4u_i \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{(z_i - z_j)(\mathbb{Z}_i - z_j)} &= (1+s_i)^2 + 4u_i T_i \\
&= \left\{ (1+s_i)^2 + 4u_i \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_i}{z_i - z_j} \frac{\bar{z}_i - \bar{z}_j}{|z_i - z_j|^2 - r_i^2}; \right. \\
&\qquad \left. \sum_{\substack{j=1\\ j\neq i}}^{n} \left| \frac{4u_i u_j}{(z_i - z_j)(|z_i - z_j|^2 - r_i^2)} \right| r_i \right\} \\
&= \{c_i, \eta_i\},
\end{aligned} \tag{18}$$

where

$$c_i = (1+s_i)^2 + 4u_i \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_i}{z_i - z_j} \frac{\bar{z}_i - \bar{z}_j}{|z_i - z_j|^2 - r_i^2}, \tag{19}$$

$$\eta_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{4|u_i||u_j|}{|z_i - z_j|(|z_i - z_j|^2 - r_i^2)} r_i. \tag{20}$$

Suppose that the angle of intersection between $\exp(\mathrm{i}\theta/2)$ and $1+s_i$ is acute angle (where θ is determined by $c_i = |c_i|\exp(\mathrm{i}\theta)$,) otherwise, the angle of intersection between — $\exp(\mathrm{i}\theta/2)$, and $1+s_i$ is an acute angle.

By definition of quadratic root of a disk (2), we can choose

$$[(1+s_i)^2 + 4u_i T_i]_*^{1/2} = \left\{ |c_i|^{1/2} \exp\left(\frac{\mathrm{i}\theta}{2}\right); |c_i|^{1/2} - (|c_i| - \eta_i)^{1/2} \right\}. \tag{21}$$

Thus,

$$
\begin{aligned}
1+s_i+[(1+s_i)^2+4u_iT_i]_*^{1/2} &= \left\{1+s_i+|c_i|^{1/2}\exp\left(\frac{\mathrm{i}\theta}{2}\right); |c_i|^{1/2}-(|c_i|-\eta_i)^{1/2}\right\}\\
&= \left\{1+s_i+|c_i|^{1/2}\exp\left(\frac{\mathrm{i}\theta}{2}\right); \frac{\eta_i}{|c_i|^{1/2}+(|c_i|-\eta_i)^{1/2}}\right\}\\
&= \{\hat{c}_i,\hat{\eta}_i\},
\end{aligned} \tag{22}
$$

where

$$\hat{c}_i = 1+s_i+|c_i|^{1/2}\exp\left(\frac{\mathrm{i}\theta}{2}\right), \tag{23}$$

$$\hat{\eta}_i = \frac{\eta_i}{|c_i|^{1/2}+(|c_i|-\eta_i)^{1/2}}. \tag{24}$$

Lemma 3. *η_i is defined by* (20), *then*

$$\eta_i < \frac{4(n-1)}{\varrho^3}\left(1+\frac{r}{\varrho}\right)^{2n-2} r^3. \tag{25}$$

Proof. From Lemma 1, and the definition of ϱ, we have

$$
\begin{aligned}
\eta_i &= \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{4|u_i|\,|u_j|}{|z_i-z_j|(|z_i-z_j|^2-r_i^2)} r_i \leqslant \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{4r_i(1+(r/\varrho))^{n-1}r_j(1+(r/\varrho))^{n-1}r_i}{|z_i-z_j|(|z_i-z_j|+r_i)\varrho}\\
&\leqslant \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{4r^3(1+(r/\varrho))^{2n-2}}{\varrho^2(|z_i-z_j|+r_i)}\\
&< \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{4r^3}{\varrho^3}\left(1+\frac{r}{\varrho}\right)^{2n-2}\\
&= \frac{4(n-1)}{\varrho^3}\left(1+\frac{r}{\varrho}\right)^{2n-2} r^3. \qquad \square
\end{aligned}
$$

Lemma 4. *Suppose that the angle of intersection between* $\exp(\mathrm{i}\theta/2)$ *and* $1+s_i$ *is an acute angle, then,*

$$|\hat{c}_i| \geqslant 1-\frac{n-1}{\varrho}\left(1+\frac{r}{\varrho}\right)^{n-1} r. \tag{26}$$

Proof. By the definition of s_i, and Lemma 1, we have

$$|s_i| = \left|\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{z_i-z_j}\right| \leqslant \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{r(1+(r/\varrho))^{n-1}}{\varrho} = \frac{n-1}{\varrho}\left(1+\frac{r}{\varrho}\right)^{n-1} r.$$

Then we have

$$|\hat{c}_i| = \left|1 + s_i + |c_i|^{1/2}\exp\left(\frac{\mathrm{i}\theta}{2}\right)\right| \geqslant |1 + s_i| \geqslant 1 - |s_i|$$
$$\geqslant 1 - \frac{n-1}{\varrho}\left(1 + \frac{r}{\varrho}\right)^{n-1} r.$$

From the definition of c_i (19), and Lemma 1, we have

$$|c_i| = \left|(1+s_i)^2 + 4u_i \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{z_i - z_j}\frac{\bar{z}_i - \bar{z}_j}{|z_i - z_j|^2 - r_i^2}\right| \geqslant (1 - |s_i|)^2 - 4|u_i|$$
$$\times \sum_{\substack{j=1\\ j\neq i}}^{n}\left|\frac{u_j(\bar{z}_i - \bar{z}_j)}{(z_i - z_j)(|z_i - z_j|^2 - r_i^2)}\right|$$
$$\geqslant \left(1 - \frac{n-1}{\varrho}\left(1+\frac{r}{\varrho}\right)^{n-1} r\right)^2 - 4r\left(1+\frac{r}{\varrho}\right)^{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{r(1+(r/\varrho))^{n-1}}{\varrho^2}$$
$$= \left(1 - \frac{n-1}{\varrho}\left(1+\frac{r}{\varrho}\right)^{n-1} r\right)^2 - \frac{4(n-1)}{\varrho^2}\left(1+\frac{r}{\varrho}\right)^{2n-2} r^2, \tag{27}$$

where

$$|s_i| = \left|\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{u_j}{z_i - z_j}\right| \leqslant \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{r(1+(r/\varrho))^{n-1}}{\varrho} = \frac{n-1}{\varrho}\left(1+\frac{r}{\varrho}\right)^{n-1} r.$$

Lemma 5. *Suppose that* $(r/\varrho)(1+(r/\varrho))^n < \frac{1}{3}$. *Then*

$$\hat{\eta}_i^2 < \frac{4(n-1)^2(1+(r/\varrho))^{4n-4}}{\varrho^6(1-(3(n-1)/\varrho)(1+(r/\varrho))^{n-1}r)}\, r^6. \tag{28}$$

Proof. From (24), and Lemma 3 and (27), we have

$$\hat{\eta}_i = \frac{\eta_i}{|c_i|^{1/2} + (|c_i| - \eta_i)^{1/2}} \leqslant \frac{\eta_i}{2(|c_i| - \eta_i)^{1/2}} \leqslant \frac{4(n-1)}{\varrho^3}\left(1+\frac{r}{\varrho}\right)^{2n-2} r^3$$
$$\times \frac{1}{2[(1 - \frac{n-1}{\varrho}(1+\frac{r}{\varrho})^{n-1}r)^2 - \frac{4(n-1)}{\varrho^3}(1+\frac{r}{\varrho})^{2n-2}r^3]}$$
$$= \frac{2(n-1)(1+\frac{r}{\varrho})^{2n-2}r^3}{\varrho^3[(1-\frac{n-1}{\varrho}(1+\frac{r}{\varrho})^{n-1}r)^2 - \frac{4(n-1)}{\varrho^2}(1+\frac{r}{\varrho})^{2n-2}r^2 - \frac{4(n-1)}{\varrho^3}(1+\frac{r}{\varrho})^{2n-2}r^3]^{1/2}}$$
$$= \frac{2(n-1)(1+\frac{r}{\varrho})^{2n-2}r^3}{\varrho^3[(1-\frac{n-1}{\varrho}(1+\frac{r}{\varrho})^{n-1}r)^2 - \frac{4(n-1)}{\varrho^2}(1+\frac{r}{\varrho})^{2n-1}r^2]^{1/2}}. \tag{29}$$

Because $(r/\varrho)(1+(r/\varrho))^n < \frac{1}{3}$,
so $(r/\varrho) < \frac{1}{3}$, $1+(r/\varrho) < \frac{4}{3}$,
When $n \geqslant 3$, $n-1 \geqslant \frac{4}{3} > 1+(r/\varrho)$,
thus

$$\frac{(n-1)^2}{\varrho^2}\left(1+\frac{r}{\varrho}\right)^{2n-2} r^2 \geqslant \frac{n-1}{\varrho^2}\left(1+\frac{r}{\varrho}\right)^{2n-1} r^2$$

and

$$\frac{n-1}{\varrho^2}\left(1+\frac{r}{\varrho}\right)^{2n-1} r^2 < \frac{n-1}{3\varrho}\left(1+\frac{r}{\varrho}\right)^{n-1} r.$$

Therefore, by (29)

$$\begin{aligned}\hat{\eta}_i^2 &\leqslant \frac{4(n-1)^2(1+\frac{r}{\varrho})^{4n-4}r^6}{\varrho^6(1-\frac{2(n-1)}{\varrho}(1+\frac{r}{\varrho})^{n-1}r+\frac{(n-1)^2}{\varrho^2}(1+\frac{r}{\varrho})^{2n-2}r^2-\frac{4(n-1)}{\varrho^2}(1+\frac{r}{\varrho})^{2n-1}r^2)}\\ &< \frac{4(n-1)^2(1+\frac{r}{\varrho})^{4n-4}r^6}{\varrho^6(1-\frac{2(n-1)}{\varrho}(1+\frac{r}{\varrho})^{n-1}r-\frac{3(n-1)}{\varrho^2}(1+\frac{r}{\varrho})^{2n-1}r^2)}\\ &< \frac{4(n-1)^2(1+\frac{r}{\varrho})^{4n-4}}{\varrho^6(1-\frac{3(n-1)}{\varrho}(1+\frac{r}{\varrho})^{n-1}r)}. \qquad \square\end{aligned}$$

From (26) and (28), we obtain

$$\begin{aligned}|\hat{c}_i|^2-\hat{\eta}_i^2 &> \left(1-\frac{(n-1)}{\varrho}\left(1+\frac{r}{\varrho}\right)^{n-1} r\right)^2 - \frac{4(n-1)^2(1+\frac{r}{\varrho})^{4n-4}}{\varrho^6(1-\frac{3(n-1)}{\varrho}(1+\frac{r}{\varrho})^{n-1}r)} r^6\\ &> 1-\frac{2(n-1)}{\varrho}\left(1+\frac{r}{\varrho}\right)^{n-1} r - \frac{4(n-1)^2(1+\frac{r}{\varrho})^{4n-4}}{\varrho^6(1-\frac{3(n-1)}{\varrho}(1+\frac{r}{\varrho})^{n-1}r)} r^6. \qquad (30)\end{aligned}$$

Therefore we have the following results:

Theorem. *Let $\mathbb{Z}_1^{(0)},\ldots,\mathbb{Z}_n^{(0)}$ be the initial disks containing the distinct roots $\zeta_1,\ldots,\zeta_n$; let the interval sequences $\{\mathbb{Z}_i^{(m)}\}_m$, with $i=1,\ldots,n$ and $m=0,1,2,\ldots$, be defined by (15), and $r^{(m)},\varrho^{(m)}$ be defined as above. Note $q_n^{(m)}=(n-1)(1+(r^{(m)}/\varrho^{(m)}))^{n-1}$. Then, under the assumption*

$$\frac{r^{(0)}}{\varrho^{(0)}}\left(1+\frac{r^{(0)}}{\varrho^{(0)}}\right)^n < \frac{1}{3}, \quad n \geqslant 3,$$

we get

$$\zeta_i \in \mathbb{Z}_i^{(m)}, \quad i=1,\ldots,n \quad \text{and} \quad m=0,1,\ldots, \qquad (31)$$

the sequence $\{r^{(m)}\}_m$ monotonically goes to zero and

$$r^{(m+1)} < \frac{4q_n^{(m)3}(1-\frac{3r^{(m)}}{\varrho^{(m)}}q_n^{(m)})^{1/2}}{(n-1)^2[\varrho^{(m)3}(1-\frac{3r^{(m)}}{\varrho^{(m)}}q_n^{(m)})^2-\frac{4(n-1)}{27}q_n^{(m)}r^{(m)3}]} r^{(m)4}. \qquad (32)$$

Proof. The proof of (31) follows from the assumption $\zeta_i \in \mathbb{Z}_i^{(0)}$, $i = 1,\dots,n$ and the inclusion isotonicification (14).

Now, we prove assertion (32). From (15), (16), (22), (28), (30), we have

$$\begin{aligned}
r_i^{(m+1)} &= \mathrm{rad}(\mathbb{Z}_i^{(m+1)}) = 2|u_i^{(m)}|\mathrm{rad}\left(\frac{1}{1+s_i^{(m)}+[(1+s_i^{(m)})^2+4u_i^{(m)}T_i^{(m)}]_*^{1/2}}\right)\\
&= 2|u_i^{(m)}|\mathrm{rad}\left(\frac{1}{\{\hat{c}_i^{(m)};\hat{\eta}_i^{(m)}\}}\right)\\
&= 2|u_i^{(m)}|\frac{\hat{\eta}_i^{(m)}}{|\hat{c}_i^{(m)}|^2-\hat{\eta}_i^{(m)2}}\\
&< 2r^{(m)}\left(1+\frac{r^{(m)}}{\varrho^{(m)}}\right)^{n-1}\\
&\quad\times\frac{\dfrac{2(n-1)\left(1+\frac{r^{(m)}}{\varrho^{(m)}}\right)^{2n-2}r^{(m)3}}{\varrho^{(m)3}\left(1-\frac{3(n-1)}{\varrho^{(m)}}\left(1+\frac{r^{(m)}}{\varrho^{(m)}}\right)^{n-1}r^{(m)}\right)^{1/2}}}{\left(1-\frac{2(n-1)}{\varrho^{(m)}}(1+\frac{r^{(m)}}{\varrho^{(m)}})^{n-1}r^{(m)}-\dfrac{4(n-1)^2\left(1+\frac{r^{(m)}}{\varrho^{(m)}}\right)^{4n-4}r^{(m)6}}{\varrho^{(m)6}\left(1-\frac{3(n-1)}{\varrho^{(m)}}\left(1+\frac{r^{(m)}}{\varrho^{(m)}}\right)^{n-1}r^{(m)}\right)}\right)}\\
&< 2r^{(m)}\left(1+\frac{r^{(m)}}{\varrho^{(m)}}\right)^{n-1}\\
&\quad\times\frac{2(n-1)(1+\frac{r^{(m)}}{\varrho^{(m)}})^{2n-2}(1-\frac{3(n-1)}{\varrho^{(m)}}(1+\frac{r^{(m)}}{\varrho^{(m)}})^{n-1}r^{(m)})^{1/2}r^{(m)3}}{\varrho^{(m)3}(1-\frac{3(n-1)}{\varrho^{(m)}}(1+\frac{r^{(m)}}{\varrho^{(m)}})^{n-1}r^{(m)})(1-\frac{2(n-1)}{\varrho^{(m)}}(1+\frac{r^{(m)}}{\varrho^{(m)}})^{n-1}r^{(m)})-\frac{4}{27}(n-1)^2(1+\frac{r^{(m)}}{\varrho^{(m)}})^{n-4}r^{(m)3}}\\
&< 2r^{(m)}\left(1+\frac{r^{(m)}}{\varrho^{(m)}}\right)^{n-1}\\
&\quad\times\frac{2q_n^{(m)}(1+\frac{r^{(m)}}{\varrho^{(m)}})^{n-1}(1-\frac{3r^{(m)}}{\varrho^{(m)}}q_n^{(m)})^{1/2}r^{(m)3}}{\varrho^{(m)3}(1-\frac{3r^{(m)}}{\varrho^{(m)}}q_n^{(m)})(1-\frac{2r^{(m)}}{\varrho^{(m)}}q_n^{(m)})-\dfrac{4(n-1)q_n^{(m)}}{27\left(1+\frac{r^{(m)}}{\varrho^{(m)}}\right)^3}r^{(m)3}}\\
&= \frac{4q_n^{(m)3}(1-\frac{3r^{(m)}}{\varrho^{(m)}}q_n^{(m)})^{1/2}}{(n-1)^2[\varrho^{(m)3}(1-\frac{3r^{(m)}}{\varrho^{(m)}}q_n^{(m)})^2-\frac{4(n-1)}{27}q_n^{(m)}r^{(m)3}]}r^{(m)4}. \qquad \square
\end{aligned}$$

Remark. In practice, if we can assure or assume that

$$u_it_i^* - s_iu_it_i^* + \mathrm{O}(r^4) \subset u_iT_i, \tag{33}$$

where $t_i^* = \sum_{j=1, j\neq i}^n u_i/(z_i - z_j)(\zeta_i - z_j)$,

then we can introduce the following interval method by the interval method (15):

$$\hat{\mathbb{Z}}_i = z_i - \frac{u_i}{1 + s_i + u_iT_i}. \tag{34}$$

In fact, by (13), that is

$$\zeta_i = z_i - \frac{2u_i}{1+s_i+((1+s_i)^2+4u_1t_i^*)^{1/2}}. \tag{35}$$

Form the definition of u_i, s_i and t_i^*, we have $u_i = O(r)$, $s_i = O(r)$, $t_i^* = O(r)$.

Thus by the Taylor expansion formula, we can obtain

$$\begin{aligned}((1+s_i)^2+4u_it_i^*)^{1/2} &= (1+2s_i+s_i^2+4u_it_i^*)^{1/2}\\ &= 1+\tfrac{1}{2}(2s_i+s_i^2+4u_it_i^*)-\tfrac{1}{8}(2s_i+s_i^2+4u_it_i^*)^2\\ &\quad+\tfrac{1}{16}(2s_i+s_i^2+4u_it_i^*)^3+\cdots\\ &= 1+s_i+2u_it_i^*-2s_iu_it_i^*+O(r^4).\end{aligned}$$

So, we have

$$\zeta_i = z_i - \frac{u_i}{1+s_i+u_it_i^*-s_iu_it_i^*+O(r^4)}. \tag{36}$$

By the assumption of (33), we have, if

$$\zeta_i \in \mathbb{Z}_i, \quad \text{then } \zeta_i \in \hat{\mathbb{Z}}_i = z_i - \frac{u_i}{1+s_i+u_iT_i}$$

and the convergence of interval method (34) is shown to be fourth order, we have the following estimates:

Let $\mathbb{Z}_1,\ldots,\mathbb{Z}_n$ be disjoint disks such that $\zeta_j \in \mathbb{Z}_j$, $j=1,\ldots,n$. Suppose that $u_it^*-s_iu_it_i^*+O(r^4)\subset u_iT_i$, and $(r/\varrho)(1+(r/\varrho))^n < \frac{1}{3}$. Then

$$\zeta_i \in \hat{\mathbb{Z}}_i, \quad i=1,\ldots,n \tag{37}$$

and

$$\hat{r} \leqslant \frac{54(n-1)(1+(r/\varrho))^{3n-3}}{\varrho^2[9\varrho-(n-1)(24(1+(r/\varrho))^{n-1}+r)r]}r^4. \tag{38}$$

Proof. The proof of (37) follows from the assumption $\zeta_j \in \mathbb{Z}_j$, $j=1,\ldots,n$ and assumption of (33), and relation (36).

Similar to the proof of (32) in the theorem, we have

$$\begin{aligned}\hat{r}_i = \operatorname{rad}(\hat{\mathbb{Z}}_i) &= |u_i|\operatorname{rad}\left(\frac{1}{1+s_i+u_iT_i}\right)\\ &= |u_i|\operatorname{rad}\left(\frac{1}{\{1+s_i+u_i\sum_{\substack{j=1\\ j\neq i}}^n \frac{u_j}{z_i-z_j}\frac{\bar{z}_i-\bar{z}_j}{|z_i-z_j|^2-r_i^2};\sum_{\substack{j=1\\ j\neq i}}^n\left|\frac{u_iu_j}{(z_i-z_j)(|z_i-z_j|^2-r_i^2)}\right|r_i\}}\right)\\ &= |u_i|\operatorname{rad}\left(\frac{1}{\{c_i^*,\eta_i^*\}}\right)\\ &= |u_i|\operatorname{rad}\left\{\frac{|\bar{c}_i^*|}{|c_i^*|^2-\eta_i^{*2}};\frac{\eta_i^*}{|c_i^*|^2-\eta_i^{*2}}\right\}\\ &= |u_i|\frac{\eta_i^*}{|c_i^*|^2-\eta_i^{*2}},\end{aligned}$$

where

$$\eta_i^* = \sum_{\substack{j=1\\ j\neq i}}^{n} \left| \frac{u_i u_j}{(z_i - z_j)(|z_i - z_j|^2 - r_i^2)} \right| r_i \leqslant \frac{(n-1)}{\varrho^3}\left(1+\frac{r}{\varrho}\right)^{2n-2} r^3,$$

$$|c_i^*| = \left| 1 + s_i + u_i \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_i}{z_i - z_j}\frac{\bar{z}_i - \bar{z}_j}{|z_i - z_j|^2 - r_i^2} \right| \geqslant 1 - |s_i| - |u_i| \sum_{\substack{j=1\\ j\neq i}}^{n} \left| \frac{u_j(\bar{z}_i - \bar{z}_j)}{(z_i - z_j)(|z_i - z_j|^2 - r_i^2)} \right|$$

$$\geqslant 1 - \frac{n-1}{\varrho}\left(1+\frac{r}{\varrho}\right)^{n-1} r - \frac{n-1}{\varrho^2}\left(1+\frac{r}{\varrho}\right)^{2n-2} r^2.$$

Thus by the assumption $(r/\varrho)(1+(r/\varrho))^2 < \frac{1}{3}$,

$$\hat{r}_i \leqslant r\left(1+\frac{r}{\varrho}\right)^{n-1} \frac{(n-1)}{\varrho^3}\left(1+\frac{r}{\varrho}\right)^{2n-2} r^3$$

$$\times \frac{1}{(1 - \frac{n-1}{\varrho}(1+\frac{r}{\varrho})^{n-1} r - \frac{n-1}{\varrho^2}(1+\frac{r}{\varrho})^{2n-2} r^2)^2 - (\frac{(n-1)}{\varrho^3}(1+\frac{r}{\varrho})^{2n-2} r^3)^2}$$

$$\leqslant \frac{54(n-1)(1+\frac{r}{\varrho})^{3n-3}}{\varrho^2[9\varrho - (n-1)(24(1+\frac{r}{\varrho})^{n-1} + r)r]} r^4.$$

That is

$$\hat{r} = \max_{i\leqslant i\leqslant n} \hat{r}_i \leqslant \frac{54(n-1)(1+(r/\varrho))^{3n-3}}{\varrho^2[9\varrho - (n-1)(24(1+(r/\varrho))^{n-1} + r)r]} r^4. \qquad \square$$

5. Numerical results

In this section, numerical experience is carried out by the interval method (15) and (34). We consider three polynomials denoted by P_1, P_2 and P_3. For each polynomial, the initial disks $\mathbb{Z}_1^{(0)}, \ldots, \mathbb{Z}_n^{(0)}$ are given.

The numerical results show that method (15) and (34) converge very fast, and have fourth-order convergence as shown in theorem. Unusually, condition (33) can be satisfied, and iteration (34) is simpler than (15).

Tables 1–3 give the radius $r_i^{(m)}$ of $\mathbb{Z}_i^{(m)}$ derived by method (15), where $\mathbb{Z}_i^{(m)} = \{z_i^{(m)}; r_i^{(m)}\}$. The numerical results of method (34) is the same as that of method (15), that is, has the same order of convergence. So, we omit the tables of method (34) here.

Example 1. Let $P_1(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300$ with zeros $\zeta_1 = -3$, $\zeta_2 = -2 + \mathrm{i}$, $\zeta_3 = -2 - \mathrm{i}$, $\zeta_4 = -1$, $\zeta_5 = 2\mathrm{i}$, $\zeta_6 = -2\mathrm{i}$, $\zeta_7 = 1$, $\zeta_8 = 2 + \mathrm{i}$, $\zeta_9 = 2 - \mathrm{i}$ (see Ref. [8, p. 68]).

Table 1
$r_i^{(m)}$ of $\mathbb{Z}_i^{(m)}$ by (15)

	$m=1$	$m=2$	$m=3$
$r_1^{(m)}$	$3.72\cdot10^{-4}$	$2.77\cdot10^{-14}$	$6.90\cdot10^{-55}$
$r_2^{(m)}$	$1.75\cdot10^{-3}$	$2.45\cdot10^{-13}$	$8.87\cdot10^{-54}$
$r_3^{(m)}$	$9.54\cdot10^{-4}$	$6.42\cdot10^{-14}$	$1.38\cdot10^{-55}$
$r_4^{(m)}$	$2.82\cdot10^{-3}$	$2.61\cdot10^{-13}$	$1.11\cdot10^{-53}$
$r_5^{(m)}$	$1.79\cdot10^{-4}$	$3.17\cdot10^{-16}$	$4.39\cdot10^{-60}$
$r_6^{(m)}$	$3.27\cdot10^{-4}$	$1.43\cdot10^{-15}$	$5.98\cdot10^{-59}$
$r_7^{(m)}$	$7.99\cdot10^{-4}$	$8.63\cdot10^{-15}$	$1.79\cdot10^{-56}$
$r_8^{(m)}$	$4.55\cdot10^{-4}$	$2.65\cdot10^{-15}$	$4.74\cdot10^{-59}$
$r_9^{(m)}$	$4.64\cdot10^{-4}$	$4.66\cdot10^{-15}$	$8.66\cdot10^{-58}$

Table 2
$r_i^{(m)}$ of $\mathbb{Z}_i^{(m)}$ by (15)

	$m=1$	$m=2$	$m=3$
$r_1^{(m)}$	$4.76\cdot10^{-3}$	$1.23\cdot10^{-11}$	$3.89\cdot10^{-45}$
$r_2^{(m)}$	$2.61\cdot10^{-3}$	$2.06\cdot10^{-12}$	$1.11\cdot10^{-47}$
$r_3^{(m)}$	$2.47\cdot10^{-3}$	$9.38\cdot10^{-13}$	$7.44\cdot10^{-49}$
$r_4^{(m)}$	$5.53\cdot10^{-3}$	$9.55\cdot10^{-12}$	$1.04\cdot10^{-45}$

Table 3
$r_i^{(m)}$ of $\mathbb{Z}_i^{(m)}$ by (15)

	$m=1$	$m=2$	$m=3$
$r_1^{(m)}$	$3.17\cdot10^{-3}$	$4.50\cdot10^{-11}$	$2.19\cdot10^{-42}$
$r_2^{(m)}$	$1.09\cdot10^{-2}$	$3.77\cdot10^{-10}$	$2.32\cdot10^{-41}$
$r_3^{(m)}$	$7.58\cdot10^{-3}$	$7.07\cdot10^{-11}$	$3.65\cdot10^{-42}$
$r_4^{(m)}$	$2.47\cdot10^{-3}$	$3.67\cdot10^{-11}$	$1.65\cdot10^{-42}$
$r_5^{(m)}$	$9.46\cdot10^{-3}$	$3.52\cdot10^{-10}$	$2.82\cdot10^{-41}$
$r_6^{(m)}$	$7.68\cdot10^{-4}$	$9.48\cdot10^{-13}$	$1.73\cdot10^{-47}$
$r_7^{(m)}$	$5.56\cdot10^{-4}$	$1.78\cdot10^{-12}$	$7.94\cdot10^{-46}$

The initial disks, containing these zeros, were chosen to be $\mathbb{Z}_i^{(0)}=\{z_i^{(0)};0.3\}$, where $z_1^{(0)}=-3.2$, $z_2^{(0)}=-2.1+0.9\mathrm{i}$, $z_3^{(0)}=-1.9-1.1\mathrm{i}$, $z_4^{(0)}=-0.9+0.2\mathrm{i}$, $z_5^{(0)}=0.1+2.1\mathrm{i}$, $z_6^{(0)}=-0.1-1.9\mathrm{i}$, $z_7^{(0)}=0.9-0.2\mathrm{i}$, $z_8^{(0)}=2.1+0.9\mathrm{i}$, $z_9^{(0)}=1.9-1.1\mathrm{i}$.

The numerical results are shown in Table 1.

Example 2. Let $P_2(z)=z^4-1$ with zeros $\zeta_1=-1$, $\zeta_2=1$, $\zeta_3=\mathrm{i}$, $\zeta_4=-\mathrm{i}$.

The initial disks, containing these zeros, were chosen to be $\mathbb{Z}_i^{(0)}=\{z_i^{(0)};0.3\}$, where $z_1^{(0)}=-0,8$, $z_2^{(0)}=0.8+0.1\mathrm{i}$, $z_3^{(0)}=0.1+1.1\mathrm{i}$, $z_4^{(0)}=-0.2-0.9\mathrm{i}$.

The numerical results are shown in Table 2.

Example 3. Let $P_3(z)=z^7+z^5-10z^4-z^3-z+10$ with zeros $\zeta_1=2$, $\zeta_2=1$, $\zeta_3=-1$, $\zeta_4=\mathrm{i}$, $\zeta_5=-\mathrm{i}$, $\zeta_6=-1+2\mathrm{i}$, $\zeta_7=-1-2\mathrm{i}$ (see Ref. [8, p. 66]).

The initial disks, containing these zeros, were chosen to be $\mathbb{Z}_i^{(0)}=\{z_i^{(0)};0.3\}$, where $z_1^{(0)}=2.2$, $z_2^{(0)}=1.2+0.1\mathrm{i}$, $z_3^{(0)}=-0.8-0.1\mathrm{i}$, $z_4^{(0)}=0.1+1.2\mathrm{i}$, $z_5^{(0)}=-0.1-0.8\mathrm{i}$, $z_6^{(0)}=-1.1+2.2\mathrm{i}$, $z_7^{(0)}=-1.1-1.8\mathrm{i}$. The numerical results are shown in Table 3.

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