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Physics Letters B 529 (2002) 241–255

www.elsevier.com/locate/npe

Two-loop superstrings. I. Main formulas[☆]

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Received 24 January 2002; accepted 28 January 2002

Editor: M. Cvetič

Abstract

An unambiguous and slice-independent formula for the two-loop superstring measure on moduli space for even spin structure is constructed from first principles. The construction uses the super-period matrix as moduli invariant under worldsheet supersymmetry. This produces new subtle contributions to the gauge-fixing process, which eliminate all the ambiguities plaguing earlier gauge-fixed formulas.

The superstring measure can be computed explicitly and a simple expression in terms of modular forms is obtained. For fixed spin structure, the measure exhibits the expected behavior under degenerations of the surface. The measure allows for a unique modular covariant GSO projection. Under this GSO projection, the cosmological constant, the 1-, 2- and 3-point functions of massless supergravitons all vanish pointwise on moduli space without the appearance of boundary terms. A certain disconnected part of the 4-point function is shown to be given by a convergent, finite integral on moduli space. A general slice-independent formula is given for the two-loop cosmological constant in compactifications with central charge $c = 15$ and $\mathcal{N} = 1$ worldsheet supersymmetry in terms of the data of the compactification conformal field theory.

In this Letter, a summary of the above results is presented with detailed constructions, derivations and proofs to be provided in a series of subsequent publications.

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1. Introduction

Despite great advances in superstring theory, multi-loop amplitudes are still unavailable, almost twenty years after the derivation of the one-loop amplitudes by Green and Schwarz for type II strings [1] and by Gross et al. for heterotic strings [2]. The main obstacle is the presence of supermoduli for worldsheets of non-trivial topology [3,4]. Considerable efforts had been made by many authors in order to overcome this obstacle, and a chaotic situation ensued, with many competing prescriptions proposed in the literature. These prescriptions drew from a variety of

[☆] Research supported in part by National Science Foundation grants PHY-98-19686 and DMS-98-00783, and by the Institute for Pure and Applied Mathematics under NSF grant DMS-98-10282.

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fundamental principles such as BRST invariance and the picture-changing formalism [3,5], descent equations and Cech cohomology [6], modular invariance [7], the light-cone gauge [8], the global geometry of the Teichmueller curve [9], the unitary gauge [10], the operator formalism [11], group theoretic methods [12], factorization [13], and algebraic supergeometry [14]. However, the basic problem was that gauge-fixing required a local gauge slice, and the prescriptions ended up depending on the choice of such slices, violating gauge invariance. At the most pessimistic end, this raised the undesirable possibility that superstring amplitudes could be ambiguous [15], and that it may be necessary to consider other options, such as the Fischler–Susskind mechanism [16].

In [17] and [18], we had suggested that the difficulties encountered in the earlier prescriptions could be the result of improper gauge-fixing procedures which did not respect worldsheet local supersymmetry. To address this difficulty, we had outlined a new gauge-fixing procedure based on projecting supergeometries to their super period matrices instead of their underlying bosonic geometries. Unlike the projection to the bosonic geometries, the projection to the super period matrix descends to a projection of superconformal structures, since the super period matrix is invariant under local worldsheet supersymmetry. It is well defined for any genus h .

In this Letter, we implement this new gauge-fixing procedure for genus $h = 2$. This is the lowest loop order where supermoduli must be confronted in all scattering amplitudes. We shall concentrate on the case of even spin structures since odd spin structure contributions are absent for the cosmological constant and scattering amplitudes with 4 or fewer states.

- We obtain a gauge-fixed formula $d\mu[\delta](\Omega)$ for the contribution to the superstring measure of each even spin structure δ , which is *independent of the choice of gauge slice*. In particular, the ambiguities plaguing the earlier prescriptions have now disappeared.
- For each δ , $d\mu[\delta](\Omega)$ transforms covariantly under modular transformations. There is a unique assignment of relative phases η_δ so that $\sum_\delta \eta_\delta d\mu[\delta](\Omega)$ is a modular form, and hence a unique way of implementing the Gliozzi–Scherk–Olive (GSO) projection.
- The superstring measure, when summed over all δ , vanishes *point by point* on moduli space, and not just up to a total derivative, as in earlier prescriptions. In particular, the cosmological constant vanishes. This is the 2-loop generalization of the Jacobi identity. Remarkably, it is not a consequence of genus 2 Riemann identities. Instead, it is equivalent to the identity, special to genus 2, that any modular form of weight 8 must be proportional to the square of the unique modular form of weight 4.
- Similarly, the 1-, 2-, 3-point functions for the scattering of the supergraviton multiplet vanish by a variety of novel identities.
- The 4-point function may be evaluated explicitly in terms of modular forms. For a certain disconnected part of the 4-point function, explicit formulas are presented here; they are manifestly finite, in the regime of purely imaginary Mandelstam variables. (As is well known [19], the other regimes are only accessible by analytic continuation.)
- Finally, we provide a simple slice independent formula for the even spin structure superstring measure and cosmological constant for general compactifications with matter central charge $c = 15$ and $\mathcal{N} = 1$ worldsheet supersymmetry.

Since the derivation of all of these results is quite lengthy, we provide here the main formulas, leaving the detailed treatment to a forthcoming series of papers.

2. The supermoduli space measure

In the Ramond–Neveu–Schwarz (RNS) formulation [20], the worldsheet for superstring propagation in 10-dimensional Minkowski space-time at loop order h is a compact surface Σ of genus h , equipped with a supergeometry E_M^A, Ω_M obeying the Wess–Zumino torsion constraints [17,21,22]. The non-chiral vacuum-to-

vacuum functional integral for fixed even spin structure δ (before the GSO projection) is

$$\mathbf{A}[\delta] = \int DE_M^A D\Omega_M \delta(T) \int DX^\mu e^{-I_m}, \tag{2.1}$$

where

$$I_m = \frac{1}{4\pi} \int d^{2|2}z E \mathcal{D}_+ X^\mu \mathcal{D}_- X^\mu, \quad E \equiv \text{sdet } E_M^A. \tag{2.2}$$

Here, $X^\mu = x^\mu + \theta\psi_+^\mu + \bar{\theta}\psi_-^\mu + i\theta\bar{\theta}F^\mu$, $\mu = 0, 1, \dots, 9$ are scalar superfields, and $\delta(T)$ indicates the torsion constraints.¹ The theory is invariant under $\text{sDiff}(M)$, super-Weyl, and $\text{sU}(1)$ local gauge transformations. The space of supergeometries E_M^A, Ω_M , modulo these symmetries is supermoduli space. In genus $h \geq 2$, it has dimension $(3h - 3|2h - 2)$. Let $m^A = (m^a|\zeta^\alpha)$, $a = 1, \dots, 3h - 3, \alpha = 1, \dots, 2h - 2$, be parameters for a local slice \mathcal{S} transversal to the orbits of the gauge group. The starting point for our considerations is the gauge-fixed expression for $\mathbf{A}[\delta]$, which is given by [17, p. 967],

$$\mathbf{A}[\delta] = \int \left| \prod_A dm^A \right|^2 \int D(B\bar{B}C\bar{C}X^\mu) \left| \prod_A \delta((H_A|B)) \right|^2 e^{-I_m - I_{gh}}. \tag{2.3}$$

The ghost superfields $B = \beta + \theta b, C = c + \theta\gamma$ have $\text{U}(1)$ weights $3/2$ and -1 , respectively, with action

$$I_{gh} = \frac{1}{2\pi} \int d^{2|2}z E (B\mathcal{D}_- C + \bar{B}\mathcal{D}_+ \bar{C}) \tag{2.4}$$

and $H_A \equiv (H_A)_-{}^z$ are the Beltrami superdifferentials tangent to the gauge slice

$$(H_A)_-{}^z \equiv (-)^{A(M+1)} E_-{}^M \frac{\partial E_M^z}{\partial m^A}. \tag{2.5}$$

The vertex operators for NSNS states do not involve superghost fields, and it suffices to consider the gauge-fixed measure in (2.3) for the scattering of these states. However, we would like to stress that (2.3) is only a preliminary first step in the construction of the desired superstring measure, since it is a *non-chiral* measure on *supermoduli* space.

3. Chiral splitting

A first step in the construction of the superstring measure is to extract from (2.3) a chiral contribution. In Wess–Zumino gauge, the supergeometry $E_m^a = e_m^a + \theta\gamma^a\chi_m - \frac{i}{2}\theta\bar{\theta}e_m^a A$ decomposes into a zweibein e_m^a , a gravitino field χ_m^a and an auxiliary field A . The expression (2.3) as well as vertex operators $V_i(k_i, \epsilon_i)$ mix fields of opposite chiralities such as $\chi_{\bar{z}}^+$ and χ_z^- , as can be seen in the components expression of I_m ,

$$I_m = \frac{1}{4\pi} \int d^2z \left(\partial_z x^\mu \partial_{\bar{z}} x^\mu - \psi_+^\mu \partial_{\bar{z}} \psi_+^\mu - \psi_-^\mu \partial_z \psi_-^\mu + \chi_{\bar{z}}^+ \psi_+^\mu \partial_z x^\mu + \chi_z^- \psi_-^\mu \partial_{\bar{z}} x^\mu - \frac{1}{2} \chi_{\bar{z}}^+ \chi_z^- \psi_+^\mu \psi_-^\mu \right). \tag{3.1}$$

They also involve the scalar fields $x^\mu(z)$ whose oscillator modes are split, but whose momentum zero modes are not split since left and right momenta coincide. As shown in [18], to obtain the chiral amplitudes, we have to introduce internal loop momenta $p_I^\mu, I = 1, \dots, h, \mu = 0, 1, \dots, 9$, and require the following effective prescription

¹ Here as well as in the superghost fields B, C below, we shall henceforth omit auxiliary fields such as F^μ , since they can be conveniently integrated out and do not play a significant role [17,18].

Table 1
Effective rules for chiral splitting

	Original	Effective chiral
Bosons	$x^\mu(z)$	$x_+^\mu(z)$
Fermions	$\psi_+^\mu(z)$	$\psi_+^\mu(z)$
Internal loop momenta	None	$\exp(p_I^\mu \oint_{B_I} dz \partial_z x_+^\mu)$
x -propagator	$\langle x^\mu(z)x^\nu(w) \rangle$	$-\delta^{\mu\nu} \ln E(z, w)$
ψ_+ -propagator	$\langle \psi_+^\mu(z)\psi_+^\nu(w) \rangle$	$-\delta^{\mu\nu} S_\delta(z, w)$
Covariant derivatives	\mathcal{D}_+	$\partial_\theta + \theta \partial_z$

for the scalar superfield correlation functions,

$$\left\langle \prod_{i=1}^N V_i(k_i, \epsilon_i) \right\rangle_{X^\mu} = \int dp_I^\mu \left| \left\langle \prod_{i=1}^N V_i^{\text{chi}}(k_i, \epsilon_i; p_I^\mu) \right\rangle_+ \right|^2. \tag{3.2}$$

Here, $\langle \dots \rangle_+$ denotes the fact that the effective rules for the contractions of the vertex operators $V_i^{\text{chi}}(k_i, \epsilon_i; p_I^\mu)$ are used, as given in Table 1. In this table, we have chosen a canonical homology $A_I, B_I, I = 1, \dots, h$ with canonical intersections $\#(A_I \cap B_J) = \delta_{IJ}$, $E(z, w)$ is the prime form, and $S_\delta(z, w)$ is the Szegő kernel. The point of the effective rules is that they only involve meromorphic notions, unlike the x -propagator $\langle x^\mu(z)x^\nu(w) \rangle$ which is given by the scalar Green's function $\delta^{\mu\nu} G(z, w)$. The superghost correlation functions are manifestly split. For the superstring measure alone, we obtain the following formula,

$$\mathbf{A}[\delta] = \int \left| \prod_A dm^A \right|^2 \int dp_I^\mu |e^{i\pi p_I^\mu \widehat{\Omega}_{IJ}} p_J^\mu \mathcal{A}[\delta]|^2 = \int \frac{|\prod_A dm^A \mathcal{A}[\delta]|^2}{(\det \text{Im } \widehat{\Omega})^5}, \tag{3.3}$$

where $\mathcal{A}[\delta]$ is the following effective chiral correlator

$$\mathcal{A}[\delta] = \left\langle \prod_A \delta(\langle H_A | B \rangle) \exp \left\{ \int \frac{d^2z}{2\pi} \chi_{\bar{z}}^+ S(z) \right\} \right\rangle_+. \tag{3.4}$$

$S(z)$ is the total supercurrent

$$S(z) = -\frac{1}{2} \psi_+^\mu \partial_z x_+^\mu + \frac{1}{2} b\gamma - \frac{3}{2} \beta \partial_z c - (\partial_z \beta) c, \tag{3.5}$$

and $\widehat{\Omega}_{IJ}$ is the super period matrix, defined by [17,18]

$$\widehat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int d^2z \int d^2w \omega_I(z) \chi_{\bar{z}}^+ \widehat{S}_\delta(z, w) \chi_{\bar{w}}^+ \omega_J(w). \tag{3.6}$$

Here Ω_{IJ} is the period matrix corresponding to the complex structure of the metric $g_{mn} = e_m^a e_n^b \delta_{ab}$ in the homology basis $\{A_I, B_I, I = 1, \dots, h\}$; $\{\omega_I(z), I = 1, \dots, h\}$, is a basis of holomorphic Abelian differentials dual to the A_I -cycles; and $\widehat{S}_\delta(z, w)$ is a modified Dirac propagator defined by

$$\partial_z \widehat{S}_\delta(z, w) + \frac{1}{8\pi} \chi_{\bar{z}}^+ \int d^2x \partial_x \ln E(z, x) \chi_{\bar{x}}^+ \widehat{S}_\delta(x, w) = 2\pi \delta(z, w). \tag{3.7}$$

The chirally split expression (3.3) is our first significant departure from the proposals of other authors in the late 1980's, in that it is the super period matrix $\widehat{\Omega}_{IJ}$ which appears as covariance of the internal loop momenta p_I^μ , and not the period matrix Ω_{IJ} . More important, we observe that a correct chiral splitting points then to the super period matrix $\widehat{\Omega}_{IJ}$ as the proper locally supersymmetric moduli for gauge-fixing.

4. Local supersymmetry and gauge-fixing

The expression for $\mathbf{A}[\delta]$ in (2.3) and (3.3) is an integral over supermoduli space. The main problem in superstring perturbation theory is how to integrate out the odd supermoduli ζ^α in (2.3) and (3.3) to reduce $\mathbf{A}[\delta]$ to a measure $d\mu[\delta]$ over moduli space

$$d\mu[\delta] = \prod_{a=1}^{3h-3} dm^a \int \prod_{\alpha=1}^{2h-2} d\zeta^\alpha \mathcal{A}[\delta]. \tag{4.1}$$

Let \mathcal{S} be a gauge slice for supermoduli, obtained by choosing a $(3h - 3)$ -dimensional gauge slice of zweibeins e_m^a , a $(2h - 2)$ -dimensional slice of gravitino sections χ_α , and setting $\chi = \sum_{\alpha=1}^{2h-2} \zeta^\alpha \chi_\alpha$. Naively, it may seem that the natural way of descending from supermoduli space to moduli space is to use the projection

$$E_M^A \rightarrow e_m^a. \tag{4.2}$$

This has been the method followed in the literature on superstring perturbation theory, but it has resulted in amplitudes which depend on the gauge slice \mathcal{S} chosen. The origin of this apparent ambiguity is the fact that the projection (4.2) is *not* invariant under local worldsheet supersymmetry. The remedy, originally proposed in [17] and carried out in the present Letter, is to use instead the projection

$$E_M^A \rightarrow \widehat{\Omega}_{IJ}, \tag{4.3}$$

where $\widehat{\Omega}_{IJ}$ is the super period matrix defined by (3.6). The correct moduli measure is obtained from the supermoduli measure by integrating along the fibers of (4.3). For genus 2, this is implemented by choosing $\{m^A\}_{A=1,2,3} = \{\widehat{\Omega}_{IJ}\}_{1 \leq I \leq J \leq 2}$ (instead of $\{\Omega_{IJ}\}_{1 \leq I \leq J \leq 2}$), and integrating in ζ^α , $\alpha = 1, 2$. We describe next some important steps in this process.

5. The moduli space measure

The change of projection from (4.2) to (4.3) leads to three modifications which require particular care. First, the Beltrami superdifferentials get modified to superdifferentials $H_A = \bar{\theta}(\mu_A - \theta\nu_A)$ (in Wess–Zumino gauge) with *both* components μ_A and ν_A usually non-zero. Second, the correlation functions in (2.3) and (3.3) were originally given in the metric g_{mn} corresponding to Ω_{IJ} . They have now to be re-expressed in a new metric \hat{g}_{mn} corresponding to $\widehat{\Omega}_{IJ}$. This deformation of metrics requires an insertion of the stress tensor integrated against a Beltrami differential $\hat{\mu}$ that represents this change of metrics. Third, we note that for given $\widehat{\Omega}_{IJ}$, the metric \hat{g}_{mn} is not unique. Thus the choice of \hat{g}_{mn} , or equivalently $\hat{\mu}$, should be viewed as an additional gauge choice, of which the final amplitude has also to be shown to be independent.

Due to the complicated nature of the Beltrami superdifferentials H_A , the superghost correlation functions in the presence of $\delta(\langle H_A|B \rangle)$ are not convenient meromorphic objects. To circumvent this problem, we change basis to super-Beltrami differentials $H_a^*(z, \theta) = \bar{\theta}\delta(z, p_a)$ and $H_\alpha^*(z, \theta) = \bar{\theta}\theta\delta(z, q_\alpha)$ which are δ -functions at points p_a and q_α , respectively. Assuming that correlation functions are considered with superghost field B -independent vertex operators only (as is always the case for NS states [23]), H_A is effectively integrated versus superholomorphic B 's. A change of basis for H_A is then carried out and produces an associated Jacobian,

$$\prod_A \delta(\langle H_A|B \rangle) = \frac{\text{sdet}\langle H_A|\Phi_C \rangle}{\text{sdet}\langle H_A^*|\Phi_C \rangle} \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \tag{5.1}$$

for an arbitrary basis of superholomorphic 3/2 forms Φ_C . By construction, all dependence on p_a and q_α cancels out in the full measure and amplitude. Upon choosing Φ_C to satisfy $\langle H_A|\Phi_C \rangle = \delta_{AC}$, we find the following expression

for the chiral measure,

$$A[\delta] = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \rangle}{\det \Phi_{IJ+}(p_a) \det \langle H_\alpha | \Phi_\beta^* \rangle} \left\{ 1 - \frac{1}{8\pi^2} \int d^2z \chi_{\bar{z}}^+ \int d^2w \chi_{\bar{w}}^+ \langle S(z)S(w) \rangle + \frac{1}{2\pi} \int d^2z \hat{\mu}_{\bar{z}}^z \langle T(z) \rangle \right\}. \tag{5.2}$$

Here, Φ_{IJ} are the odd superholomorphic 3/2 differentials corresponding to the covectors $d\widehat{\Omega}_{IJ}$ on supermoduli space, Φ_{IJ+} is the θ component of $\Phi_{IJ} = \theta\Phi_{IJ+} + \Phi_{IJ0}$, and $\Phi_\beta^* = \theta\Phi_{\beta+}^* + \Phi_{\beta0}^*$ is the basis for the even superholomorphic 3/2 differentials normalized by $\Phi_{\beta0}^*(q_\alpha) = \delta_{\alpha\beta}$ and $\Phi_{\beta+}^*(p_a) = 0$.

Taking all this into account, and expanding the finite-dimensional determinants in powers of χ , we arrive at the following formula

$$A[\delta] = i \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \rangle}{\det(\omega_I\omega_J(p_a)) \cdot \det \langle \chi_\alpha | \psi_\beta^* \rangle} \{ 1 + \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4 + \mathcal{X}_5 + \mathcal{X}_6 \}, \tag{5.3}$$

where all correlation functions are now written with respect to the $\widehat{\Omega}_{IJ}$ complex structure.² The various terms $\{\mathcal{X}_i\}_{i=1}^6$ in (5.3) have the following origins. The term \mathcal{X}_1 is the familiar contribution arising from two supercurrent insertions. All the other terms are more subtle and incorporate the effect of using $\widehat{\Omega}_{IJ}$ as supermoduli invariant. The term \mathcal{X}_2 arises from the stress tensor insertion; the term \mathcal{X}_3 arises when passing from the metric g_{mn} to the metric \hat{g}_{mn} in $\det \Phi_{IJ+}(p_a)$ and $\det \langle H_\alpha | \Phi_\beta^* \rangle$; the terms \mathcal{X}_4 and \mathcal{X}_5 arise from the remaining χ -dependence of $\det \Phi_{IJ+}(p_a)$; and the term \mathcal{X}_6 arises from the remaining χ -dependence of $\det \langle H_\alpha | \Phi_\beta^* \rangle$. The term $\mathcal{X}_2 + \mathcal{X}_3$ thus contains all the effects of passing from the metric g_{mn} to the metric \hat{g}_{mn} via the Beltrami differential $\hat{\mu}$. Using its expression below in terms of the holomorphic differential $T^{IJ}\omega_I(z)\omega_J(w)$, it will be manifest that the measure depends only on the moduli of \hat{g}_{mn} and not on the slice chosen.

More specifically, the quantities ψ_β^* are the holomorphic 3/2 differentials normalized at the points q_α by $\psi_\beta^*(q_\alpha) = \delta_{\beta\alpha}$, and the Green's functions $G_2(z, w)$ and $G_{3/2}(z, w)$ are of tensor type $(2, -1)$ and $(3/2, -1/2)$, respectively, in z and w , and normalized so that $G_2(p_a, w) = 0$ and $G_{3/2}(q_\alpha, w) = 0$. The terms \mathcal{X}_i , $i = 1, \dots, 6$ are then defined as follows,

$$\begin{aligned} \mathcal{X}_1 &= -\frac{1}{8\pi^2} \int d^2z \chi_{\bar{z}}^+ \int d^2w \chi_{\bar{w}}^+ \langle S(z)S(w) \rangle, \\ \mathcal{X}_2 + \mathcal{X}_3 &= +\frac{1}{16\pi^2} \int d^2z \int d^2w \chi_{\bar{z}}^+ \chi_{\bar{w}}^+ T^{IJ} \omega_I(z) S_\delta(z, w) \omega_J(w), \\ \mathcal{X}_4 &= +\frac{1}{8\pi^2} \int d^2w \partial_{p_a} \partial_w \ln E(p_a, w) \chi_{\bar{w}}^+ \int d^2u S_\delta(w, u) \chi_{\bar{u}}^+ \varpi_a^*(u), \\ \mathcal{X}_5 &= +\frac{1}{16\pi^2} \int d^2u \int d^2v S_\delta(p_a, u) \chi_{\bar{u}}^+ \partial_{p_a} S_\delta(p_a, v) \chi_{\bar{v}}^+ \varpi_a(u, v), \\ \mathcal{X}_6 &= +\frac{1}{16\pi^2} \int d^2z \chi_\alpha^*(z) \int d^2w G_{3/2}(z, w) \chi_{\bar{w}}^+ \int d^2v \chi_{\bar{v}}^+ \Lambda_\alpha(w, v). \end{aligned} \tag{5.4}$$

The sections $\chi_\beta^*(z)$ are the linear combinations of the $\chi_\alpha(z)$ normalized by $\langle \chi_\beta^* | \psi_\alpha^* \rangle = \delta_{\beta\alpha}$ and $T^{IJ}\omega_I(w)\omega_J(w)$ is the holomorphic quadratic differential defined by

$$T^{IJ} \omega_I \omega_J(w) = \left\langle T(w) \prod_{a=1}^3 b(p_a) \prod_{\alpha=1}^2 \delta(\beta(q_\alpha)) \right\rangle \left\langle \prod_{a=1}^3 b(p_a) \prod_{\alpha=1}^2 \delta(\beta(q_\alpha)) \right\rangle^{-1}$$

² Henceforth, the original Ω_{IJ} will no longer enter, and to simplify notations we denote $\widehat{\Omega}_{IJ}$ by Ω_{IJ} .

$$\begin{aligned}
 & -2 \sum_{a=1}^3 \partial_{p_a} \partial_w \ln E(p_a, w) \varpi_a^*(w) \\
 & + \int d^2z \chi_\alpha^*(z) \left(-\frac{3}{2} \partial_w G_{3/2}(z, w) \psi_\alpha^*(w) - \frac{1}{2} G_{3/2}(z, w) (\partial \psi_\alpha^*)(w) \right. \\
 & \quad \left. + G_2(w, z) \partial_z \psi_\alpha^*(z) + \frac{3}{2} \partial_z G_2(w, z) \psi_\alpha^*(z) \right). \tag{5.5}
 \end{aligned}$$

Here, $T(z)$ is the total stress tensor

$$T(z) = -\frac{1}{2} \partial_z x^\mu \partial_z x^\mu + \frac{1}{2} \psi_+^\mu \partial_z \psi_+^\mu + c \partial_z b - (\partial_z c) b - 2\beta \partial_z \gamma - \frac{3}{2} \gamma \partial_z \beta + \frac{3}{2} (\partial_z c) b \tag{5.6}$$

and Λ_α is defined by

$$\Lambda_\alpha(w, v) = 2G_2(w, v) \partial_v \psi_\alpha^* + 3\partial_v G_2(w, v) \psi_\alpha^*(v). \tag{5.7}$$

Finally, ϖ_a^* and ϖ_a are holomorphic 1-forms in u and v defined by $\varpi_a^*(u) = \varpi_a(u, p_a)$ and the determinants of 3×3 matrices

$$\begin{aligned}
 \varpi_a(u, v) &= \frac{\det\{\omega_I \omega_J(p_b[u, v; a])\}}{\det\{\omega_I \omega_J(p_b)\}}, \\
 \omega_I \omega_J(p_b[u, v; a]) &= \begin{cases} \omega_I \omega_J(p_b), & b \neq a, \\ \frac{1}{2}(\omega_I(u)\omega_J(v) + \omega_I(v)\omega_J(u)), & b = a. \end{cases} \tag{5.8}
 \end{aligned}$$

The above gauge-fixed amplitude (5.3) is independent of the points p_a and q_α . Furthermore, it satisfies the crucial requirement of invariance under infinitesimal deformations of the gauge slice \mathcal{S} , produced by $\delta_\xi(\chi_\alpha)_{\bar{z}}^+ = -2\partial_{\bar{z}} \xi_\alpha^+$,

$$\delta_\xi \left(\int \prod_{\alpha=1}^2 d\zeta^\alpha \mathcal{A}[\delta] \right) = 0 \tag{5.9}$$

with ξ_α^+ two arbitrary spinor fields, generators of local supersymmetry transformations.

6. Slice independence and absence of ambiguities

We specialize now to gauge slices \mathcal{S} given by δ functions

$$\chi_{\bar{z}}^+ = \zeta^1 \delta(z, x_1) + \zeta^2 \delta(z, x_2), \tag{6.1}$$

where x_α , $\alpha = 1, 2$ are two arbitrary fixed points on Σ . It may be shown explicitly that for each Ω_{IJ} , (5.3) is a holomorphic scalar in all points p_a , q_α and x_α , and thus independent thereof. As a result, the gauge-fixed chiral measure (5.3) is not just invariant under infinitesimal deformations of gauge slices, but more globally, independent of the gauge slices \mathcal{S} themselves, at least when they are δ -functions.

It is natural to let q_α coincide with x_α . In this limit, the positions of the supercurrent insertions $S(x_\alpha)$ tend to those of the superghost insertion $\delta(\beta(q_\alpha))$. In the early literature on superstring perturbation theory [5], the picture changing operator $Y(z)$ had been naively identified with $Y(z) = \delta(\beta(z))S(z)$, but the difficulties inherent to taking this product at coincident points had been obscure. Thanks to the expression (5.3) and (5.4), we see now that these difficulties cannot be ignored. Indeed, the corresponding term \mathcal{X}_1 fails to admit a limit as $x_\alpha \rightarrow q_\alpha$, and the correct limit is more subtle: it requires the contribution of the finite-dimensional determinant $\det\langle H_\alpha | \Phi_\beta^* \rangle$ in \mathcal{X}_6 , and it is only the sum $\mathcal{X}_1 + \mathcal{X}_6$ which admits a finite limit.

With the subtleties taken into account, the gauge-fixed amplitude $\mathcal{A}[\delta]$ can be expressed in the form (5.3), with the following simpler expressions for the terms \mathcal{X}_i , $i = 1, \dots, 6$,

$$\begin{aligned}
 \mathcal{X}_1 + \mathcal{X}_6 &= \frac{\xi^1 \xi^2}{16\pi^2} \left[-10 S_\delta(q_1, q_2) \partial_{q_1} \partial_{q_2} \ln E(q_1, q_2) - \partial_{q_1} G_2(q_1, q_2) \partial \psi_1^*(q_2) + \partial_{q_2} G_2(q_2, q_1) \partial \psi_2^*(q_1) \right. \\
 &\quad \left. + 2G_2(q_1, q_2) \partial \psi_1^*(q_2) f_{3/2}^{(1)}(q_2) - 2G_2(q_2, q_1) \partial \psi_2^*(q_1) f_{3/2}^{(2)}(q_1) \right], \\
 \mathcal{X}_2 &= \frac{\xi^1 \xi^2}{16\pi^2} \omega_I(q_1) \omega_J(q_2) S_\delta(q_1, q_2) \left[\partial_I \partial_J \ln \frac{\vartheta[\delta](0)^5}{\vartheta[\delta](D_\beta)} + \partial_I \partial_J \ln \vartheta(D_b) \right], \\
 \mathcal{X}_3 &= \frac{\xi^1 \xi^2}{8\pi^2} S_\delta(q_1, q_2) \sum_a \varpi_a(q_1, q_2) [B_2(p_a) + B_{3/2}(p_a)], \\
 \mathcal{X}_4 &= \frac{\xi^1 \xi^2}{8\pi^2} S_\delta(q_1, q_2) \sum_a [\partial_{p_a} \partial_{q_1} \ln E(p_a, q_1) \varpi_a^*(q_2) + \partial_{p_a} \partial_{q_2} \ln E(p_a, q_2) \varpi_a^*(q_1)], \\
 \mathcal{X}_5 &= \frac{\xi^1 \xi^2}{16\pi^2} \sum_a [S_\delta(p_a, q_1) \partial_{p_a} S_\delta(p_a, q_2) - S_\delta(p_a, q_2) \partial_{p_a} S_\delta(p_a, q_1)] \varpi_a(q_1, q_2). \tag{6.2}
 \end{aligned}$$

Here $D_b = \sum_{a=1}^3 p_a - 3\Delta$, $D_\beta = \sum_{\alpha=1}^2 q_\alpha - 2\Delta$; the terms $B_{3/2}$ and B_2 are given by

$$\begin{aligned}
 B_2(w) &= -27T_1(w) + \frac{1}{2} f_2(w)^2 - \frac{3}{2} \partial_w f_2(w) - 2 \sum_a \partial_{p_a} \partial_w \ln E(p_a, w) \varpi_a^*(w), \\
 B_{3/2}(w) &= 12T_1(w) - \frac{1}{2} f_{3/2}(w)^2 + \partial_w f_{3/2}(w) \\
 &\quad + \int d^2z \chi_\alpha^*(z) \left(-\frac{3}{2} \partial_w G_{3/2}(z, w) \psi_\alpha^*(w) - \frac{1}{2} G_{3/2}(z, w) \partial_w \psi_\alpha^*(w) \right. \\
 &\quad \left. + G_2(w, z) \partial_z \psi_\alpha^*(z) + \frac{3}{2} \partial_z G_2(w, z) \psi_\alpha^*(z) \right), \tag{6.3}
 \end{aligned}$$

the expression $-T_1(z)$ is the vev of the chiral scalar boson stress tensor defined by $E(z, w) = (z - w) + (z - w)^3 T_1(w) + \mathcal{O}(z - w)^4$. The expressions $f_{3/2}(w)$, $f_2(w)$ are given by

$$f_n(w) = \omega_I(w) \partial_I \ln \vartheta[\delta](D_n) + \partial_w \ln \left(\sigma(w)^{2n-1} \prod_{i=1}^{2n-1} E(w, z_i) \right) \tag{6.4}$$

with $\{z_i\} = \{p_a\}$ for $n = 2$, and $\{z_i\} = \{q_\alpha\}$ for $n = 3/2$, and

$$\begin{aligned}
 f_{3/2}^{(1)}(q_2) &= \omega_I(q_1) \partial_I \ln \vartheta[\delta](q_2 - q_1 + D_\beta) + \partial_{q_1} \ln (E(q_1, q_2)^2 \sigma(q_1)^2), \\
 f_{3/2}^{(2)}(q_1) &= \omega_I(q_2) \partial_I \ln \vartheta[\delta](q_1 - q_2 + D_\beta) + \partial_{q_2} \ln (E(q_2, q_1)^2 \sigma(q_2)^2). \tag{6.5}
 \end{aligned}$$

The corresponding chiral measure $d\mu[\delta](\Omega)$ is independent of p_a and q_α . This is of course a consequence of the independence of the earlier formulas from p_a , q_α and x_α . But it can also be verified directly from (6.2). This provides a direct proof of the absence of ambiguities in the gauge-fixed expression measure $d\mu[\delta](\Omega)$.

7. The measure as a modular form

It remains to express the moduli measure in terms of modular forms. In genus $h = 2$, there are 10 even spin structures, denoted δ , and 6 odd spin structures, denoted ν . Each even spin structure δ can be written as

$\delta = \nu_1 + \nu_2 + \nu_3$, where the ν_i 's are odd and pairwise distinct. The mapping $\{\nu_1, \nu_2, \nu_3\} \rightarrow \delta$ is 2 to 1, with $\{\nu_1, \nu_2, \nu_3\}$ and $\{\nu_1, \dots, \nu_6\} \setminus \{\nu_1, \nu_2, \nu_3\}$ corresponding to the same even spin structure. In the hyperelliptic representation, the surface Σ is given by $s^2 = \prod_{k=1}^6 (x - u_k)$ with 6 branch points u_k . Each odd spin structure ν corresponds to a unique branch point u_ν . Each even spin structure $\delta = \nu_1 + \nu_2 + \nu_3$ corresponds then to a partition of the 6 branch points into two sets of 3 branch points each, namely $\{u_{\nu_1}, u_{\nu_2}, u_{\nu_3}\}$ and $\{u_k\}_{k=1}^6 \setminus \{u_{\nu_1}, u_{\nu_2}, u_{\nu_3}\}$.

Define $\mathcal{E}_6[\delta](\Omega)$ by the following combination of ϑ -constants, $\vartheta[\delta](\Omega) \equiv \vartheta[\delta](0, \Omega)$,

$$\mathcal{E}_6[\delta](\Omega) \equiv \sum_{1 \leq i < j \leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \vartheta[\nu_i + \nu_j + \nu_k]^4(\Omega) \tag{7.1}$$

which depends only on $\delta = \nu_1 + \nu_2 + \nu_3$. Here $\langle \nu_i | \nu_j \rangle = \exp 4\pi i (\nu'_i \nu''_j - \nu'_j \nu''_i)$ is the relative signature of the spin structures $\nu_i = (\nu'_i | \nu''_i)$, $\nu_j = (\nu'_j | \nu''_j)$. We shall also need the well-known modular form of weight 10, defined by $\Psi_{10}(\Omega) \equiv \prod_\delta \vartheta[\delta](\Omega)^2$. The modular transformation properties of the spin structures are given by [24],

$$\begin{pmatrix} \tilde{\delta}' \\ \tilde{\delta}'' \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix} + \frac{1}{2} \text{diag} \begin{pmatrix} CD^T \\ AB^T \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbf{Z}) \tag{7.2}$$

while the period matrix transforms as $\tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}$, $\det \text{Im } \tilde{\Omega} = |\det(C\Omega + D)|^{-2} (\det \text{Im } \Omega)$, and the ϑ constants obey

$$\begin{aligned} \vartheta[\tilde{\delta}]^4(\tilde{\Omega}) &= \epsilon^4 \det(C\Omega + D)^2 \vartheta[\delta]^4(\Omega), \\ \mathcal{E}_6[\tilde{\delta}](\tilde{\Omega}) &= \epsilon^4 \det(C\Omega + D)^6 \mathcal{E}_6[\delta](\Omega), \\ \Psi_{10}(\tilde{\Omega}) &= \det(C\Omega + D)^{10} \Psi_{10}(\Omega). \end{aligned} \tag{7.3}$$

Here, ϵ depends on both δ and the modular transformation, and satisfies $\epsilon^8 = 1$.³

Recall that the gauge-fixed amplitude (5.3), (6.2) has been shown to be independent of the points p_a and q_a . We may then choose the points p_a to make up either one of the two sets of 3 branch points defining δ . It turns out that the most convenient choice for q_a is to constrain them by the following *split gauge condition*

$$S_\delta(q_1, q_2) = 0. \tag{7.4}$$

The latter choice implies $\widehat{\Omega}_{IJ} = \Omega_{IJ}$, although no direct use is made of this fact.

With (6.2), the above choice of p_a and (7.4), the chiral amplitude (5.3) for the spin structure δ can be evaluated explicitly. We obtain in this way one of the main results of this Letter, which gives the contribution of each even spin structure δ to the superstring measure

$$d\mu[\delta](\Omega) = \prod_{I \leq J} d\Omega_{IJ} \int \prod_\alpha d\zeta^\alpha \mathcal{A}[\delta] = \frac{1}{16\pi^6} \prod_{I \leq J} d\Omega_{IJ} \frac{\mathcal{E}_6[\delta](\Omega) \vartheta[\delta]^4(\Omega)}{\Psi_{10}(\Omega)}. \tag{7.5}$$

Using the modular transformation properties of the measure $\prod_{I \leq J} d\tilde{\Omega}_{IJ} = \det(C\Omega + D)^{-3} \prod_{I \leq J} d\Omega_{IJ}$, and the above modular transformation rules, we obtain as an immediate consequence that the measure $d\mu[\delta]$ is modular covariant, i.e.,

$$d\mu[\tilde{\delta}](\tilde{\Omega}) = \det(C\Omega + D)^{-5} d\mu[\delta](\Omega), \tag{7.6}$$

without any multiplicative phase factors arising.

³ It is important to remark that $\mathcal{E}_6[\delta](\Omega)$ is not a modular form since it depends on δ and the sign factor ϵ^4 arises in its transformation laws. $\mathcal{E}_6[\delta](\Omega)$ is thus different from the modular form $\psi_6(\Omega)$ of weight 6, which is obtained by summing 60 (syzygous) products of three ϑ^4 each, see [24].

8. The GSO projection and cosmological constant

To implement the GSO projection [26], we have to sum over spin structures. Given the above modular transformation laws of the measure, there is a unique choice of relative phase factors (namely, all $\eta_\delta = 1$) leading to a modular form,

$$d\mu(\Omega) \equiv \sum_{\delta} d\mu[\delta](\Omega). \tag{8.1}$$

In genus 1, GSO phases were related to the sign factors arising in the unique genus 1 Riemann relation. In genus 2, however, there is a different Riemann relation for each of the 6 odd spin structures ν ,

$$\sum_{\delta} \langle \nu | \delta \rangle \vartheta^4[\delta](\Omega) = 0, \tag{8.2}$$

and there is neither a unique nor a natural choice that leads to modular invariance. Therefore, the uniqueness and naturality of the relative phases in (8.1) (and hence of the GSO projection in the even spin structure sector) should be viewed as a major advantage over any mechanism for modular invariance based on Riemann identities.

Since the right-hand side of (8.1) is now known to be a modular form, it can be shown to vanish identically in Ω ,

$$\sum_{\delta} \mathcal{E}_6[\delta](\Omega) \vartheta^4[\delta](\Omega) = 0 \tag{8.3}$$

by examining its behavior along the divisor of Riemann surfaces with nodes. This identity does not follow from the Riemann identities. Rather, it is equivalent to the genus 2 identity that a modular form of weight 8 must be proportional to the square of the unique modular form of weight 4. Altogether, we have obtained a proof from first principles that the two-loop cosmological constant Λ is given by an integral over moduli space which vanishes point by point, and hence Λ must also vanish. For the type II superstrings,

$$\Lambda = \frac{1}{2^8 \pi^{12}} \int (\det \text{Im } \Omega)^{-5} \left| \prod_{I \leq J} d\Omega_{IJ} \frac{\sum_{\delta} \mathcal{E}_6[\delta](\Omega) \vartheta^4[\delta](\Omega)}{\Psi_{10}(\Omega)} \right|^2 = 0. \tag{8.4}$$

We have an analogous expression for the heterotic strings which also vanishes using (8.3). Finally, the asymptotic behavior of the measure as Ω approaches the boundary of moduli space for a separating degeneration (the non-separating case is analogous) is obtained by decomposing the period matrix and the spin structure consistently with the separation,

$$\Omega = \begin{pmatrix} \tau_1 & \tau \\ \tau & \tau_2 \end{pmatrix}, \quad \delta = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{or} \quad \delta = \begin{pmatrix} \nu_0 \\ \nu_0 \end{pmatrix}. \tag{8.5}$$

Here, $\tau \rightarrow 0$ in the degeneration while the genus 1 moduli $\tau_{1,2}$ remain finite; μ_1 and μ_2 are one of the three even and ν_0 is the unique odd spin structures on each separated genus 1 surface. The separating degeneration limit of the measure is then given by

$$\begin{aligned} d\mu \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} (\Omega) &\rightarrow \frac{1}{2^{10} \pi^8 \tau^2} \langle \mu_1 | \nu_0 \rangle \langle \mu_2 | \nu_0 \rangle \frac{\vartheta_1[\mu_1](\tau_1)^4 \vartheta_1[\mu_2](\tau_2)^4}{\eta(\tau_1)^{12} \eta(\tau_2)^{12}} d\tau_1 d\tau_2 d\tau, \\ d\mu \begin{bmatrix} \nu_0 \\ \nu_0 \end{bmatrix} (\Omega) &\rightarrow \frac{3\tau^2}{2^6 \pi^4} d\tau_1 d\tau_2 d\tau. \end{aligned} \tag{8.6}$$

The first 9 spin structures (with even spin structures μ_1 and μ_2 on each genus 1 part) exhibit the tachyon pole on each genus 1 part, and the tachyon and massless intermediate state divergences, as expected.⁴ The last spin structure

⁴ This behavior for fixed δ coincides with the one of the bosonic string [17,25], as expected.

(with odd spin structure ν_0 on each genus 1 part) has no tachyon and no massless intermediate state divergences, as expected.

9. Scattering amplitudes

The vertex operators for the scattering of N massless bosons are given by

$$\prod_{i=1}^N V(k_i, \epsilon_i) = \prod_{i=1}^N \int d^2|z_i E(z_i) \epsilon_i^{\mu_i} \bar{\epsilon}_i^{\bar{\mu}_i} \mathcal{D}_+ X^{\mu_i} \mathcal{D}_- X^{\bar{\mu}_i} e^{ik_i \mu_i X^{\mu_i}}(z_i). \tag{9.1}$$

As in the case of the measure, the superstring scattering amplitudes require a GSO summation over spin structures of the conformal blocks of $\langle \prod_{i=1}^N V(k_i, \epsilon_i) \rangle_X$ in the X^μ superconformal field theory. The following formulas together with (8.3) are the proper analogues in genus 2 of the Riemann identities in genus 1, and may be used to carry out the required summations,

$$\begin{aligned} \sum_{\delta} \mathcal{E}_6[\delta](\Omega) \vartheta[\delta](\Omega)^4 S_{\delta}(z_1, z_2)^2 &= 0, \\ \sum_{\delta} \mathcal{E}_6[\delta](\Omega) \vartheta[\delta](\Omega)^4 S_{\delta}(z_1, z_2) S_{\delta}(z_2, z_3) S_{\delta}(z_3, z_1) &= 0. \end{aligned} \tag{9.2}$$

The 0-, 1-, 2- and 3-point functions in both the type II and the heterotic strings are then found to vanish pointwise on moduli space and without the appearance of boundary terms.

The 4-point function receives contributions from two distinct parts. The first arises from the connected part of the correlators

$$\left\langle S(z) S(w) \prod_{i=1}^4 V(k_i, \epsilon_i)^{\text{chi}} \right\rangle \quad \text{and} \quad \left\langle T(z) \prod_{i=1}^4 V(k_i, \epsilon_i)^{\text{chi}} \right\rangle. \tag{9.3}$$

The second arises from the disconnected part

$$\langle S(z) S(w) \rangle \left\langle \prod_{i=1}^4 V(k_i, \epsilon_i)^{\text{chi}} \right\rangle \quad \text{and} \quad \langle T(z) \rangle \left\langle \prod_{i=1}^4 V(k_i, \epsilon_i)^{\text{chi}} \right\rangle \tag{9.4}$$

of these correlators and combines with the gauge fixing determinants into a contribution proportional to the measure $d\mu[\delta](\Omega)$. The connected part is more complicated and requires an independent treatment to appear in a later publication.

The disconnected part (for example for the type II superstrings) is given by

$$\left\langle \prod_{i=1}^4 V(\epsilon_i, k_i) \right\rangle = g_s^2 \delta(k) \int \frac{|\prod_{I \leq J} d\Omega_{IJ}|^2}{(\det \text{Im } \Omega)^5} \prod_{i=1}^4 \int_{\Sigma} d^2 z_i |\mathcal{F}|^2 \exp\left(-\sum_{i < j} k_i \cdot k_j G(z_i, z_j)\right). \tag{9.5}$$

Here, g_s is the string coupling, the scalar Green's function is given by

$$G(z, w) = -\log|E(z, w)|^2 + 2\pi \text{Im} \int_z^w \omega_I (\text{Im } \Omega)^{-1}_{IJ} \text{Im} \int_z^w \omega_J \tag{9.6}$$

while k is the total momentum, and \mathcal{F} is a holomorphic 1-form in each z_i , given by

$$\mathcal{F} = C_S \mathcal{S}(1234) + \sum_{(i,j,k)=\text{perm}(2,3,4)} C_T(1i|jk) \mathcal{T}(1i|jk). \tag{9.7}$$

The combinations C_S and C_T are kinematical factors, which depend only on the polarization vectors ϵ_i and the external momenta k_i through the gauge invariant combinations $f_i^{\mu\nu} \equiv \epsilon_i^\mu k_i^\nu - \epsilon_i^\nu k_i^\mu$ and are given by

$$\begin{aligned} C_S &= f_1^{\mu\nu} f_2^{\nu\mu} f_3^{\rho\sigma} f_4^{\sigma\rho} + f_1^{\mu\nu} f_2^{\rho\sigma} f_3^{\nu\mu} f_4^{\sigma\rho} + f_1^{\mu\nu} f_2^{\rho\sigma} f_3^{\sigma\rho} f_4^{\nu\mu} - 4 f_1^{\mu\nu} f_2^{\nu\rho} f_3^{\rho\sigma} f_4^{\sigma\mu} \\ &\quad - 4 f_1^{\mu\nu} f_2^{\rho\sigma} f_3^{\nu\rho} f_4^{\sigma\mu} - 4 f_1^{\mu\nu} f_2^{\nu\rho} f_3^{\sigma\mu} f_4^{\rho\sigma}, \\ C_T(ij|kl) &= f_i^{\mu\nu} f_j^{\rho\sigma} f_k^{\nu\mu} f_l^{\sigma\rho} - f_i^{\mu\nu} f_j^{\rho\sigma} f_k^{\sigma\rho} f_l^{\nu\mu} + 2 f_i^{\mu\nu} f_j^{\nu\sigma} f_k^{\sigma\rho} f_l^{\rho\mu} - 2 f_i^{\mu\nu} f_j^{\nu\sigma} f_k^{\rho\mu} f_l^{\sigma\rho}. \end{aligned} \quad (9.8)$$

The kinematical combination C_S coincides with the unique kinematical invariant of the NS 4-point function encountered at tree and 1-loop level, which is often expressed in terms of the rank 8 tensor t (see [1,27]),

$$C_S = -8 t_{\kappa_1 \lambda_1 \kappa_2 \lambda_2 \kappa_3 \lambda_3 \kappa_4 \lambda_4} f_1^{\kappa_1 \lambda_1} f_2^{\kappa_2 \lambda_2} f_3^{\kappa_3 \lambda_3} f_4^{\kappa_4 \lambda_4}. \quad (9.9)$$

Finally, the forms \mathcal{S} and \mathcal{T} are given by

$$\begin{aligned} \mathcal{S}(1234) &= -\frac{1}{192\pi^6 \Psi_{10}} \omega_I(z_1) \omega_J(z_2) \omega_K(z_3) \omega_L(z_4) \sum_{\delta} \mathcal{E}_6[\delta] \vartheta[\delta]^3 \partial_I \partial_J \partial_K \partial_L \vartheta[\delta](\Omega), \\ \mathcal{T}(ij|kl) &= -\frac{1}{8\pi^2} \omega_{[1}(z_i) \omega_{2]}(z_j) \omega_{[1}(z_k) \omega_{2]}(z_l). \end{aligned} \quad (9.10)$$

The δ -sum for the \mathcal{T} -term was carried out explicitly, and no Ψ_{10} appears in its contribution. \mathcal{S} and C_S are totally symmetric, while \mathcal{T} and C_T are odd under the interchange of $i \leftrightarrow j$ or $k \leftrightarrow l$. As a result, the \mathcal{T} -term is novel at 2 loops and could not exist at 1 loop.

The disconnected part of the 4-point function for massless bosons, calculated above, is finite.⁵ Recall that the modular form $\Psi_{10}(\Omega)$ vanishes of second order along the divisor of separating nodes. This corresponds to the propagation of a tachyon, and was responsible for divergence in the bosonic string [25]. Here however, the vector

$$\sum_{\delta} \mathcal{E}_6[\delta](\Omega) \vartheta^3[\delta](\Omega) \partial_I \partial_J \partial_K \partial_L \vartheta[\delta](\Omega), \quad (9.11)$$

also vanishes of second order along the divisor of separating nodes, rendering the superstring amplitude finite. The \mathcal{A} -term is manifestly finite.

In the low energy limit, the exponential factor of the scalar Green's function in (9.5) tends to 1. It is instructive to identify the kinematical factors that emerge from the integration over the 4 vertex insertion points z_i of $|\mathcal{F}|^2$ in (9.5) in the type II superstrings (analogous expressions may be derived for the heterotic strings). The first contribution is from the product $C_S \overline{C}_S$, and yields the well-known ttR^4 term of four Riemann tensors contracted with two copies of the rank 8 tensor t of (9.9) as obtained in [27]. As argued in the preceding paragraph, this contribution is given by a convergent integral. The second contribution is from the products $C_S \overline{C}_T$; it vanishes in view of the complete symmetry in the points z_i in \mathcal{S} and the antisymmetry in two pairs of points in \mathcal{T} . The third contribution is from the product $C_T \overline{C}_T$ for which the z_i integrals may be carried out using the Riemann bilinear relations. The resulting kinematical factors is again a quadrilinear in the Riemann tensor and is proportional to

$$\begin{aligned} C_T \overline{C}_T &\rightarrow + (R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu})^2 - R_{\alpha\beta\mu\nu} R^{\gamma\delta\mu\nu} R^{\alpha\beta\rho\sigma} R_{\gamma\delta\rho\sigma} + 4 R_{\alpha\beta\mu\nu} R^{\gamma\delta\mu\nu} R^{\beta}_{\gamma\rho\sigma} R_{\delta}^{\alpha\rho\sigma} \\ &\quad - 4 R^{\alpha\beta\mu\nu} R_{\delta\alpha\mu\nu} R_{\beta\gamma\rho\sigma} R^{\gamma\delta\rho\sigma} + 4 R^{\alpha\beta\mu\nu} R_{\beta\gamma\nu\rho} R^{\gamma\delta\rho\sigma} R_{\delta\alpha\sigma\mu} \\ &\quad - 4 R^{\alpha\beta\mu\nu} R_{\beta\gamma\nu\rho} R_{\delta\alpha}^{\rho\sigma} R^{\gamma\delta}_{\sigma\mu}. \end{aligned} \quad (9.12)$$

While it is possible that this term, which arose from the *disconnected contributions* in (9.4), will be cancelled by similar contributions arising from the *connected contributions* in (9.3), the above contribution to the low energy

⁵ This is the case at least when the Mandelstam variables $k_i \cdot k_j$ are purely imaginary. As is now well known [19], finiteness for general $k_i \cdot k_j$ cannot be read off directly, but has to be established by analytic continuation.

effective action has at least one remarkable property: the integral over moduli space becomes simply the volume of moduli space with respect to the $Sp(4, \mathbf{Z})$ invariant volume form $\prod_{I \leq J} |d\Omega_{IJ}|^2 (\det \text{Im } \Omega)^{-3}$. We note that the problem of loop corrections in type II superstrings and their contribution to low energy effective actions has witnessed a resurgence of interest recently (see, for example, [28]).

10. Compactification with worldsheet supersymmetry

Our methods extend easily to compactifications of some of the space–time directions to a manifold C , under the following basic assumptions

- the compactification only modifies the matter conformal field theory, leaving the superghost part unchanged;
- the compactification respects $\mathcal{N} = 1$ local worldsheet supersymmetry, so that the super-Virasoro algebra with matter central charge $c = 15$ is preserved.

Under these conditions, the superstring measure is independent of any choices of gauge slice, and a simple prescription for its calculation can be given in terms of the OPE of two supercurrents. Denote by C , respectively, M the fact that the corresponding object is considered on the space–time manifold C , respectively, Minkowski space M . In particular, the chiral partition functions for the matter parts of C and M will be denoted by Z_C and Z_M , respectively. Then we have the following result,

$$\begin{aligned} \mathcal{A}_C[\delta] = \mathcal{A}_M[\delta] \frac{Z_C}{Z_M} \left\{ 1 - \frac{1}{8\pi^2} \int d^2z \int d^2w \chi_{\bar{z}}^+ \chi_{\bar{w}}^+ [\langle S_C(z) S_C(w) \rangle_C - \langle S_M(z) S_M(w) \rangle_M] \right. \\ \left. + \frac{1}{2\pi} \int d^2z \hat{\mu}_{\bar{z}}^z [\langle T_C(z) \rangle_C - \langle T_M(z) \rangle_M] \right\}. \end{aligned} \tag{10.1}$$

Here, $\hat{\mu}_{\bar{z}}^z$ is a Beltrami differential shifting the complex structure from Ω_{IJ} to $\widehat{\Omega}_{IJ}$. The terms $S_C(z)$, $S_M(z)$, $T_C(z)$, $T_M(z)$ are the supercurrents and stress tensors. The expressions $\langle S_C(z) S_C(w) \rangle$ and $\langle T(z) \rangle$ are the chiral correlation functions of $S(z)$ and $T(z)$, or more precisely, the superconformal blocks of the corresponding correlation functions. They are related by the OPE,

$$\begin{aligned} S_C(z) S_C(w) &= \frac{1}{4} \frac{T_C(z) + T_C(w)}{z - w} + (z - w) \mathcal{O}_C(w) + \mathcal{O}(z - w)^2, \\ S_M(z) S_M(w) &= \frac{1}{4} \frac{T_M(z) + T_M(w)}{z - w} + (z - w) \mathcal{O}_M(w) + \mathcal{O}(z - w)^2 \end{aligned} \tag{10.2}$$

for some non-universal operators \mathcal{O}_C and \mathcal{O}_M . The expression $\mathcal{A}_C[\delta]$ is independent of $\hat{\mu}_{\bar{z}}^z$ within its conformal class since $\langle T_C(z) \rangle_C - \langle T_M(z) \rangle_M$ is a holomorphic 2-form. The argument for the supercurrents is similar. Indeed, since the ghost parts of $S_C(z)$ and $S_M(z)$ coincide, all the singularities in z and w with the insertion points p_a and q_a are the same, and cancel between the C and the M contributions. Thus the only possible singularity in the SS correlator is when $z \rightarrow w$, and these contribute to the supersymmetry variation $\delta_\xi \chi_{\bar{z}}^+ = -2\partial_{\bar{z}} \xi^+$. Just as in the flat Minkowski case, this singularity is cancelled precisely by the variation $\delta_\xi \hat{\mu}_{\bar{z}}^z = \xi^+ \chi_{\bar{z}}^+$, in view of the OPE. Thus $\langle S_C(z) S_C(w) \rangle_C - \langle S_M(z) S_M(w) \rangle_M$ is singularity free, and $\mathcal{A}_C[\delta]$ is slice independent, just as $\mathcal{A}[\delta]$ was.

It is then straightforward to evaluate $\mathcal{A}_C[\delta]$, for example, by collapsing all points p_a to a single point, or by going to the gauge (7.4). In particular, in the gauge (7.4), we find

$$\mathcal{A}_C[\delta] = \frac{Z_C}{Z_M} \left\{ \mathcal{Z} + \frac{\zeta^1 \zeta^2}{16\pi^6} \frac{\Xi_6[\delta] \vartheta[\delta](0)}{\Psi_{10}} - \frac{\zeta^1 \zeta^2}{4\pi^2} \mathcal{Z} \langle S_C(q_1) S_C(q_2) \rangle \right\}, \tag{10.3}$$

where \mathcal{Z} is the basic Minkowski space–time matter–ghost correlator

$$\mathcal{Z} = \left\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \right\rangle / \det(\omega_I \omega_J(p_a)) \cdot \det\langle \chi_\alpha | \psi_\beta^* \rangle. \quad (10.4)$$

The model of [29] is given by compactification on an orbifold. The above formulas provide an explicit and consistent framework for the calculation of the cosmological constant in this model and others.

Acknowledgements

We are happy to acknowledge conversations with Zvi Bern, Per Kraus, Boris Pioline and Pierre Vanhove. We are especially grateful to Costas Bachas and Edward Witten for their continued interest in this work and for their encouragement throughout. We wish to thank the Aspen Center for Physics, the Ecole Normale Supérieure, the Institut Henri Poincaré, and the National Center for Theoretical Science in Hsin-Chu, Taiwan for their hospitality while part of this work was being carried out.

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