Iterative approximation of a solution of multi-valued variational-like inclusion in Banach spaces: A $P$-$\eta$-proximal-point mapping approach

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Abstract

In this paper, we introduce a class of $P$-$\eta$-accretive mappings, an extension of $\eta$-$m$-accretive mappings [C.E. Chidume, K.R. Kazmi, H. Zegeye, Iterative approximation of a solution of a general variational-like inclusion in Banach spaces, Int. J. Math. Math. Sci. 22 (2004) 1159–1168] and $P$-accretive mappings [Y.-P. Fang, N.-J. Huang, $H$-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces, Appl. Math. Lett. 17 (2004) 647–653], in real Banach spaces. We prove some properties of $P$-$\eta$-accretive mappings and give the notion of proximal-point mapping, termed as $P$-$\eta$-proximal-point mapping, associated with $P$-$\eta$-accretive mapping. Further, using $P$-$\eta$-proximal-point mapping technique, we prove the existence of solution and discuss the convergence analysis of iterative algorithm, for multi-valued variational-like inclusions in real Banach space. The theorems presented in this paper extend and improve many known results in the literature.

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1. Introduction

Variational inequality theory has emerged as a powerful tool for wide class of unrelated problems arising in various branches of physical, engineering, pure and applied sciences in a unified and general framework, see, for example, [6,7]. Variational inequalities have been extended and generalized in different directions by using novel and innovative techniques and ideas, both for their own sake and for their applications. An important and useful generalization of variational(-like) and quasi-variational(-like) inequality is a variational(-like) inclusion.

In 1994, Hassouni and Moudafi [8] introduced and studied a class of variational inclusions and developed a perturbed iterative algorithm for the variational inclusions. Adly [1], Chidume et al. [3], Ding and Luo [4], Huang [9], Kazmi [11], Noor [14] have obtained some important extensions of the results [8]. One of the most important and interesting problem in the theory of variational inequality is the development of methods which provide an efficient and implementable algorithm for solving variational inequality and its generalizations. These methods include projection method and its variant forms, linear approximation, descent and Newton’s method, and the methods based on auxiliary principle technique. The method based on proximal-point mapping technique is a generalization of projection method and has been widely used to study the existence of solution and to develop iterative algorithms for finding the approximate solution of variational(-like) inclusions, see, for example, [1,3–5,8–12,14]. We remark that most of the work carried out in this direction has been done in the setting of Hilbert spaces.

Recently, Chidume, Kazmi and Zegeye [3] and Fang and Huang [5] introduced and studied the classes of $\eta$-$m$-accretive mappings and $P$-accretive mappings, respectively, and discuss the convergence analysis of iterative algorithms for multi-valued variational-like inclusions and variational inclusions, respectively, in Banach spaces.

Motivated by recent work going in this direction, we define a class of $P$-$\eta$-accretive mappings in real Banach space, which includes the classes of $\eta$-$m$-accretive mappings and $P$-accretive mappings. We study some properties of $P$-$\eta$-accretive mappings and give the notion of $P$-$\eta$-proximal-point mapping, which is a natural extension of the concepts of proximal-point mapping associated with $\eta$-$m$-accretive mappings [3] and $P$-accretive mappings [5]. Further, we prove that $P$-$\eta$-proximal-point mapping is Lipschitz continuous. Furthermore, we consider a multi-valued variational-like inclusion (in short, MVLI) involving $P$-$\eta$-accretive mapping. Using $P$-$\eta$-proximal-point mapping technique, we develop an iterative algorithm for MVLI and show that approximate solution obtained by the iterative algorithm for MVLI converges strongly to the exact solution of MVLI in Banach space. The theorems presented in this paper extend and improve many known results in the literature, see, for example, [3–5] and the relevant references cited therein.

2. $P$-$\eta$-proximal-point mapping

Throughout this paper, we assume that $E$ is a real Banach space equipped with norm $\| \cdot \|$; $E^*$ is the topological dual space of $E$ equipped with norm $\| \cdot \|$; $CB(E)$ is the family of all nonempty closed and bounded subsets of $E$; $2^E$ is the power set of $E$; $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y) \right\}, \quad A, B \in CB(E);$$
⟨·, ·⟩ is the dual pair between $E$ and $E^*$, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \| x \|^2, \| x \| = \| f \| \}, \quad x \in E.$$  

We observe immediately that if $E \equiv H$, a Hilbert space, then $J$ is the identity map on $H$. In sequel, we shall denote a selection of normalized duality mapping by $j$.

First, we define the following concepts.

**Definition 2.1.** Let $\eta : E \times E \to E$ be a mapping. Then a mapping $P : E \to E$ is said to be

(i) $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ such that

$$\langle Px - Py, j\eta(x, y) \rangle \geq 0, \quad \forall x, y \in E;$$

(ii) strictly $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ such that

$$\langle Px - Py, j\eta(x, y) \rangle > 0, \quad \forall x, y \in E,$$

and equality holds if and only if $x = y$;

(iii) $\delta$-strongly $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ and $\delta > 0$ such that

$$\langle Px - Py, j\eta(x, y) \rangle \geq \delta \| x - y \|^2, \quad \forall x, y \in E.$$

**Definition 2.2.** [2] A mapping $\eta : E \times E \to E$ is said to be $\tau$-Lipschitz continuous, if $\exists \tau > 0$ such that

$$\| \eta(x, y) \| \leq \tau \| x - y \|, \quad \forall x, y \in E.$$

**Definition 2.3.** [3] Let $\eta : E \times E \to E$ be a single-valued mapping. Then a multi-valued mapping $M : E \to 2^E$ is said to be

(i) $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ such that

$$\langle u - v, j\eta(x, y) \rangle \geq 0, \quad \forall x, y \in E \text{ and } \forall u \in Mx, \; v \in My;$$

(ii) strictly $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ such that

$$\langle u - v, j\eta(x, y) \rangle > 0, \quad \forall x, y \in E \text{ and } \forall u \in Mx, \; v \in My,$$

and equality holds if and only if $x = y$;

(iii) $\gamma$-strongly $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ and $\gamma > 0$ such that

$$\langle u - v, j\eta(x, y) \rangle \geq \gamma \| x - y \|^2, \quad \forall x, y \in E \text{ and } \forall u \in Mx, \; v \in My;$$

(iv) $\eta$-$m$-accretive, if $M$ is $\eta$-accretive and $(I + \rho M)(E) = E$ for any $\rho > 0$, where $I$ stands for identity mapping.

**Definition 2.4.** Let $\eta : E \times E \to E$ and $P : E \to E$ be nonlinear mappings. Then a multi-valued mapping $M : E \to 2^E$ is said to be $P$-$\eta$-accretive, if $M$ is $\eta$-accretive and $(P + \rho M)(E) = E$ for any $\rho > 0$.

The following theorem gives some properties of $P$-$\eta$-accretive mappings.
Theorem 2.1. Let $\eta : E \times E \to E$ be a mapping; let $P : E \to E$ be a strictly $\eta$-accretive mapping and let $M : E \to 2^E$ be a $P$-$\eta$-accretive multi-valued mapping. Then

(a) $\langle u - v, j\eta(x, y) \rangle \geq 0, \forall (v, y) \in \text{Graph}(M)$ implies $(u, x) \in \text{Graph}(M)$;
(b) the mapping $(P + \rho M)^{-1}$ is single-valued for all $\rho > 0$.

Proof. (a) Suppose, on the contrary, that there exists $(u_0, x_0) \notin \text{Graph}(M)$ such that

$$\langle u_0 - v, j\eta(x_0, y) \rangle \geq 0, \forall (v, y) \in \text{Graph}(M).$$

Since $M$ is $P$-$\eta$-accretive, we have $(P + \rho M)(E) = E$, and hence there exists $(u_1, x_1) \in \text{Graph}(M)$ such that

$$Px_1 + \rho u_1 = Px_0 + \rho u_0.$$  

Now, first set $(v, y) = (u_1, x_1)$ in (2.1) and then, from the resultant inequality (2.2) and from the fact that $\rho$ is positive, we obtain

$$0 \leq \rho\langle u_0 - u_1, j\eta(x_0, x_1) \rangle = \langle Px_1 - Px_0, j\eta(x_0, x_1) \rangle,$$

which implies that

$$\langle Px_0 - Px_1, j\eta(x_0, x_1) \rangle \leq 0.$$

But $P$ is strictly $\eta$-accretive, so we have

$$\langle Px_0 - Px_1, j\eta(x_0, x_1) \rangle = 0,$$

which yields $x_0 = x_1$ and hence from (2.2), we get $u_1 = u_0$, a contradiction. This completes the proof of (a).

(b) For any given $z \in E$ and a constant $\rho > 0$, let $x, y \in (P + \rho M)^{-1}(z)$. Then $\rho^{-1}(z - Px) \in Mx$ and $\rho^{-1}(z - Py) \in My$. Now

$$0 = \rho\langle \rho^{-1}(z - Px) - \rho^{-1}(z - Py), j\eta(x, y) \rangle + \langle Px - Py, j\eta(x, y) \rangle \geq \langle Px - Py, j\eta(x, y) \rangle,$$

using $\eta$-accretiveness of $M$. Since $P$ is strictly $\eta$-accretive, from above inequality, we have $x = y$. This implies $(P + \rho M)^{-1}$ is single-valued. This completes the proof of (b).

By Theorem 2.1, we can define $P$-$\eta$-proximal point mapping for a $P$-$\eta$-accretive mapping $M$ as follows:

$$J_M^\rho(z) = (P + \rho M)^{-1}(z), \quad \forall z \in E,$$

where $\rho > 0$ is a constant, $\eta : E \times E \to E$ is a mapping and $P : E \to E$ is a strictly $\eta$-accretive mapping.

Next we prove that $P$-$\eta$-proximal-point mapping is Lipschitz continuous.

Theorem 2.2. Let $P : E \to E$ be a $\delta$-strongly $\eta$-accretive mapping. Let $\eta : E \times E \to E$ be a $\tau$-Lipschitz continuous mapping and $M : E \to 2^E$ be a $P$-$\eta$-accretive mapping. Then $P$-$\eta$-proximal-point mapping $J_M^\rho$ is $\frac{\tau}{\delta}$-Lipschitz continuous, i.e.,

$$\| J_M^\rho(x) - J_M^\rho(y) \| \leq \frac{\tau}{\delta} \| x - y \|, \quad \forall x, y \in E.$$
Proof. Let \(x, y \in E\). From definition of \(J^M_\rho\), we have \(J^M_\rho(x) = (P + \rho M)^{-1}(x)\). This implies that
\[
\rho^{-1}(x - P(J^M_\rho(x))) \in M(J^M_\rho(x)).
\]
Similarly, we have
\[
\rho^{-1}(y - P(J^M_\rho(y))) \in M(J^M_\rho(y)).
\]
Since \(M\) is \(\eta\)-accretive, we obtain
\[
0 \leq \rho^{-1}\{(x - P(J^M_\rho(x))) - (y - P(J^M_\rho(y)))\, j\eta(J^M_\rho(x), J^M_\rho(y))\}
\]
\[
- \rho^{-1}\{P(J^M_\rho(x)) - P(J^M_\rho(y))\, j\eta(J^M_\rho(x), J^M_\rho(y))\}.
\]
Since \(\rho > 0\), \(P\) is \(\delta\)-strongly \(\eta\)-accretive and \(\eta\) is \(\tau\)-Lipschitz continuous, then from the preceding inequality, we have
\[
\delta \|J^M_\rho(x) - J^M_\rho(y)\|^2 \leq \tau \|x - y\| \|J^M_\rho(x) - J^M_\rho(y)\|.
\]
This implies that
\[
\|J^M_\rho(x) - J^M_\rho(y)\| \leq \frac{\tau}{\delta} \|x - y\|, \quad \forall x, y \in E,
\]
and this completes the proof. \(\square\)

3. Multi-valued variational-like inclusion and iterative algorithm

Let \(N, \eta, M, S, T\) and \(g\), MVLIP (3.1) reduces to various classes of variational inclusions and variational inequalities, see, for example, [3–5,8,9,11], studied by many authors in \(Hilbert spaces\).

Assume that \(\text{domain}(P) \cap g(E) \neq \emptyset\).

The following lemma which will be used in the sequel, is an immediate consequence of the definition of \(J^M_{\rho, \eta}(\cdot, x)\).

Lemma 3.1. \(x \in E, u \in Sx, v \in Tx\) is a solution of GVIP (3.1) if and only if \((x, u, v)\) satisfies the relation
\[
gx = J^M_{\rho, \eta}(\cdot, x)(P \circ g(x) - \rho N(u, v)),
\]
where \(J^M_{\rho, \eta}(\cdot, x) = (P + \rho M(\cdot, x))^{-1}; \rho > 0\) is a constant and \(P \circ g\) denotes \(P\) composition \(g\).

Using Lemma 3.1 and Nadler’s technique [13], we develop an iterative algorithm for finding the approximate solution of MVLIP (3.1) as follows.
Iterative Algorithm 3.1. Let \( \eta, N : E \times E \to E \), \( g : E \to E \), and \( S, T : E \to CB(E) \) be such that for each \( x \in E \), \( Q(x_0) \subseteq g(E) \), where \( Q : E \times E \to 2^E \) be a multi-valued mapping defined by

\[
Q(x) = \bigcup_{u \in Sx} \bigcup_{v \in Tx} (J^M_{\rho,x}(P \circ g(x) - \rho N(u,v))),
\]

where \( M : E \times E \to 2^E \) be a multi-valued mapping such that for each \( x \in E \), \( M(\cdot,x) \) is \( P, \eta \)-accretive.

For given \( x_0 \in E \), \( u_0 \in Sx_0 \), \( v_0 \in Tx_0 \), let

\[
w_0 = J^M_{\rho,x_0}(P \circ g(x_0) - \rho N(u_0,v_0)) \in Q(x_0) \subseteq g(E).
\]

Hence there exists \( x_1 \in E \) such that \( w_0 = g(x_1) \). Since \( u_0 \in Sx_0 \subseteq CB(E) \) and \( v_0 \in Tx_0 \), then by Nadler [13], there exist \( u_1 \in Sx_1 \), and \( v_1 \in Tx_1 \) such that

\[
\|u_1 - u_0\| \leq (1 + (1 + 1)^{-1}) H(Sx_1, Sx_0),
\]

\[
\|v_1 - v_0\| \leq (1 + (1 + 1)^{-1}) H(Tx_1, Tx_0).
\]

Let

\[
w_1 = J^M_{\rho,x_1}(P \circ g(x_1) - \rho N(u_1,v_1)) \in Q(x_1) \subseteq g(E).
\]

Hence there exists \( x_2 \in E \) such that \( w_1 = g(x_2) \). By induction, we can define iterative sequences \( \{x_n\}, \{u_n\}, \{v_n\} \) as follows:

\[
g(x_{n+1}) = J^M_{\rho,x_n}(P \circ g(x_n) - \rho N(u_n,v_n)),
\]

\[
u_n \in Sx_n: \|u_{n+1} - u_n\| \leq (1 + (1 + n)^{-1}) H(Sx_{n+1}, Sx_n),
\]

\[
v_n \in Tx_n: \|v_{n+1} - v_n\| \leq (1 + (1 + n)^{-1}) H(Tx_{n+1}, Tx_n),
\]

where \( n = 0, 1, 2, \ldots \) and \( \rho > 0 \) is a constant.

4. Convergence analysis

First, we define the following concepts.

Definition 4.1. Let \( \eta : E \times E \to E \) be a single-valued mapping and let \( S, T : E \to 2^E \) be two multi-valued mappings. A mapping \( N : E \times E \to E \) is said to be

(i) \( \gamma \)-strongly \( \eta \)-accretive with respect to \( S \) and \( T \), if \( \exists \eta(x_1, x_2) \in J\eta(x_1, x_2) \) and \( \gamma > 0 \) such that

\[
\{N(u_1, v_1) - N(u_2, v_2), J\eta(x_1, x_2)\} \ni \alpha \|x_1 - x_2\|^2,
\]

\[
\forall x_1, x_2 \in E, \ u_1 \in Sx_1, \ v_1 \in Tx_1, \ u_2 \in Sx_2, \ v_2 \in Tx_2;
\]

(ii) \( (\alpha, \beta) \)-Lipschitz continuous, if \( \exists \alpha, \beta > 0 \) such that

\[
\|N(x_1, y_1) - N(x_2, y_2)\| \leq \alpha \|x_1 - x_2\| + \beta \|y_1 - y_2\|, \ \forall x_1, x_2, y_1, y_2 \in E.
\]

Remark 4.1. The concepts of \( \gamma \)-strongly \( \eta \)-accretiveness with respect to \( S \) and \( T \) and \( (\alpha, \beta) \)-Lipschitz continuity of mapping \( N \) are more general than the concepts used in Theorem 4.1 of Chidume et al. [3].
Next, we recall the following concept and result.

**Definition 4.2.** A multi-valued mapping $T : E \to 2^E$ is said to be $\alpha$-$H$-Lipschitz continuous, if $\exists \alpha > 0$ such that

$$H(Tx, T\eta) \leq \alpha \|x - \eta\| , \quad \forall x, \eta \in E .$$

**Lemma 4.1.** [15] Let $E$ be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, \eta \in E$,

$$\|x + \eta\|^2 \leq \|x\|^2 + 2\{\eta, j(x + \eta)\}, \quad \forall j(x + \eta) \in J(x + \eta). \quad (4.1)$$

Now, we prove the existence of solution of MVLIP (3.1) and discuss the convergence analysis of Iterative Algorithm 3.1.

**Theorem 4.1.** Let $E$ be a real Banach space and $\eta : E \times E \to E$ be $\tau$-Lipschitz continuous mapping. Let $S, T : E \to CB(E)$ and $g : E \to E$ be $\sigma$-$H$-Lipschitz continuous, $k$-$H$-Lipschitz continuous, $\xi$-Lipschitz continuous mappings, respectively, and let $(g - I) : E \to E$ be $\nu$-strongly accretive mapping. Let $P : E \to E$ be $\delta$-strongly $\eta$-accretive and $P \circ g$ be $\mu$-Lipschitz continuous mappings. Let $N : E \times E \to E$ be $(\alpha, \beta)$-Lipschitz continuous and $\gamma$-strongly $\eta$-accretive mapping with respect to $S$ and $T$. Let $M : E \times E \to 2^E$ be such that for each fixed $x \in E$, $M(., x)$ is $P$-$\eta$-accretive mapping, and for each $x \in E$, let $Q(x) \subseteq g(E)$, where $Q$ is defined by (3.3). Suppose that there exists $\lambda > 0$ such that, for each $x_1, x_2, z \in E$,

$$\|J^M(., x_1)\| - J^M(., x_2)\| \leq \lambda \|x_1 - x_2\|, \quad (4.2)$$

and, for $\rho > 0$,

$$\rho - \frac{\gamma - (\mu + \tau)L}{2L^2} \leq \frac{(\gamma - (\mu + \tau)L)^2 - 2L^2[\mu^2 - (\frac{L}{s} - \lambda)^2]}{2L^2}, \quad (4.3)$$

$$\gamma > (\mu + \tau)L + \sqrt{2L}\sqrt{\mu^2 - \left(\frac{t}{s} - \lambda\right)^2}; \quad \mu > \left(\frac{t}{s} - \lambda\right), \quad t > s, \quad (4.4)$$

where $t := \sqrt{2\nu + 1}$, $s := \frac{\tau}{\delta}$, $L := (\alpha \sigma + \beta k)$.

Then the iterative sequences $\{x_n\}, \{u_n\}$ and $\{v_n\}$ generated by Iterative Algorithm 3.1, converge strongly to $x^*$, $u^*$ and $v^*$, respectively, and $(x^*, u^*, v^*)$ is a solution of MVLIP (3.1).

**Proof.** Since $(g - I)$ is $\nu$-strongly accretive, by using Lemma 4.1, we have the following estimate:

$$\|x_{n+2} - x_{n+1}\|^2 = \|g(x_{n+2} - g x_{n+1} + x_{n+2} - x_{n+1})\|^2 \leq \|g x_{n+2} - g x_{n+1}\|^2 + 2\{(g - I)x_{n+2} - (g - I)x_{n+1}, j(x_{n+2} - x_{n+1})\} \leq \|g x_{n+2} - g x_{n+1}\|^2 - 2\nu \|x_{n+2} - x_{n+1}\|^2 ,$$

which implies that

$$\|x_{n+2} - x_{n+1}\| \leq \frac{1}{\sqrt{2\nu + 1}} \|g x_{n+2} - g x_{n+1}\|. \quad (4.5)$$

Now, by using (3.4), (4.2) and Theorem 2.2, we have
\[ \| g x_{n+2} - g x_{n+1} \| \]
\[ \leq \| J^M_{\rho}(\cdot;x_{n+1})(P \circ g(x_{n+1}) - \rho N(u_{n+1}, v_{n+1})) \]
\[ - J^M_{\rho}(\cdot;x_{n+1})(P \circ g(x_n) - \rho N(u_n, v_n)) \]
\[ + \| J^M_{\rho}(\cdot;x_{n+1})(P \circ g(x_n) - \rho (u_n, v_n)) - J^M_{\rho}(\cdot;x_n)(P \circ g(x_n) - \rho N(u_n, v_n)) \| \]
\[ \leq \frac{\tau}{\delta} \| P \circ g(x_{n+1}) - P \circ g(x_n) - \rho [N(u_{n+1}, v_{n+1}) - N(u_n, v_n)] \|
\[ + \lambda \| x_{n+1} - x_n \|. \quad (4.6) \]

Since \( N \) is \( \gamma \)-strongly \( \eta \)-accretive with respect to \( S \) and \( T \), \( P \circ g \) is \( \mu \)-Lipschitz continuous and \( \eta \) is \( \tau \)-Lipschitz continuous, using Lemma 4.1,
\[ \| P \circ g(x_{n+1}) - P \circ g(x_n) - \rho [N(u_{n+1}, v_{n+1}) - N(u_n, v_n)] \|^2 \]
\[ \leq \| P \circ g(x_{n+1}) - P \circ g(x_n) \|^2 - 2\rho [N(u_{n+1}, v_{n+1}) - N(u_n, v_n)], j \eta(x_{n+1}, x_n) \]
\[ - 2\rho [N(u_{n+1}, v_{n+1}) - N(u_n, v_n), j(P \circ g(x_{n+1}) - P \circ g(x_n)) \]
\[ - \rho (N(u_{n+1}, v_{n+1}) - N(u_n, v_n)) - j \eta(x_{n+1}, x_n) \]
\[ \leq \| P \circ g(x_{n+1}) - P \circ g(x_n) \|^2 - 2\rho \gamma \| x_{n+1} - x_n \|^2 \]
\[ + 2\rho \| N(u_{n+1}, v_{n+1}) - N(u_n, v_n) \| \| P \circ g(x_{n+1}) - P \circ g(x_n) \|
\[ + \rho \| N(u_{n+1}, v_{n+1}) - N(u_n, v_n) \| + \| \eta(x_{n+1}, x_n) \| \]
\[ \leq \mu^2 \| x_{n+1} - x_n \|^2 - 2\rho \gamma \| x_{n+1} - x_n \|^2 \]
\[ + 2\rho \| N(u_{n+1}, v_{n+1}) - N(u_n, v_n) \| \| \mu \| x_{n+1} - x_n \|
\[ + \rho \| N(u_{n+1}, v_{n+1}) - N(u_n, v_n) \| + \tau \| x_{n+1} - x_n \|. \quad (4.7) \]

Now, by \((a, \beta)\)-Lipschitz continuity of \( N \) and \( H \)-Lipschitz continuity of \( S, T \), we obtain
\[ \| N(u_{n+1}, v_{n+1}) - N(u_n, v_n) \|
\[ \leq \alpha \| u_{n+1} - u_n \| + \beta \| v_{n+1} - v_n \|
\[ \leq \alpha (1 + (1 + n)^{-1}) H(Sx_{n+1}, Sx_n) + \beta (1 + (1 + n)^{-1}) H(Tx_{n+1}, Tx_n) \]
\[ \leq L(1 + (1 + n)^{-1}) \| x_{n+1} - x_n \|, \quad (4.8) \]

where \( L := (a \sigma + \beta k) \).

From (4.7) and (4.8), we get
\[ \| P \circ g(x_{n+1}) - P \circ g(x_n) - \rho [N(u_{n+1}, v_{n+1}) - N(u_n, v_n)] \|^2 \]
\[ \leq (\mu^2 - 2\rho \gamma + 2\rho L(1 + (1 + n)^{-1})[\mu + \tau + \rho L(1 + (1 + n)^{-1})]) \]
\[ \times \| x_{n+1} - x_n \|^2. \quad (4.9) \]

From (4.6) and (4.9), we get
\[ \| g x_{n+2} - g x_{n+1} \|
\[ \leq \left( \frac{\tau}{\delta} \sqrt{\mu^2 - 2\rho \gamma + 2\rho L(1 + (1 + n)^{-1})[\mu + \tau + \rho L(1 + (1 + n)^{-1})]} + \lambda \right) \]
\[ \times \| x_{n+1} - x_n \|. \quad (4.10) \]
From (4.5) and (4.10), we get
\[ \|x_{n+2} - x_{n+1}\| \leq \theta_n \|x_{n+1} - x_n\|, \]  
(4.11)

where
\[ \theta_n := \frac{1}{\sqrt{2\nu + 1}} \times \left[ \frac{\tau}{\delta} \sqrt{\mu^2 - 2\rho\gamma + 2\rho L(1 + (1 + n)^{-1})[\mu + \tau + \rho L(1 + (1 + n)^{-1})]} + \lambda \right]. \]  
(4.12)

Letting \( n \to \infty \), we obtain that \( \theta_n \to \theta \), where
\[ \theta := \frac{1}{\sqrt{2\nu + 1}} \left[ \frac{\tau}{\delta} \sqrt{\mu^2 - 2\rho(\gamma + L(\mu + \tau)) + 2\rho^2 L^2} + \lambda \right]. \]  
(4.13)

Since \( \theta < 1 \) by conditions (4.3), (4.4), hence \( \theta_n < 1 \) for \( n \) sufficiently large. Therefore (4.11) implies that \( \{x_n\} \) is a Cauchy sequence in \( E \), and hence there exists \( x^* \in E \) such that \( x_n \to x^* \) as \( n \to \infty \).

Since \( S \) is \( \sigma \)-\( \text{H-Lipschitz} \) continuous, from (3.5), we get
\[ \|u_{n+1} - u_n\| \leq (1 + (1 + n)^{-1}) H(Sx_{n+1}, Sx_n) \leq \sigma (1 + (1 + n)^{-1}) \|x_{n+1} - x_n\|. \]  
(4.14)

Since \( \{x_n\} \) is a Cauchy sequence, it follows from (4.14) that \( \{u_n\} \) is a Cauchy sequence. Similarly, Lipschitz continuity of \( T \) and \( g \) implies that \( \{v_n\} \) and \( \{g(x_n)\} \) are Cauchy sequences. Hence, there exist \( u^*, v^* \in E \) such that \( g x_n \to g x^*, u_n \to u^*, \) and \( v_n \to v^* \) as \( n \to \infty \). Furthermore,
\[ d(u^*, Sx^*) \leq \|u^* - u_n\| + d(u_n, Sx_n) + H(Sx_n, Sx^*) \leq \|u^* - u_n\| + \sigma \|x_n - x^*\| \to 0, \text{ as } n \to \infty, \]
and hence \( u^* \in Sx^* \). Similarly, we can show \( v^* \in Tx^* \).

Finally, we define
\[ w^* = J^M_{\rho}(P \circ g(x^*)) - \rho N(u^*, v^*). \]

Now, we estimate that
\[ \|g x_{n+2} - w^*\| \]
\[ \leq \frac{\tau}{\delta} \|P \circ g(x_{n+1}) - P \circ g(x^*) - \rho [N(u_{n+1}, v_{n+1}) - N(u^*, v^*)]\| + \lambda \|x_{n+1} - x^*\| \]
\[ \leq \frac{\tau}{\delta} \left[ \mu \|x_{n+1} - x_n\| + \rho \alpha \|u_{n+1} - u^*\| + \rho \beta \|v_{n+1} - v^*\| \right] + \lambda \|x_{n+1} - x^*\| \]
\[ \to 0, \text{ as } n \to \infty. \]

Thus,
\[ g(x^*) = w^* = J^M_{\rho}(P \circ g(x^*)) - \rho N(u^*, v^*). \]

By Lemma 3.1, it follows that \( (x^*, u^*, v^*) \) is a solution of MVLIP (3.1), and this completes the proof. \( \square \)
Remark 4.2.

(i) Theorem 2.1 generalizes Lemma 2.6 [3] and Theorems 2.1–2.2 [5]; Theorem 2.2 generalizes Lemma 2.8 [3] and Theorem 2.3 [5], and Theorem 4.1 generalizes Theorem 4.1 [3] and Theorem 3.8 [5].

(ii) In the spirit of Theorem 2.1 [10], the results concerning for solving monotone multi-valued variational inclusions (inequalities) established by many authors (see, for example, [14] and the relevant reference cited therein) are actually for single-valued variational inclusions despite of involving multi-valued mappings. Therefore, such methods used for studying the existence of solution and the convergence criteria of the iterative algorithms for monotone multi-valued variational inclusions (inequalities) need improvement. Very recently Chidume, Kazmi and Zegeye [2,3] and Kazmi [12] improved the methods which are applicable for multi-valued variational-like inclusions (inequalities). The technique developed in this paper modify the methods developed in Chidume et al. [3] and Kazmi [12], and can also be applied for multi-valued variational inclusions (inequalities).

References