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Convergence of a splitting scheme applied to the Ruijgrok–Wu model of the Boltzmann equation

Hailiang Liu[a,*], Jinghua Wang[b], Gerald Warnecke[a]

[a] *IAN, Otto-von-Guericke-Universität Magdeburg, PSF 4120, D-39016 Magdeburg, Germany*

[b] *Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China*

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Abstract

This paper deals with upwind splitting schemes for the Ruijgrok–Wu model (Physica A 113 (1982) 401–416) of the kinetic theory of rarefied gases in the fluid-dynamic scaling. We prove the stability and the convergence for these schemes. The relaxation limit is also investigated and the limit equation is proved to be a first-order quasi-linear conservation law. The loss of quasi-monotonicity of the present model makes it necessary to give a more careful analysis of its structure. We also obtain global error estimates in the spaces $W^{s,p}$ for $-1 \leqslant s \leqslant 1/p$, $1 \leqslant p \leqslant \infty$ and pointwise error estimates for the approximate solution. The proof naturally uses the framework introduced by Nessyahu and Tadmor (SIAM J. Numer Anal. 29 (1992) 1505–1519) due to the convexity of the flux function. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we propose a numerical splitting scheme for the Ruijgrok–Wu (R–W) model derived from the Boltzmann equation. This model was introduced by Ruijgrok and Wu [20]. In this model, the gas is composed by two kinds of particles that move parallel to the x-axis with constant and equal speeds c, either in the positive x-direction with a density u, or in the negative x-direction

* Correspondence address UCLA, Department of Mathematics, 405 Hilgard Avenue, Los Angeles, CA90095-1555, USA.

E-mail addresses: hliu@math.ucla.edu (H. Liu), jwang@iss06.iss.ac.cn (J. Wang), gerald.warnecke@mathematik.uni-magdeburg.de (G. Warnecke).

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with a density v. The space–time evolution of the densities $u=u(x,t), v=v(x,t)$ is described by the semi-linear system

$$\begin{aligned}&\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=-\alpha u+\beta v+\gamma uv,\\&\frac{\partial v}{\partial t}-c\frac{\partial v}{\partial x}=\alpha u-\beta v-\gamma uv,\end{aligned}\tag{1.1}$$

where $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, with given nonnegative initial data $u(x,0)=u_0(x)$, $v(x,0)=v_0(x)$, where α, β, $\gamma\geqslant 0$ are three parameters. The equilibrium curve contains two branches separated by its two asymptotics $v=\alpha/\gamma$ and $u=-\beta/\gamma$.

We are interested in the fluid-dynamic limit associated with the kinetic system (1.1). This will be achieved by rescaling $\alpha\to(\alpha/\varepsilon), \beta\to(\beta/\varepsilon), \gamma\to(\gamma/\varepsilon)$. System (1.1) becomes

$$\begin{aligned}&\frac{\partial u^\varepsilon}{\partial t}+c\frac{\partial u^\varepsilon}{\partial x}=-\frac{1}{\varepsilon}[\alpha u^\varepsilon-\beta v^\varepsilon-\gamma u^\varepsilon v^\varepsilon],\\&\frac{\partial v^\varepsilon}{\partial t}-c\frac{\partial v^\varepsilon}{\partial x}=\frac{1}{\varepsilon}[\alpha u^\varepsilon-\beta v^\varepsilon-\gamma u^\varepsilon v^\varepsilon],\end{aligned}\tag{1.2}$$

where $\varepsilon>0$ is the relaxation parameter. The macroscopic variables of the system for this model are the mass density $\rho^\varepsilon=u^\varepsilon+v^\varepsilon$ and the flux $j^\varepsilon=c(u^\varepsilon-v^\varepsilon)$. Since u^ε and v^ε can be expressed in terms of ρ^ε and j^ε, system (1.2) is equivalent to the following system for the mass density and the flux:

$$\begin{aligned}&\frac{\partial}{\partial t}\rho^\varepsilon+\frac{\partial}{\partial x}j^\varepsilon=0,\\&\frac{\partial j^\varepsilon}{\partial t}+c^2\frac{\partial}{\partial x}\rho^\varepsilon=-\frac{1}{\varepsilon}B(\rho^\varepsilon,j^\varepsilon),\end{aligned}\tag{1.3}$$

where using

$$F(\rho):=c\left[\rho^2+2\frac{\beta-\alpha}{\gamma}\rho+\left(\frac{\beta+\alpha}{\gamma}\right)^2\right]^{1/2}-\frac{\beta+\alpha}{\gamma}c\tag{1.4}$$

and

$$G(\rho):=c\left[\rho^2+2\frac{\beta-\alpha}{\gamma}\rho+\left(\frac{\beta+\alpha}{\gamma}\right)^2\right]^{1/2}+\frac{\beta+\alpha}{\gamma}c,\tag{1.5}$$

we have

$$\begin{aligned}B(\rho,j)&:=(\alpha+\beta)j-(\beta-\alpha)c\rho-\frac{\gamma c}{2}\rho^2+\frac{\gamma}{2c}j^2\\&=\frac{\gamma}{2c}[j-F(\rho)][j+G(\rho)].\end{aligned}$$

Using the special form of $G(\rho)$ Gabetta and Perthame [6] has shown that $j+G(\rho)$ is always positive. Therefore, in the zero relaxation limit ($\varepsilon\to 0^+$), system (1.3) can be approximated to leading order by the equation

$$j=F(\rho)\tag{1.6}$$

and

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}F(\rho)=0, \tag{1.7}$$

see [6]. A state satisfying (1.6) will be called a *local equilibrium*. A rigorous justification of this limit for $\varepsilon\to 0$ in (1.3) was given by Gabetta and Perthame [6], who also investigated the diffusive limit to a viscous Burgers' equation. The diffusive limit necessitates a diffusive scaling and the reduced equations are of parabolic type, see, e.g., refs. [9,13,26]. In fact, this kind of limit from a hyperbolic system to a hyperbolic system of fewer equations has drawn much attention due to the work of Liu [14] as well as Chen et al. [4]. A total variation diminishing (TVD) bound was proved for the special case $B=j-F(\rho)$ in [16], for the higher space dimensional case see [11]. On the other hand, this approximation is strongly connected with the study of fluid-dynamical limits, see, e.g., [3] or [19].

From a numerical point of view, hyperbolic conservation laws with stiff source terms were extensively studied in [2,7,8], see also [23]. One feature of the R–W model is that if the initial values are nonnegative, then so are the solutions. Special care must be taken to assure that this nonnegativity preserving property also holds at the discrete level. In the present paper we construct a first-order numerical scheme approximating (1.2). This is done by considering a fractional-step scheme, where the homogeneous (linear) part is treated by using an explicit scheme and then the nonlinear source term is treated by solving exactly an ODE in each time interval. Based on this idea Natalini [1] proposed a class of schemes for the case $B=j^{\varepsilon}-F(\rho^{\varepsilon})$ and proved the convergence of these schemes. The convergence for first-order relaxation schemes introduced in [10] was obtained by Yong [25]. See also [24] for the convergence of some second-order relaxation schemes. The main argument in these investigations uses the fact that the systems have some monotonicity properties allowing comparison properties. One of the main differences to previous models is that the R–W model is not a quasi-monotone system in the sense of Natalini-Hanouzet [17], and thus needs a more careful analysis of its structure.

This paper is organized as follows. In Section 2 we recall some analytical results on (1.2), or (1.3), obtained in [6] and then introduce the numerical schemes for (1.2) that we want to study. Section 3 is devoted to the proof of stability with respect to ε of the schemes in the L^{∞}, L^1 and BV norms. In Section 4 we prove that for a fixed time step, the approximate solutions of (1.3) converge as $\varepsilon\to 0$, to the numerical approximations by a TVD, L^{∞}-stable discretization of the limit conservation law (1.7). Convergence to solutions of (1.7) as Δx and ε tend to zero is proved in Section 5, in which some convergence rate estimates are also obtained.

Notation. Let $\mathrm{BV}=\mathrm{BV}(\mathbb{R})$ denote the subspace of $L^1_{\mathrm{loc}}(\mathbb{R})$ consisting of functions with bounded variation, i.e.,

$$\mathrm{BV}=\{u\in L^1_{\mathrm{loc}}(\mathbb{R}),\mathrm{TV}(u)<\infty\},$$

where

$$\mathrm{TV}(u)=\sup_{h\neq 0}\int_{\mathbb{R}}\frac{|u(x+h)-u(x)|}{|h|}\,\mathrm{d}x.$$

The L^1-norm is denoted by $\|\cdot\|_1$. For grid functions the total variation is defined by

$$\mathrm{TV}(u)=\sum_{i\in\mathbb{Z}}|u_i-u_{i-1}|$$

and $\|\cdot\|_1$ denotes the discrete L^1-norm, $\|\cdot\|_\infty$ the discrete L^∞-norm that are defined as

$$\|u\|_1 = \Delta x \sum_{i\in\mathbb{Z}} |u_i|, \quad \|u\|_\infty = \sup_{i\in\mathbb{Z}} |u_i|.$$

We shall use $\mathrm{TV}(u,v)$ to denote $\mathrm{TV}(u)+\mathrm{TV}(v)$, similarly $\|(u,v)\|_1$, $\|(u,v)\|_\infty$ and the like will also be used.

2. Preliminaries and numerical scheme

First, we specify the assumptions for model (1.2) under which some analytical properties of problems (1.2) were obtained in [6]. Then the discretization of the model and the initial data are discussed.

Let us consider the following conditions:

(H_1) *The initial functions* $(u_0^\varepsilon, v_0^\varepsilon) \in \mathrm{BV}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ *are nonnegative and there exist constants* M_∞, M_1 and M_0 *not depending on* ε *such that the data are uniformly bounded*

$$\|(u_0^\varepsilon, v_0^\varepsilon)\|_\infty \leqslant M_\infty, \quad \|(u_0^\varepsilon, v_0^\varepsilon)\|_{L^1} \leqslant M_1, \quad \mathrm{TV}(u_0^\varepsilon, v_0^\varepsilon) \leqslant M_0.$$

(H_2) *The initial functions* $(u_0^\varepsilon, v_0^\varepsilon)$ *converge to* (u_0, v_0) *in* $L^1_{\mathrm{loc}}(\mathbb{R})^2$ *as* $\varepsilon \to 0^+$.

Under hypothesis (H_1), the initial value problem for (1.2) has a unique global weak solution, satisfying $u^\varepsilon(x,t) \geqslant 0$, $v^\varepsilon(x,t) \geqslant 0$ and belonging to $\mathrm{BV}(\mathbb{R})$ for each $t > 0$, see [6]. It turns out that for $\varepsilon \to 0^+$ the family of solutions $\rho^\varepsilon = u^\varepsilon + v^\varepsilon$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ towards the entropy solution $\rho = \rho(x,t)$ of the scalar problem (1.7). Here we consider entropy solutions in the sense of Kruzkov [12].

Now, we begin to discuss the discretization of system (1.2) and (1.3). We derive first-order accurate and stable discretizations that have the nonnegativity preserving property. Let the spatial grid points be $x_{i+1/2} = (j + 1/2)\Delta x$, $i \in \mathbb{Z}$ with uniform mesh length Δx. The discrete time levels $t^n = n\Delta t$ with $n \in \mathbb{N}$ are also spaced uniformly with the time step Δt.

As usual we denote by $(u_j^{n,\varepsilon}, v_j^{n,\varepsilon})$ the nodal values $(u^{\Delta x,\varepsilon}(j\Delta x, n\Delta t), v^{\Delta x,\varepsilon}(j\Delta x, n\Delta t))$ of our approximate solutions. Our numerical approximations are taken to be step functions $(u^{\Delta x,\varepsilon}, v^{\Delta x,\varepsilon})$ that are piecewise constant on each rectangle $I_i \times [t_n, t_{n+1}[$ with $I_i := [x_{i-1/2}, x_{i+1/2}[$. Consider a solution for some finite length of time $T = N\Delta t$. We can write

$$(u^{\Delta x,\varepsilon}, v^{\Delta x,\varepsilon})(x,t) = \sum_{n=0}^{N} \sum_{i\in\mathbb{Z}} (u_i^{n,\varepsilon}, v_i^{n,\varepsilon}) \chi_{I_i}(x) \chi_{[t_n,t_{n+1}[}(t) \tag{2.1}$$

with $\chi_{I_i}(x)$ denoting the characteristic function of the interval I_i. From now on we will drop the superscript ε for $(u_i^{n,\varepsilon}, v_i^{n,\varepsilon})$ and all other terms unless they are really needed for clarity of presentation.

Set

$$u_i^0 = \frac{1}{\Delta x} \int_{I_i} u_0^\varepsilon(x)\,\mathrm{d}x, \qquad v_i^0 = \frac{1}{\Delta x} \int_{I_i} v_0^\varepsilon(x)\,\mathrm{d}x.$$

Thus denoting by u_Δ^0, v_Δ^0 the associated step functions on $\mathbb{R}$ we have

$$\|(u_\Delta^0, v_\Delta^0) - (u_0^\varepsilon, v_0^\varepsilon)\|_1 := \|(u_\Delta^0 - u_0^\varepsilon)\|_1 + \|(v_\Delta^0 - v_0^\varepsilon)\|_1 \leqslant \Delta x\, \mathrm{TV}\ (u_0^\varepsilon, v_0^\varepsilon) \leqslant M_0 \Delta x.$$

It is easy to deduce that the discrete initial data satisfy the bounds

$$\begin{aligned} &\text{(i) } \|(u^0_\Delta, v^0_\Delta)\|_\infty \leqslant M,\\ &\text{(ii) } \|(u^0_\Delta, v^0_\Delta)\|_1 \leqslant M, \\ &\text{(iii) } \mathrm{TV}(u^0_\Delta, v^0_\Delta) \leqslant M \end{aligned} \tag{2.2}$$

where $M = \max\{M_0, M_1, M_\infty\}$.

Now, we turn to the discretization of system (1.2). For the numerical approximation system (1.2) may be split into a linear hyperbolic part and a system of ordinary differential equations. For given data $(u^n_\Delta, v^n_\Delta) =: U^n_\Delta$, let $U^{n+1/2}_\Delta$ be an approximate solution at time t_{n+1} of the system

$$\begin{aligned} &\partial_t u + c\partial_x u = 0,\\ &\partial_t v - c\partial_x v = 0 \end{aligned} \tag{2.3}$$

with initial data

$$U(t_n, x) = U^n_\Delta(x).$$

Since (2.3) is a linear hyperbolic system in diagonal form, it is straightforward to apply the upwind scheme. Doing this we get

$$\begin{aligned} u_i^{n+1/2} &= u_i^n - \lambda(u_i^n - u_{i-1}^n),\\ v_i^{n+1/2} &= v_i^n + \lambda(v_{i+1}^n - v_i^n). \end{aligned} \tag{2.4}$$

It is well known that it is a consistent monotone scheme if $\lambda = c(\Delta t/\Delta x)$ satisfies the CFL condition $\lambda \leqslant 1$. The nonlinear part is treated by solving exactly, on the time interval $[t_n, t_{n+1}]$, the problem

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}t} &= -\frac{1}{\varepsilon}(\alpha w - \beta z - \gamma wz),\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{1}{\varepsilon}(\alpha w - \beta z - \gamma wz) \end{aligned} \tag{2.5}$$

for the initial data

$$(w(t_n), z(t_n)) = (u_i^{n+1/2}, v_i^{n+1/2}), \quad i \in \mathbb{Z}.$$

We denote by S_t the exact solution operator to the initial value problem for a solution $U(t) = (w(t), z(t))$, $t \geqslant \tau$, with given initial values $U(\tau)$ at time $\tau \geqslant 0$, i.e., we write $U(t) = S_t(\tau, U(\tau))$.

The specific nature of the source terms leads to an explicit expression for $(u_i^{n+1}, v_i^{n+1}) = (w,z)(t_{n+1})$ in terms of $(u_i^{n+1/2}, v_i^{n+1/2})$.

Lemma 2.1. *System* (2.5) *has an explicit solution for each iteration. With* $\rho = u + v$ *we introduce*

$$\mu_1(\rho) := \frac{1}{2}\left[\sqrt{\rho^2 + 2\frac{\beta-\alpha}{\gamma}\rho + \left(\frac{\beta+\alpha}{\gamma}\right)^2} + \rho - \frac{\alpha+\beta}{\gamma}\right],$$

$$\mu_2(\rho) := \frac{1}{2}\left[\sqrt{\rho^2 + 2\frac{\beta-\alpha}{\gamma}\rho + \left(\frac{\beta+\alpha}{\gamma}\right)^2} - \rho + \frac{\alpha+\beta}{\gamma}\right]$$

and define the following function:

$$H(u,v):=\frac{\varepsilon}{\gamma\Delta t}\left[1-\exp\left(-\frac{\gamma}{\varepsilon}\Delta t(\mu_1+\mu_2)\right)\right]\frac{\alpha u-\beta v-\gamma uv}{(u+\mu_2)-(u-\mu_1)\exp[-(\gamma\Delta t/\varepsilon)(\mu_1+\mu_2)]}.$$

The solution to (2.5) *is given as*

$$\begin{aligned} u_i^{n+1}&=u_i^{n+1/2}-\frac{\Delta t}{\varepsilon}H(u_i^{n+1/2},v_i^{n+1/2}),\\ v_i^{n+1}&=v_i^{n+1/2}+\frac{\Delta t}{\varepsilon}H(u_i^{n+1/2},v_i^{n+1/2}).\end{aligned} \tag{2.6}$$

Proof. Solve (2.5) in $[t_n,t_{n+1}]$ with initial data $(w(t_n),z(t_n))=(u_i^{n+1/2},v_i^{n+1/2})$. First, adding the two equations (2.5) shows that $w(t)+z(t)$ is constant; with the above initial data we have $w(t)+z(t)\equiv\rho_i^{n+1/2}$.

Thus the first equation can be written as

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t}&=-\frac{\gamma}{\varepsilon}\left[w^2+\left(\frac{\alpha+\beta}{\gamma}-\rho_i^{n+1/2}\right)w-\frac{\beta}{\gamma}\rho_i^{n+1/2}\right]\\ &=-\frac{\gamma}{\varepsilon}[(w-\mu_1(\rho_i^{n+1/2}))(w+\mu_2(\rho_i^{n+1/2}))],\end{aligned} \tag{2.7}$$

where μ_i $(i=1,2)$ are defined as above.

Thus (2.7) can be solved explicitly and we have the solution

$$w(t)=\frac{\mu_1+b\mu_2\exp[-(\gamma/\varepsilon)(\mu_1+\mu_2)(t-t_n)]}{1-b\mu_2\exp[-(\gamma/\varepsilon)(\mu_1+\mu_2)(t-t_n)]},\quad b=\frac{u_i^{n+1/2}-\mu_1}{u_i^{n+1/2}+\mu_2}.$$

By $v_i^{n+1}=\rho_i^{n+1}-u_i^{n+1}$, after some simple calculations, we have

$$\begin{aligned} u_i^{n+1}&=u_i^{n+1/2}-\frac{1}{\gamma}\left[1-\exp\left(-\frac{\gamma}{\varepsilon}\Delta t(\mu_1+\mu_2)\right)\right]G(u_i^{n+1/2},v_i^{n+1/2}),\\ v_i^{n+1}&=v_i^{n+1/2}+\frac{1}{\gamma}\left[1-\exp\left(-\frac{\gamma}{\varepsilon}\Delta t(\mu_1+\mu_2)\right)\right]G(u_i^{n+1/2},v_i^{n+1/2}),\end{aligned}$$

where

$$G(u,v)=\frac{\alpha u-\beta v-\gamma uv}{(u+\mu_2)-(u-\mu_1)\exp[-(\gamma\Delta t/\varepsilon)(\mu_1+\mu_2)]}. \qquad \square$$

In the next section we shall give various estimates for scheme (2.4), (2.6) and prove convergence for fixed $\varepsilon>0$. To study the limit as $\varepsilon\to 0$, we rewrite the above schemes (2.4)–(2.6) in the macroscopic variables ρ_i^n and j_i^n. This gives

$$\begin{aligned}\rho_i^{n+1/2}&=\rho_i^n-\frac{\lambda}{2c}(j_{i+1}^n-j_{i-1}^n)+\frac{\lambda}{2}(\rho_{i+1}^n-2\rho_i^n+\rho_{i-1}^n),\\ j_i^{n+1/2}&=j_i^n-\frac{c\lambda}{2}(\rho_{i+1}^n-\rho_{i-1}^n)+\frac{\lambda}{2}(j_{i+1}^n-2j_i^n+j_{i-1}^n)\end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \rho_i^{n+1} &= \rho_i^{n+1/2}, \\ j_i^{n+1} &= j_i^{n+1/2} - \frac{2c\Delta t}{\varepsilon}\tilde{H}(\rho_i^{n+1/2}, j_i^{n+1/2}), \end{aligned} \tag{2.9}$$

where $\tilde{H}(\rho_i^{n+1/2}, j_i^{n+1/2}) = H(u_i^{n+1/2}, v_i^{n+1/2})$.

Note that we can solve the ODE (2.5) for the source term explicitly, due to the fact that the macroscopic variable ρ remains unchanged in the relaxation step.

3. Estimates for the numerical schemes

In this section we will give various estimates for schemes (2.4)–(2.6) starting with given non-negative initial data of bounded variation satisfying (2.2).

3.1. Nonnegativity

For given nonnegative initial data $(u_0^\varepsilon, v_0^\varepsilon)$ we prove here the nonnegativity for the numerical approximation given by scheme (2.4), (2.6). Denote $U := (u, v)$, write $U \geqslant 0$ if $u \geqslant 0$ and $v \geqslant 0$. Since $U_0^\varepsilon \geqslant 0$,

$$U_i^0 = \frac{1}{\Delta x}\int_{I_i} (u_0^\varepsilon(x), v_0^\varepsilon(x))\,\mathrm{d}x = \frac{1}{\Delta x}\int_{I_i} U_0^\varepsilon(x)\,\mathrm{d}x \geqslant 0, \quad i \in \mathbb{Z}.$$

Theorem 3.1. *Suppose $\lambda < 1$ is satisfied and $U_i^0 = (u_i^0, v_i^0) \geqslant 0$ for any $i \in \mathbb{Z}$, then for any $i \in \mathbb{Z}$, and $n \in \mathbb{Z}^+$ we have*

$$U_i^n \geqslant 0. \tag{3.1}$$

Proof. Since $U_i^0 \geqslant 0$ it suffice to show that if $U_i^k \geqslant 0$, for $0 \leqslant k < n$, $i \in \mathbb{Z}$ then $U_i^{k+1} > 0$. In fact due to the fact that the upwind scheme (2.4) is monotone we have $U_i^{k+1/2} \geqslant 0$. Using the solution operator for (2.5) and Lemma A.1 in the appendix we find

$$U(t) = S_t(0, U_i^{k+1/2}) \geqslant 0 \quad \text{for any } t \in [t_k, t_{k+1}].$$

In particular, one gets

$$U_i^{k+1} = S_{t_{k+1}}(t_k, U_i^{k+1/2}) \geqslant 0$$

which completes the proof of Theorem 3.1. □

3.2. L^∞ bound for v_j^n

Theorem 3.2. *For any $i \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, we have using $(\cdot)_+ = \max(\cdot, 0)$*

$$\sup_{i\in\mathbb{Z}} \left(v_i^n - \frac{\alpha}{\gamma}\right)_+ \leqslant \left\| \left(v^0 - \frac{\alpha}{\gamma}\right)_+ \right\|_\infty \exp\left(-\frac{\beta n \Delta t}{\varepsilon}\right). \tag{3.2}$$

Proof. Observe that the second equation of system (2.5) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(z(t)-\frac{\alpha}{\gamma}\right)=-\frac{\gamma}{\varepsilon}w(t)\left(z(t)-\frac{\alpha}{\gamma}\right)-\frac{\beta}{\varepsilon}\left(z(t)-\frac{\alpha}{\gamma}\right)-\frac{\alpha\beta}{\varepsilon\gamma}. \tag{3.3}$$

Note that for any differentiable function the distributional derivative of $(f)_+$ on $]0,\infty[$ satisfies

$$\left\langle\frac{\mathrm{d}(f)_+}{\mathrm{d}t},\phi\right\rangle=-\langle(f)_+,\phi_t\rangle=-\int_{f(t)>0}f\phi_t\,\mathrm{d}t=\langle\chi_{f(t)>0}f_t,\phi\rangle$$

for any test function $\phi\in C_0^\infty(]0,\infty[)$. Since, for $t\in[t_n,t_{n+1}]$,

$$(w,z)(t)=S_t(t_n,U_i^{n+1/2})\geqslant 0,$$

we obtain by multiplication Eq. (3.3) by $\chi_{\{z(t)-\alpha/\gamma>0\}}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(z(t)-\frac{\alpha}{\gamma}\right)_+\leqslant-\frac{\beta}{\varepsilon}\left(z(t)-\frac{\alpha}{\gamma}\right)_+.$$

Thus, using $z(t_n)=v_i^{n+1/2}$, we have

$$\left(z(t)-\frac{\alpha}{\gamma}\right)_+\leqslant\left\|\left(v^{n+1/2}-\frac{\alpha}{\gamma}\right)_+\right\|_\infty\exp\left(-\frac{\beta(t-t_n)}{\varepsilon}\right)\quad\text{for } t\in[t_n,t_{n+1}].$$

Noting that $v_i^{n+1/2}=(1-\lambda)v_i^n+\lambda v_{i+1}^n$ as a convex combination for $\lambda\leqslant 1$ leads to

$$\sup_{i\in\mathbb{Z}}\left(v_i^{n+1/2}-\frac{\alpha}{\gamma}\right)_+\leqslant\sup_{i\in\mathbb{Z}}\left(v_i^n-\frac{\alpha}{\gamma}\right)_+=\left\|\left(v^n-\frac{\alpha}{\gamma}\right)_+\right\|_\infty.$$

Therefore, $v_i^{n+1}:=z(t_{n+1})$ for $i\in\mathbb{Z}$, satisfies

$$\begin{aligned}\left(v_i^{n+1}-\frac{\alpha}{\gamma}\right)_+&\leqslant\left\|\left(v^{n+1/2}-\frac{\alpha}{\gamma}\right)_+\right\|_\infty\exp\left(-\frac{\beta\Delta t}{\varepsilon}\right)\\&\leqslant\left\|\left(v^n-\frac{\alpha}{\gamma}\right)_+\right\|_\infty\exp\left(-\frac{\beta\Delta t}{\varepsilon}\right),\end{aligned}$$

where we have used the monotonicity property of the linear scheme (2.4). By induction one finds that

$$\sup_{i\in\mathbb{Z}}\left(v_i^n-\frac{\alpha}{\gamma}\right)_+\leqslant\left\|\left(v^0-\frac{\alpha}{\gamma}\right)_+\right\|_\infty\exp\left(-\frac{\beta n\Delta t}{\varepsilon}\right)$$

which completes the proof of Theorem 3.2. □

Remark. Theorem 3.2 yields the estimate $\lim_{\varepsilon\to 0^+}v_i^n\leqslant\alpha/\gamma$.

3.3. L^1 stability

Theorem 3.3 (Stability). *For any $i\in\mathbb{Z}$ and $n\in\mathbb{N}$, let U_i^n and $\tilde{U}_i^n$ be solutions of schemes* (2.4)–(2.6), *corresponding to the initial data U_i^0 and $\tilde{U}_i^0$, respectively, where U_i^0 and $\tilde{U}_i^0$ satisfy* (2.2).

We set $|U_i^0|_1 = |u_i^0| + |v_i^0|$, *the usual* 1-*norm on* $\mathbb{R}^2$. *Then if* $n > 0$, *there exists a constant* C *not depending on* ε, $C = \exp((2\gamma/\beta)\|(v^0 - \alpha/\gamma)_+\|_\infty)$, *such that for any given positive integer* $I \in \mathbb{Z}^+$,

$$\sum_{|i|\leqslant I} |U_i^n - \tilde{U}_i^n|_1 \Delta x \leqslant C \sum_{|i|\leqslant I+n} |U_i^0 - \tilde{U}_i^0|_1 \Delta x. \tag{3.4}$$

Proof. By Lemma A.2 in the appendix with

$$(w_0, z_0) = (u_i^{n+1/2}, v_i^{n+1/2}) \quad \text{and} \quad (\tilde{w}_0, \tilde{z}_0) = (\tilde{u}_i^{n+1/2}, \tilde{v}_i^{n+1/2}),$$

one has

$$\begin{aligned} &|u_i^{n+1} - \tilde{u}_i^{n+1}| + |v_i^{n+1} - \tilde{v}_i^{n+1}| \\ &\quad \leqslant \exp\left(\frac{2\gamma}{\beta} \left\| \left(v^{n+1/2} - \frac{\alpha}{\gamma}\right)_+ \right\|_\infty \left[1 - \exp\left(-\frac{\beta\Delta t}{\varepsilon}\right)\right]\right) [|u_i^{n+1/2} - \tilde{u}_i^{n+1/2}| + |v_i^{n+1/2} - \tilde{v}_i^{n+1/2}|]. \end{aligned}$$

We define

$$T(n, I) := \sum_{|i|\leqslant I} [|u_i^n - \tilde{u}_i^n| + |v_i^n - \tilde{v}_i^n|].$$

The monotone scheme (2.4) being an L^1-contraction [5], implies that

$$T(n + \tfrac{1}{2}, I) \leqslant T(n, I + 1).$$

Noting that

$$\sup_{i\in\mathbb{Z}} \left(v_i^{n+1/2} - \frac{\alpha}{\gamma}\right)_+ \leqslant \left\| \left(v^n - \frac{\alpha}{\gamma}\right)_+ \right\|_\infty.$$

Then a recursive argument gives

$$\begin{aligned} T(n+1, I) &\leqslant \exp\left(\frac{2\gamma}{\beta} \left\| \left(v^n - \frac{\alpha}{\gamma}\right)_+ \right\|_\infty \left[1 - \exp\left(-\frac{\beta\Delta t}{\varepsilon}\right)\right]\right) T(n, I+1) \\ &\leqslant \exp\left[\frac{2\gamma}{\beta} \left(\left\| \left(v^0 - \frac{\alpha}{\gamma}\right)_+ \right\|_\infty + \cdots + \left\| \left(v^n - \frac{\alpha}{\gamma}\right)_+ \right\|_\infty \right) \right. \\ &\qquad \left. \cdot \left(1 - \exp\left(-\frac{\beta\Delta t}{\varepsilon}\right)\right)\right] T(0, I+n+1). \end{aligned}$$

Now using (3.2) in Theorem 3.2 this gives

$$\begin{aligned} T(n+1, I) &\leqslant \exp\left[\frac{2\gamma}{\beta} \left\| \left(v^0 - \frac{\alpha}{\gamma}\right)_+ \right\|_\infty \left(1 + \exp\left(-\frac{\beta\Delta t}{\varepsilon}\right) + \cdots + \exp\left(-\frac{\beta n\Delta t}{\varepsilon}\right)\right) \right. \\ &\qquad \left. \cdot \left(1 - \exp\left(-\frac{\beta\Delta t}{\varepsilon}\right)\right)\right] T(0, I+n+1) \\ &\leqslant \exp\left(\frac{2\gamma}{\beta} \left\| \left(v^0 - \frac{\alpha}{\gamma}\right)_+ \right\|_\infty \right) T(0, I+n+1). \end{aligned} \tag{3.5}$$

Thus Theorem 3.3 is proved. □

It is well known that L^1-stability implies the following BV estimate.

Theorem 3.4. *Let* $U_\Delta^n=(u^n,v^n)$ *be a numerical solution to* (2.4), (2.6). *We have for* $n=1,\dots,N$

$$\mathrm{TV}(U_\Delta^n)\leqslant C\,\mathrm{TV}(U_\Delta^0) \tag{3.6}$$

with $C=\exp((2\gamma/\beta)\|(v^0-\frac{\alpha}{\gamma})_+\|_\infty)$.

Proof. In (3.4) we choose $\tilde{U}_\Delta^n=(u_{i-1}^n,v_{i-1}^n)$, $U_\Delta^n=(u_i^n,v_i^n)$, then take $I\to\infty$ to obtain

$$\mathrm{TV}(U_\Delta^{n+1})=\sum_{i\in\mathbb{Z}}[|u_i^{n+1}-u_{i-1}^{n+1}|+|v_i^{n+1}-v_{i-1}^{n+1}|]$$
$$\leqslant \exp\left(\frac{2\gamma}{\beta}\left\|\left(v^0-\frac{\alpha}{\gamma}\right)_+\right\|_\infty\right)\mathrm{TV}(U_\Delta^0)=C\,\mathrm{TV}(U_\Delta^0).$$

The proof is complete. □

3.4. Local equilibrium estimate

Now, we turn to showing that the flux j_i^n is close to $F(\rho_i^n)$ for all $n\in\mathbb{Z}^+$ and $i\in\mathbb{Z}$. To this end we consider the equivalent scheme in macroscopic variables (ρ,j), (2.8), (2.9), where (2.9) is obtained by solving the following ODE in $t\in[t_n,t_{n+1}]$, i.e.,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=0,$$
$$\frac{\mathrm{d}j}{\mathrm{d}t}=-\frac{1}{\varepsilon}B(\rho,j), \tag{3.7}$$

$$(\rho,j)(t=t_n)=(\rho_i^{n+1/2},j_i^{n+1/2}),\quad i\in\mathbb{Z},\ n=1,\dots,N$$

with

$$B(\rho,j)=\frac{\gamma}{2c}[j-F(\rho)][j+G(\rho)],$$

as well as F, G as defined in (1.4), (1.5). A direct computation shows that $|F'(\rho)|\leqslant c$ and $F(0)=0$.

Theorem 3.5. *Suppose that the initial data* (u_i^0,v_i^0) *satisfy* (2.2). *Let* $(\rho_\Delta^n,j_\Delta^n)$ *be numerical approximations generated by scheme* (2.8), (2.9) *with respect to the initial data* $\rho^0=u^0+v^0$, $j^0=c(u^0-v^0)$. *Then for all* $n=1,\dots,N$,

$$j_i^n+G(\rho_i^n)\geqslant\frac{2\beta c}{\gamma},\quad i\in\mathbb{Z} \tag{3.8}$$

and

$$\|j^n-F(\rho^n)\|_1\leqslant\exp\left(-\frac{\beta n\Delta t}{\varepsilon}\right)\|j^0-F(\rho^0)\|_1+\frac{4c\varepsilon}{\beta}\exp\left(\frac{2\gamma}{\beta}\left\|\left(v^0-\frac{\alpha}{\gamma}\right)_+\right\|_\infty\right)\mathrm{TV}(u^0,v^0). \tag{3.9}$$

Proof. In view of definition (1.5), inequality (3.8) can be proved as by Gabetta and Perthame [6]. To prove (3.9), let us first consider the relaxation step (3.7). Set

$$Z(t) = j(t) - F(\rho(t)).$$

From (3.7), using $\mathrm{d}\rho/\mathrm{d}t = 0$, we have

$$\begin{aligned}\frac{\mathrm{d}Z}{\mathrm{d}t} &= \frac{\mathrm{d}[j(t) - F(\rho(t))]}{\mathrm{d}t} = \frac{\mathrm{d}j}{\mathrm{d}t}\\ &= -\frac{\gamma}{2c\varepsilon}[j - F(\rho)][j + G(\rho)] = -\frac{\gamma}{2c\varepsilon}Z(t)[j + G(\rho)].\end{aligned}$$

Standard regularization of the signum function, which we omit here, yields that

$$\frac{\mathrm{d}|Z(t)|}{\mathrm{d}t} = -\frac{\gamma}{2c\varepsilon}|Z(t)|[j + G(\rho)] \leqslant -\frac{\beta}{\varepsilon}|Z(t)|,$$

where (3.8) is used. From this

$$|Z(t)| \leqslant |Z(\tau)| \exp\left(-\frac{\beta}{\varepsilon}(t-\tau)\right), \quad t > \tau \tag{3.10}$$

follows. Thus $Z(t_{n+1}) = Z_i^{n+1}$ is bounded from above by $Z(t = t_n) = Z_i^{n+1/2}$ in the following manner:

$$|Z_i^{n+1}| \leqslant |Z_i^{n+1/2}| \exp\left(-\frac{\beta}{\varepsilon}\Delta t\right). \tag{3.11}$$

Next, we estimate $|Z_i^{n+1/2}|$ in terms of $|Z_i^n|$ in the convection step (2.4). By definition and using for some intermediate value ξ_i^n between $\rho_i^{n+1/2}$ and ρ_i^n

$$|F'(\xi_i^n)| = \left|\frac{F(\rho_i^{n+1/2}) - F(\rho_i^n)}{\rho_i^{n+1/2} - \rho_i^n}\right| \leqslant c,$$

we have

$$\begin{aligned}Z_i^{n+1/2} - Z_i^n &= j_i^{n+1/2} - j_i^n - [F(\rho_i^{n+1/2}) - F(\rho_i^n)]\\ &= -c\frac{\lambda}{2}(\rho_{i+1}^n - \rho_{i-1}^n) + \frac{\lambda}{2}(j_{i+1}^n - 2j_i^n + j_{i-1}^n)\\ &\quad -F'(\xi_i^n)\left[\frac{\lambda}{2}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n) - \frac{\lambda}{2c}(j_{i+1}^n - j_{i-1}^n)\right]\\ &\leqslant c\lambda[|\rho_{i+1}^n - \rho_i^n| + |\rho_i^n - \rho_{i-1}^n|] + \lambda[|j_{i+1}^n - j_i^n| + |j_i^n - j_{i-1}^n|].\end{aligned}$$

Summing up the above inequalities over $i \in \mathbb{Z}$, and noting that $\rho = u + v$ and $j = c(u - v)$, we have, by Theorem 3.4,

$$\begin{aligned}\sum_i |Z_i^{n+1/2}| &\leqslant \sum_i |Z_i^n| + 4c\lambda \sum_i [|u_{i+1}^n - u_i^n| + |v_{i+1}^n - v_i^n|]\\ &\leqslant \sum_i |Z_i^n| + 4c\lambda \exp\left(\frac{2\gamma}{\beta}\left\|\left(v^0 - \frac{\alpha}{\gamma}\right)_+\right\|_\infty\right)\mathrm{TV}(U_\Delta^0).\end{aligned} \tag{3.12}$$

The combination of (3.11) with (3.12) yields

$$\sum_i |Z_i^{n+1}| \leqslant \sum_i |Z_i^{n+1/2}|\exp\left(-\frac{\beta}{\varepsilon}\Delta t\right)$$
$$\leqslant \left[\sum_i |Z_i^n| + 4c\lambda\exp\left(\frac{2\gamma}{\beta}\left\|\left(v^0-\frac{\alpha}{\gamma}\right)_+\right\|_\infty\right)\mathrm{TV}(U_\Delta^0)\right]\exp\left(-\frac{\beta}{\varepsilon}\Delta t\right). \quad (3.13)$$

Setting

$$H(n) := \sum_i |j_i^n - F(\rho_i^n)|\Delta x = \|Z^n\|_1$$

and

$$\omega := 4c\exp\left(\frac{2\gamma}{\beta}\left\|\left(v^0-\frac{\alpha}{\gamma}\right)_+\right\|_\infty\right)\mathrm{TV}(U_\Delta^0) > 0,$$

we obtain from (3.13) the estimate

$$H(n+1)\exp\left(\frac{\beta}{\varepsilon}\Delta t\right) \leqslant H(n) + \omega\Delta t.$$

By a recursive argument this yields

$$H(n+1) \leqslant \exp\left(-\frac{\beta}{\varepsilon}(n+1)\Delta t\right)H(0) + \frac{1-\exp(-((n+1)/\varepsilon)\Delta t)}{1-\exp(-(\beta/\varepsilon)\Delta t)}\exp\left(-\frac{\beta}{\varepsilon}\Delta t\right)\omega\Delta t$$
$$\leqslant \exp\left(-\frac{\beta}{\varepsilon}(n+1)\Delta t\right)H(0) + \frac{\omega}{\beta}\varepsilon.$$

This implies (3.9). □

Based on the L^1-stability and the locally equilibrium estimate (3.9), we show that the difference approximations are L_1(locally) Lipschitz continuous in time t.

Theorem 3.6. *If* $0 < \lambda \leqslant 1$, $\lambda = c\Delta t/\Delta x$, *then there exists a positive constant* $L(\varepsilon) > 0$, *independent of* Δt *and* Δx *such that if* $k > p > 0$,

$$\|U^k - U^p\|_1 \leqslant L(\varepsilon)(k-p)\Delta t. \quad (3.14)$$

If $\|j^0 - F(\rho^0)\|_1 = \mathrm{O}(\varepsilon)$, *then* $L(\varepsilon) = L$ *is independent of* ε. *Furthermore*,

$$\|\rho^k - \rho^p\|_1 \leqslant L(k-p)\Delta t. \quad (3.15)$$

Proof. To prove (3.14), we need to estimate $\|U^{n+1} - U^n\|_1$. Using Theorem 3.3 with $\tilde{u}_i^{n+1} = u_i^n$ we obtain

$$\|U^{n+1} - U^n\|_1 \leqslant \exp\left(\frac{2\gamma}{\beta}\left\|\left(v^0-\frac{\alpha}{\gamma}\right)_+\right\|_\infty\right)\|U^1 - U^0\|_1. \quad (3.16)$$

By scheme (2.4),

$$\|U^{1/2}-U^0\|_1=\sum_i[|u_i^{1/2}-u_i^0|+|v_i^{1/2}-v_i^0|]\Delta x$$
$$\leqslant \lambda \mathrm{TV}(U^0)\Delta x=\mathrm{TV}(U^0)\Delta t. \tag{3.17}$$

Noting that $\rho_i^{n+1}=\rho_i^{n+1/2}$, to estimate $\|U^1-U^{1/2}\|_1$ one only needs to estimate $\|j^1-j^{1/2}\|_1$. In fact by (3.7) with $j(t_0)=j_i^{1/2}$ and $j(\Delta t)=j_i^1$ we have

$$j_i^1-j_i^{1/2}=\int_0^{\Delta t}\frac{\mathrm{d}j(t)}{\mathrm{d}t}\,\mathrm{d}t=\int_0^{\Delta t}-\frac{\gamma}{2c\varepsilon}[j(t)-F(\rho(t))][j(t)+G(\rho(t))]\,\mathrm{d}t.$$

Thus, setting $Z(t)=j(t)-F(\rho)$ we obtain

$$\sum_i|j_i^1-j_i^{1/2}|\Delta x\leqslant\frac{C}{\varepsilon}\sum_i\left(\int_0^{\Delta t}|Z(t)\,\mathrm{d}t\right)\Delta x$$

with

$$C=\frac{\gamma}{2c}\max_{0\leqslant t\leqslant\Delta t}[j(t)+G(\rho(t))].$$

By (3.10) and (3.12) one gets

$$\sum_i|j_i^1-j_i^{1/2}|\Delta x\leqslant\frac{C}{\varepsilon}\sum_i|Z_i^{1/2}|\Delta x\int_0^{\Delta t}\exp\left(-\frac{\beta\tau}{\varepsilon}\right)\,\mathrm{d}\tau$$
$$\leqslant\frac{C_1}{\beta}\left[1-\exp\left(-\frac{\beta}{\varepsilon}\Delta t\right)\right]\left(\sum_i|Z_i^0|\Delta x+4cC_2\,\mathrm{TV}(U^0)\Delta t\right)$$
$$\leqslant\begin{cases}C(\varepsilon)\Delta t & \text{if } \|Z^0\|_1=\mathrm{O}(1),\\ C_3\Delta t & \text{if } \|Z^0\|_1=\mathrm{O}(\varepsilon),\ C_3 \text{ is independent of } \varepsilon,\end{cases}$$

which combined with $\rho_i^1=\rho_i^{1/2}$ implies

$$\|U^1-U^{1/2}\|_1\leqslant C(\varepsilon)\Delta t. \tag{3.18}$$

Hence estimates (3.16)–(3.18) yield

$$\|U^{n+1}-U^n\|_1\leqslant C(\varepsilon)\Delta t,$$
$$\|U^k-U^p\|_1\leqslant\sum_{n=p}^{k-1}\|U^{n+1}-U^n\|_1\leqslant C(\varepsilon)(k-p)\Delta t,\quad p<k$$

which proves (3.14). Estimate (3.15) follows from (3.16) and (3.18) and the fact that $\rho_i^1=\rho_i^{1/2}$. □

Consider the family of approximate solutions $(U_\Delta^\varepsilon(x,t))_{\Delta t>0}$ defined in (2.1) as

$$U_\Delta^\varepsilon(x,t)=(u_\Delta^\varepsilon,v_\Delta^\varepsilon)(x,t)=\sum_{n=0}^N\sum_{i\in\mathbb{Z}}(u_i^n,v_i^n)\chi_{[x_{i-1/2},x_{i+1/2}[}(x)\chi_{[t_n,t_{n+1}[}(t), \tag{3.19}$$

the corresponding solutions in macroscopic variables are

$$\begin{aligned}(\rho_\Delta^\varepsilon, j_\Delta^\varepsilon)(x,t) &= (u_\Delta^\varepsilon + v_\Delta^\varepsilon), c\,(u_\Delta^\varepsilon - v_\Delta^\varepsilon))\,(x,t)\\ &= \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}(u_i^{n,\varepsilon} + v_i^{n,\varepsilon}, c(u_i^{n,\varepsilon} - v_i^{n,\varepsilon}))\chi_{[x_{i-1/2},x_{i+1/2}[}(x)\chi_{[t_n,t_{n+1}[}(t)\\ &= \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}(\rho_i^n, j_i^n)\chi_{I_i}(x)\chi_{[t_n,t_{n+1}[}(t),\end{aligned} \tag{3.20}$$

where (ρ_i^n, j_i^n) are the solutions of schemes (2.8), (2.9) with initial data defined through (u_i^0, v_i^0).

Let us present our main estimates as follows.

Theorem 3.7. *Suppose* (u^0, v^0) *satisfy* (2.2). *Let* $U_\Delta^\varepsilon(x,t)$ *be the numerical solution generated by* (2.4), (2.6) *with respect to the initial data* $U_i^0 = (u_i^0, v_i^0)$. *Then there exists a constant* C_0 *not depending on* ε, $C_0 \geqslant \exp((2\gamma/\beta)\|(v^0 - \alpha/\gamma)_+\|_\infty)$, *such that the following estimates hold*:

$$0 \leqslant u_\Delta^\varepsilon \leqslant C_0, \quad 0 \leqslant v_\Delta^\varepsilon \leqslant \frac{\alpha}{\gamma} + \left\|\left(v^0 - \frac{\alpha}{\gamma}\right)_+\right\|_\infty, \tag{3.21}$$

$$\mathrm{TV}(U_\Delta^\varepsilon) \leqslant C_0\,\mathrm{TV}(U_0^\varepsilon); \tag{3.22}$$

$$\|U_\Delta^\varepsilon\|_1 \leqslant C_0 \|U_0^\varepsilon\|_1; \tag{3.23}$$

$$\|U_\Delta^\varepsilon(t) - U_\Delta^\varepsilon(t')\|_1 \leqslant C(\varepsilon)(\Delta t + |t - t'|). \tag{3.24}$$

Here $C(\varepsilon) = C$ *is independent of* ε *if* $\|j_0^\varepsilon - F(\rho_0^\varepsilon)\|_1 = \mathrm{O}(\varepsilon)$.

For the solution in macroscopic variables $(\rho_\Delta^\varepsilon, j_\Delta^\varepsilon)(x,t)$, we have

Theorem 3.8. *Under the assumptions in Theorem* 3.7, *there exists a positive constant* C_1 *not depending on* ε, *such that we obtain the estimates*

$$0 \leqslant \rho_\Delta^\varepsilon \leqslant C_4, \tag{3.25}$$

$$\mathrm{TV}(\rho_\Delta^\varepsilon) \leqslant C_4 \mathrm{TV}(U_0^\varepsilon), \tag{3.26}$$

$$\|\rho_\Delta^\varepsilon\|_1 \leqslant C_4 \|U_0^\varepsilon\|_1, \tag{3.27}$$

$$\|\rho_\Delta^\varepsilon(t) - \rho_\Delta^\varepsilon(t')\|_1 \leqslant C_4(\Delta t + |t - t'|), \tag{3.28}$$

$$\|j_\Delta^\varepsilon - f(\rho_\Delta^\varepsilon)\|_1 \leqslant C_4\varepsilon, \text{ if } \|j_0^\varepsilon - F(\rho_0^\varepsilon)\|_1 = \mathrm{O}(\varepsilon). \tag{3.29}$$

Proof. Using $\rho_\Delta^\varepsilon = u_\Delta^\varepsilon + v_\Delta^\varepsilon$, estimates in Theorem 3.7, and (3.9), one proves Theorem 3.8 by just choosing C_4 as

$$C_4 = \max\left\{\exp\left(\frac{2\gamma}{\beta}\left\|\left(v_0^\varepsilon - \frac{\alpha}{\gamma}\right)_+\right\|_\infty\right), \frac{4c}{\beta}\mathrm{TV}(U^\varepsilon)\exp\left(\frac{2\gamma}{\beta}\left\|\left(v_0^\varepsilon - \frac{\alpha}{\gamma}\right)_+\right\|_\infty\right), C_0\right\}. \qquad \square$$

The above estimates ensure the convergence result in the following section.

4. Convergence of the numerical scheme

First, we prove the convergence of $((u^\varepsilon_\Delta, v^\varepsilon_\Delta)(x,t))_{\Delta t>0}$ for fixed ε.

Theorem 4.1. *Let* $\varepsilon > 0$ *and suppose that* (H_1) *holds. For any* $T > 0$, *let the CFL number* $\lambda = c\Delta t/\Delta x$ *be constant. As* $\Delta t \to 0$, *the sequence* $U^{(\varepsilon)}_\Delta = (u^\varepsilon_\Delta, v^\varepsilon_\Delta)$ *converges in* $L^\infty(0,T;L^1(\mathbb{R})^2)$ *to the unique solution* $U^{(\varepsilon)}$ *of* (1.2) *with initial data* $U^\varepsilon_0(x) = (u^{(\varepsilon)}_0(x), v^{(\varepsilon)}_0(x))$ *satisfying* (H_1), $U^{(\varepsilon)} \in C^0([0,T];L^1(\mathbb{R})^2) \cap L^\infty(0,T;L^\infty(\mathbb{R})^2)$ *and the following estimates hold*:

$$0 \leqslant u^{(\varepsilon)}(x,t) \leqslant C\,\mathrm{TV}(u^\varepsilon_0), \tag{4.1}$$

$$0 \leqslant v^\varepsilon(x,t) \leqslant \frac{\alpha}{\gamma} + \left\| \left(v^\varepsilon_0 - \frac{\alpha}{\gamma} \right)_+ \right\|_\infty \exp\left(-\frac{\beta t}{\varepsilon} \right), \tag{4.2}$$

$$\mathrm{TV}(u^\varepsilon(\cdot,t), v^\varepsilon(\cdot,t)) \leqslant \exp\left(\frac{2\gamma}{\beta} \left\| \left(v^\varepsilon_0 - \frac{\alpha}{\gamma} \right)_+ \right\|_\infty \right) \mathrm{TV}(u^\varepsilon_0, v^\varepsilon_0) \quad \text{for } t \in [0,T], \tag{4.3}$$

$$\|(u^\varepsilon(\cdot,t), v^\varepsilon(\cdot,t))\|_1 \leqslant \exp\left(\frac{2\gamma}{\beta} \left\| \left(v^\varepsilon_0 - \frac{\alpha}{\gamma} \right)_+ \right\|_\infty \right) \|(u^\varepsilon_0, v^\varepsilon_0)\|_1, \tag{4.4}$$

$$\|(U^\varepsilon(\cdot,t) - U^\varepsilon(\cdot,t'))\| \leqslant C(\varepsilon)|t-t'|, \quad \forall t,\ t' \in [0,T]. \tag{4.5}$$

Proof. Consider $\varepsilon > 0$ fixed. Equipped with Theorem 3.7 we may apply standard arguments related to Helley's compactness principle to claim that there exists a subsequence $\Delta t_k \to 0$ such that $(u^\varepsilon_{\Delta t_k}, v^\varepsilon_{\Delta t_k})$ tends to a limit pair $(u^\varepsilon, v^\varepsilon)(x,t)$ bounded almost everywhere in $\mathbb{R}^+ \times \mathbb{R}$. This limit pair $(u^\varepsilon, v^\varepsilon)$ satisfies (4.1)–(4.5), which is derived easily from the estimates in Theorem 3.7. Since scheme (2.4), (2.6) are conservative and consistent difference schemes of system (1.2) the limit functions $(u^\varepsilon, v^\varepsilon)$ are weak solutions of this system in the sense of distributions by the Lax–Wendroff theorem. This fact combined with the uniqueness of weak solutions [6] implies the convergence of the whole sequence. □

Since $\rho = u + v$, $j = c(u - v)$ give a one to one linear mapping from (u,v) to (ρ,j), we also have the convergence of $((\rho^\varepsilon_\Delta,\ j^\varepsilon_\Delta)(x,t))_{\Delta t>0}$ for fixed ε.

Theorem 4.2. *We consider the assumptions of Theorem* 4.1. *As* $\Delta t \to 0$, *the sequence* $(\rho^\varepsilon_\Delta, j^\varepsilon_\Delta)$ *converges in* $L^\infty(0,T;L^1(\mathbb{R})^2)$ *to the unique solution* $(\rho^{(\varepsilon)}, j^{(\varepsilon)})\,(x,t)$ *of* (1.3) *with initial data* $(\rho^\varepsilon_0, j^\varepsilon_0) = (u^\varepsilon_0 + v^\varepsilon_0, c(u^\varepsilon_0 - v^\varepsilon_0))(x)$ *with*

$$(\rho^\varepsilon, j^\varepsilon) \in C^0([0,T];L^1(\mathbb{R})^2) \cap L^\infty(0,T;L^\infty(\mathbb{R})^2).$$

We obtain the following estimates:

$$0 \leqslant \rho^{(\varepsilon)}(x,t) \leqslant \tilde{M}, \quad |j^\varepsilon(x,t)| \leqslant \tilde{M}, \tag{4.6}$$

$$\mathrm{TV}(\rho^{\varepsilon}) \leqslant C_1 \mathrm{TV}(U_0^{\varepsilon}), \tag{4.7}$$

$$\|\rho^{\varepsilon}(\cdot,t) - \rho^{\varepsilon}(\cdot,t')\|_1 \leqslant C_1 |t-t'| \tag{4.8}$$

and

$$\|j^{\varepsilon} - F(\rho^{\varepsilon})\|_1 \leqslant C_1 \varepsilon, \quad \textit{if } \|j_0^{\varepsilon} - F(\rho_0^{\varepsilon})\|_1 = \mathrm{O}(\varepsilon). \tag{4.9}$$

Next, we investigate the behavior of the above numerical schemes as the relaxation parameter ε tends to zero. For fixed $\Delta t > 0$, letting $\varepsilon \to 0$, (3.29) implies for $n = 0, \dots, N$

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + \frac{1}{2\Delta x}(j_{i+1}^n - j_{i-1}^n) - \frac{c}{2\Delta x}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n) = 0, \quad j_i^n = F(\rho_i^n). \tag{4.10}$$

Here the estimates in Theorem 3.8 allow us to prove the convergence and stability properties of the relaxed scheme (4.10).

Theorem 4.3. *Suppose that* (2.2) *holds and fix* $\Delta t > 0$. *As* $\varepsilon \to 0$ *the solution of* (2.8)–(2.9) $\rho_{\Delta}^{n,\varepsilon}$, *converges in* $L^{\infty}(0,T; L^1_{\mathrm{loc}}(\mathbb{R}))$ *to a limit* ρ_{Δ}^n, *and* $j_{\Delta}^{n,\varepsilon} \to F(\rho_{\Delta}^n) = j_{\Delta}^n$ *in* $L^{\infty}([v,T];\ L^1_{\mathrm{loc}}(\mathbb{R}))$ *as* $\varepsilon \to 0^+$ *and* $v > 0$. *Moreover, if* $\|j_0^{\varepsilon} - F(\rho_0^{\varepsilon})\|_1 = \mathrm{O}(\varepsilon)$ *one can take* $v = 0$. *The limit* $(\rho_{\Delta}^n, j_{\Delta}^n)$ *satisfies the estimates*

$$0 \leqslant \rho_{\Delta}(t) \leqslant \tilde{M}, \quad \forall t \in [0,T], \tag{4.11}$$

$$\mathrm{TV}(\rho_{\Delta}^n) \leqslant C_1 \mathrm{TV}(U_{\Delta}^0), \tag{4.12}$$

$$j_{\Delta}^n = F(\rho_{\Delta}^n), \quad 0 \leqslant n \leqslant N, \tag{4.13}$$

$$\|\rho_{\Delta}(t) - \rho_{\Delta}(t')\|_1 \leqslant C_1(\Delta t + |t-t'|), \quad \forall t,\ t' \in [0,t]. \tag{4.14}$$

Proof. By Theorem 3.8, for $n \in \{0,\dots,N\}$, the sequence $\{\rho_{\Delta}^{n,\varepsilon}\}$ is bounded in $L^1(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$. Then there exists a sequence $\varepsilon_k \to 0$ such that $\rho_{\Delta}^{n,\varepsilon_k}$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ to $\rho_{\Delta}^n = \sum_i \rho_i^n \chi_{[x_{i-1/2}, x_{i+1/2}[}(x)$ and $\rho_i^{n,\varepsilon_k} \to \rho_i^n$.

Define

$$\rho_{\Delta}(x,t) = \sum_{n=0}^{N} \sum_{i \in \mathbb{Z}} \rho_i^n \chi_{[x_{i-1/2}, x_{i+1/2}[}(x) \chi_{[t_n, t_{n+1}[}(t),$$

we have

$$\rho_{\Delta}^{\varepsilon_k}(x,t) \to \rho_{\Delta}(x,t) \quad \text{in } L^{\infty}(0,T; L^1_{\mathrm{loc}}(\mathbb{R})).$$

Estimates (4.11)–(4.14) are an immediate consequence of the estimates in Theorem 3.8. If $\|j^{\varepsilon,0} - F(\rho^{\varepsilon,0})\|_1 = \mathrm{O}(\varepsilon)$, then (3.29) implies

$$\|j_{\Delta}^{\varepsilon} - F(\rho_{\Delta}^{\varepsilon})\|_1 \leqslant C_1 \varepsilon.$$

As $\varepsilon \to 0$, we have $j_\Delta^{n,\varepsilon} \to F(\rho_\Delta^n)$ in $L^\infty([0,T], L^1_{\mathrm{loc}}(\mathbb{R}))$. Then taking the limit $\varepsilon \to 0$ for the equation of (2.9), we get

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} = -\frac{1}{2\Delta x}(j_{i+1}^n - j_{i-1}^n) + \frac{c}{2\Delta x}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n),$$

$$j_i^n = F(\rho_i^n), \quad n \geqslant 0.$$

Thus ρ_Δ^n is unique and the whole sequence converges. □

Substituting $F(\rho_i^n)$ for j_i^n in the first equation of (4.10), we get the scheme

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} + \frac{1}{2\Delta x}(F(\rho_{i+1}^n) - F(\rho_{i-1}^n)) - \frac{c}{2\Delta x}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n) = 0. \tag{4.15}$$

Since $|F'(\xi)| \leqslant c$, the relaxed scheme (4.15) associated with (2.4)–(2.6) is a monotone and consistent scheme, which is consistent with any entropy condition [5]. This fact allows us to prove the convergence of relaxed scheme (4.15) towards the entropy solution of the initial value problem for the conservation law, i.e.,

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}F(\rho) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{4.16}$$

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}. \tag{4.17}$$

We use the entropy conditions of Kruzkov [12].

Definition 1. A function $\rho \in L^\infty(\mathbb{R} \times [0,T])$ is an entropy solution of (4.16) and (4.17) if for any $d \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{R} \times [0,T])$, $\phi \geqslant 0$, we have

$$\iint \left(\frac{\partial}{\partial t}\phi|\rho - d| + \mathrm{sgn}(\rho - d)(F(\rho) - F(d))\frac{\partial}{\partial x}\phi \right) \mathrm{d}x\, \mathrm{d}t \geqslant 0.$$

Theorem 4.4. *Suppose* (2.2) *is satisfied. The numerical relaxed solution* $\rho_\Delta(x,t)$ *related to* (2.4)–(2.6) *converges in* $L^\infty(0,T;L^1(\mathbb{R}))$ *to the unique entropy solution of* (4.16), (4.17) *with* $\rho(t{=}0){=}\rho_0$, *as* $\Delta t \to 0$ *and* $\Delta t/\Delta x$ *is kept constant.*

5. Error estimates

In the previous sections we have proved that the difference approximation (2.8), (2.9) converges to the entropy solution of the scalar conservation law

$$\frac{\partial}{\partial t}[\rho(x,t)] + \frac{\partial}{\partial x}[F(\rho(x,t))] = 0, \quad t > 0, \; x \in \mathbb{R} \tag{5.1}$$

with initial data $(u_0^\varepsilon, v_0^\varepsilon)$ satisfying (H_1) and (H_2). From expression (1.4) for F we have

$$F''(\rho) = \frac{4c\alpha\beta}{\gamma^2(\rho^2 + 2[(\beta-\alpha)/\gamma]\rho + (\beta+\alpha/\gamma)^2)^{3/2}} \geqslant a > 0.$$

This facts allows us to apply the Lip′ theory developed in [18,22] to investigate the convergence rate of the approximate solution generated by (2.8) and (2.9).

Consider the approximation (u_i^n, v_i^n) of (1.2) generated by scheme (2.4)–(2.6), then the macroscopic variables (ρ_i^n, j_i^n) are generated by scheme (2.8)–(2.9). Under the assumptions of Theorem 4.3 and for any fixed Δt the limit for $\varepsilon \to 0$ of $\rho_i^{n,\varepsilon}$ converges to the solution (ρ_i^n) of the relaxed scheme

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + \frac{1}{2\Delta x}(F(\rho_{i+1}^n) - F(\rho_{i-1}^n)) + \frac{c}{2\Delta x}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n) = 0. \tag{5.2}$$

This is a first-order scheme for the conservation law (5.1). Therefore, the numerical relaxed solution $\{\rho_j^n\}$ generated by (5.2) converges in $L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$ to the entropy solution of (5.1).

In order to apply the result obtained in [18, Theorem 2.1], we extend our grid solution $(\rho_i^{n,\varepsilon}, j_i^{n,\varepsilon})$ to a piecewise bilinear function

$$(\rho^{\Delta,\varepsilon}(x,t), j^{\Delta,\varepsilon}(x,t)) = \sum_{i\in\mathbb{Z}, n\in\mathbb{Z}^+} (\rho_i^{n,\varepsilon}, j_i^{n,\varepsilon}) \Lambda_i^n(x,t),$$

where $\Lambda_i^n(x,t) := \Lambda_i(x)\Lambda^n(t)$ with

$$\Lambda_i(x) = \frac{1}{\Delta x}\min(x - x_{i-1}, x_{i+1} - x)_+,$$

$$\Lambda^n(t) = \frac{1}{\Delta t}\min(t - t_{i-1}, t_{i+1} - t)_+.$$

From now on we assume that the initial data $(u_0^\varepsilon, v_0^\varepsilon)$ are compactly supported and Lip^+-bounded, i.e.,

$$\|(u_0^\varepsilon, v_0^\varepsilon)\|_{\mathrm{Lip}^+} := \|(u_0^\varepsilon, \|_{\mathrm{Lip}^+} + \|v_0^\varepsilon)\|_{\mathrm{Lip}^+} \leqslant C. \tag{5.3}$$

Here, $\|\cdot\|_{\mathrm{Lip}^+}$ denotes the usual Lip^+-seminorm

$$\|w(x)\|_{\mathrm{Lip}^+} \equiv \operatorname*{esssup}_{x\neq y} \left(\frac{w(x) - w(y)}{x - y}\right)^+, \quad (\cdot)^+ = \max(\cdot, 0).$$

We let $\|w\|_{\mathrm{Lip}'(\mathbb{R})}$ denote the Lip-dual seminorm defined as

$$\sup_\psi \frac{(\phi - \hat{\phi}_0, \psi)}{\|\psi\|_{\mathrm{Lip}(\mathbb{R})}}, \quad \text{where } \hat{\phi}_0 = \int_{\mathrm{supp}\,\phi} \phi \,\mathrm{d}x.$$

A discrete lip^+-seminorm is defined for discrete functions w as

$$\|w\|_{\mathrm{lip}^+} := \max_{j\in\mathbb{Z}} \left(\frac{w_{j+1} - w_j}{\Delta x}\right)^+.$$

Let us recall that entropy solutions of (5.1) are Lip^+-bounded, e.g., [18,21],

$$\|\rho(\cdot,t)\|_{\mathrm{Lip}^+} \leqslant \|\rho(\cdot,0)\|_{\mathrm{Lip}^+}, \quad t \geqslant 0. \tag{5.4}$$

We, therefore, concentrate on Lip^+-stable approximations, i.e., approximate solutions $\rho^{\Delta,\varepsilon}(x,t)$ satisfying

$$\|\rho^{\Delta,\varepsilon}(x,t)\|_{\mathrm{Lip}^+} \leqslant C, \quad t \geqslant 0. \tag{5.5}$$

We shall use the results of [18, Theorem 2.1], which assert that Lip'-consistency and Lip^+-stability imply a convergence of which the rate may be quantified in terms of the Lip'-size of the truncation error.

We begin with the question of Lip^+-stability. By the definition of the discrete initial values we have

$$\rho_{i+1}^0-\rho_i^0=\frac{1}{\Delta x}\int_{I_i}(\rho_0^\varepsilon(x+\Delta x)-\rho_0^\varepsilon(x))\,\mathrm{d}x\leqslant\|\rho_0^\varepsilon\|_{\mathrm{Lip}^+}\Delta x$$
$$\leqslant(\|u_0^\varepsilon\|_{\mathrm{Lip}^+}+\|v_0^\varepsilon\|_{\mathrm{Lip}^+})\Delta x\leqslant C\Delta x.$$

Thus, we obtain

$$\|\rho_\Delta^0\|_{\mathrm{lip}^+}\leqslant\frac{\rho_{i+1}^0-\rho_i^0}{\Delta x}\leqslant C, \tag{5.6}$$

which leads to the bound

$$\frac{u_{i+1}^0-u_i^0}{\Delta x}=\frac{1}{2}\frac{\rho_{i+1}^0-\rho_i^0}{\Delta x}+\frac{1}{2c}\frac{j_{i+1}^0-j_i^0}{\Delta x}$$
$$=\frac{1}{2}\left(1+\frac{F'(\xi_i^0)}{c}\right)\frac{(\rho_{i+1}^0-\rho_i^0)}{\Delta x}\leqslant C, \tag{5.7}$$

where $F'(\xi_i^0)(\rho_{i+1}^0-\rho_i^0)=F(\rho_{i+1}^0)-F(\rho_i^0)$ and $|F'(\xi_i^0))|\leqslant c$ are used. Similarly, we get the bound

$$\frac{v_{i+1}^0-v_i^0}{\Delta x}\leqslant C$$

and obtain the following lemma.

Lemma 5.1. *Suppose* $\|(u_i^0,v_i^0)\|_{\mathrm{lip}^+}\leqslant C$, *and* $v_i^0\leqslant\alpha/\gamma$ *then the approximations* $\{u_i^n,v_i^n\}$ *generated by* (2.4)–(2.6) *satisfy*

$$\|u_i^n\|_{\mathrm{lip}^+}+\|v_i^n\|_{\mathrm{lip}^+}\leqslant 2C. \tag{5.8}$$

Proof. The lip^+-stability (5.8) can be proved by the same method as that in [15]. Therefore, we omit the proof. □

Next, we turn to the question of Lip'-consistency.

Lemma 5.2 (Lip'-consistency). *The approximation generated by* (2.8), (2.9) *satisfies the following truncation error estimate*

$$\|\rho_t^{\Delta,\varepsilon}+F(\rho^{\Delta,\varepsilon})_x\|_{\mathrm{Lip}'(\mathbb{R},[0,T])}\leqslant C_T(\Delta x+\varepsilon), \tag{5.9}$$

where C_T is a positive constant depending on T.

Proof. Let N denote the number of time steps in $[0,T]$, i.e., $T=t^N=N\Delta t$.
Set

$$Z_i^n=F(\rho_i^n)-j_i^n,\quad \text{for } (i,n)\in\mathbb{Z}\times\{1,\ldots,N\}.$$

Then it follows from (2.9), dropping ε for simplicity, that

$$\Delta x(\rho_i^{n+1}-\rho_i^n) = -\frac{\Delta t}{2}[F(\rho_{i+1}^n)-F(\rho_{i-1}^n)] + \frac{\Delta x}{2}[\lambda(\rho_{i+1}^n-\rho_i^n)-\lambda(\rho_i^n-\rho_{i-1}^n)] + \frac{\Delta t}{2}[Z_{i+1}^n - Z_{i-1}^n]. \tag{5.10}$$

Let $(\cdot,\cdot)$ stand for the usual L^2 inner product. Following [18] we have that there exists a bounded piecewise-constant function

$$D^n(x) = \sum_i D_{i+1/2}^n \chi_{i+1/2}(x), \quad \chi_{i+1/2}(x) = \begin{cases} 1, & x \in [x_i, x_{i+1}], \\ 0, & \text{others}, \end{cases}$$

such that

$$\left(\frac{\partial}{\partial t}\rho^\Delta(x,t), \Lambda_i^n\right)_{\Delta x,t} + \left(\frac{\partial}{\partial x}F(\rho^\Delta(x,t)), \Lambda_i^n(x,t)\right)_{x,\Delta t}$$
$$= \frac{\Delta t}{2}\left(\frac{\partial}{\partial t}\rho^\Delta(x,t), \frac{\partial}{\partial t}\Lambda_i^n(x,t)\right)_{\Delta x,t} - \frac{\Delta x}{2}\left(\frac{\partial}{\partial x}\rho^\Delta(x,t), \frac{\partial}{\partial x}\Lambda_i^n(x,t)\right)_{D(x),\Delta t} + \frac{\Delta t}{2}(Z_{i+1}^n - Z_{i-1}^n), \tag{5.11}$$

where

$$D_{i+1/2}^n = c - \Delta\rho_{i+1/2}(t^n)\int_{-1/2}^{1/2}\left(\frac{1}{4}-\xi^2\right)F''(\rho_{i+1/2}(\xi,t^n))\,\mathrm{d}\xi,$$

$$\rho_i(t) = \sum_n \rho_i^n \Lambda^n(t), \quad \Delta\rho_{i+1/2}(t) = \rho_{i+1}(t) - \rho_i(t),$$

$$\rho_{i+1/2}(\xi,t) = \tfrac{1}{2}[\rho_i(t)+\rho_{i+1}(t)] + \xi\Delta\rho_{i+1/2}(t).$$

For arbitrary $\phi \in C_0^\infty$ we set $t_n = t$ and define the piecewise bilinear interpolant $\hat{\phi}(x,t) = \sum_{i,n}\phi(x_i,t_n)\Lambda_i^n(x,t)$. Then we may write (5.11) as

$$(\partial_t\rho^{\Delta,\varepsilon}(x,t) + \partial_x F(\rho^{\Delta,\varepsilon}(x,t)), \phi)_{x,t} = \sum_{k=1}^4 T_k^{\Delta x} + T^\varepsilon$$

with

$$T_1^{\Delta x} = -\frac{\Delta x}{2}(\partial_x\rho^\Delta, \partial_x\hat{\phi})_{D(x),\Delta t},$$

$$T_2^{\Delta x} = \frac{\Delta t}{2}(\partial_t\rho^{\Delta x}, \partial_t\hat{\phi})_{\Delta x,t},$$

$$T_3^{\Delta x} = (\partial_t\rho^\Delta(x,t), \phi)_{x,t} - (\partial_t\rho^{\Delta x}, \hat{\phi})_{\Delta x,t},$$

$$T_4^{\Delta x} = (\partial_x F(\rho^\Delta), \phi)_{x,t} - (\partial_x F(\rho^\Delta)_x, \hat{\phi})_{x,\Delta t},$$

$$T^\varepsilon = \sum_{i\in\mathscr{Z}}\sum_{n=0}^N \phi(x_i,t_n)\frac{\Delta t}{2}(Z_{i+1}^n - Z_{i-1}^n).$$

The estimates on $T_k^{\Delta x}$ for $k = 1,2,3,4$ were obtained in [18, (3.6) and (3.7)], i.e.,

$$\sum_{k=1}^{4} |T_k^{\Delta x}| \leqslant \text{Const.}\, \Delta x \|\rho^{\Delta}\|_{L^1([0,T],\mathrm{BV}_x)} \|\phi\|_{\mathrm{Lip}(\mathbb{R}\times[0,T])}. \tag{5.12}$$

It remains to estimate T^{ε}, which comes from the relaxation term. Using summation by parts

$$\begin{aligned} |T^{\varepsilon}| &= \left| \sum_{i\in\mathbb{Z}} \sum_{n\in N} \frac{\Delta t}{2} Z_{i+1}^n (\phi(x_i,t_n) - \phi(x_{i+2},t_n) \right| \\ &\leqslant \Delta t \|\phi(\cdot,t)\|_{\mathrm{Lip}(\mathbb{R},[0,T])} \sum_{i,n} |Z_{i+1}^n| \Delta x. \end{aligned}$$

By (3.9), we have $\sum_i |Z_i^n|\Delta x \leqslant C\varepsilon$ if $\sum_i |Z_i^0|\Delta x = \mathrm{O}(\varepsilon)$. This combined with the above estimate leads to

$$|T^{\varepsilon}| \leqslant CT\varepsilon \|\phi(\cdot,t)\|_{\mathrm{Lip}(\mathbb{R},[0,T])}. \tag{5.13}$$

Equipped with estimates (5.12), (5.13) we have

$$|(\partial_t \rho^{\Delta,\varepsilon}(x,t) + \partial x F(\rho^{\Delta,\varepsilon}), \phi)_{x,t}| \leqslant C_T(\Delta x + \varepsilon)\|\phi\|_{\mathrm{Lip}(\mathbb{R},[0,T])}$$

which implies (5.9). □

Furthermore, we show that $\rho^{\Delta,\varepsilon}$ is also Lip′-consistent with the initial data. We first note that $\rho^{\Delta,\varepsilon}(x,t)$ are clearly conservative, for by our choice of the discrete initial data,

$$\int \rho^{\Delta,\varepsilon}(x,t)\,\mathrm{d}x = \frac{\Delta x}{2}\sum_i (\rho_i^n + \rho_{i+1}^n) = \frac{\Delta x}{2}\sum_i (\rho_i^0 + \rho_{i+1}^0) = \int \rho_0^{\varepsilon}(x)\,\mathrm{d}x. \tag{5.14}$$

Moreover, these initial conditions are Lip′-consistent. In fact, we have

$$\begin{aligned} |(\rho^{\Delta,\varepsilon}(x,0) - \rho_0^{\varepsilon}(x), \phi(x))| &= |(\rho^{\Delta,\varepsilon}(x,0) - \rho_0^{\varepsilon}(x)), \phi(x) - \phi(x_{i+1/2}))| \\ &\leqslant \Delta x \|\phi\|_{\mathrm{Lip}(\mathbb{R})} \sum_i \int_{x_i}^{x_{i+1}} |\rho^{\Delta,\varepsilon}(x,0) - \rho_0^{\varepsilon}(x)|\,\mathrm{d}x \\ &\leqslant C(\Delta x)^2 \|\rho_0^{\varepsilon}(x)\|_{\mathrm{BV}} \|\phi\|_{\mathrm{Lip}(\mathbb{R})}. \end{aligned}$$

This yields

$$\|\rho^{\Delta x,\varepsilon}(x,0) - \rho_0^{\varepsilon}(x)\|_{\mathrm{Lip}'(\mathbb{R})} \leqslant C\|\rho_0^{\varepsilon}\|_{\mathrm{BV}}(\Delta x)^2. \tag{5.15}$$

Now, we can use results of Nessyahu and Tadmor [18, Theorem 2.1] and get

$$\begin{aligned} \|\rho^{\Delta,\varepsilon}(\cdot,T) - \rho(\cdot,T)\|_{\mathrm{Lip}'(\mathbb{R})} &\leqslant C_T[\|\rho^{\Delta x,\varepsilon}(\cdot,T) - \rho_0(x)\|_{\mathrm{Lip}'(\mathbb{R})} + \|\rho_t^{\Delta,\varepsilon} + F(\rho^{\Delta,\varepsilon})_x\|_{\mathrm{Lip}'(\mathbb{R},[0,T])}] \\ &\leqslant C_T(\Delta x + \varepsilon) = \mathrm{O}(\Delta x + \varepsilon). \end{aligned} \tag{5.16}$$

The Lip′ error estimate (5.16) may now be interpolated into the $W^{s,p}$-error estimates as shown in [18, Corollary 2.2, 2.4].

Our error estimate result is summarized in the following.

Theorem 5.3. *Consider the convex scalar conservation law* (5.1) *with* Lip$^+$*-bounded initial data* $\rho_0^\varepsilon = u_0^\varepsilon + v_0^\varepsilon$ *and* $v_0^\varepsilon \leqslant \alpha/\gamma$. *Then our difference-relaxation approximation with discrete initial data* (ρ_i^0, j_i^0) *and*

$$\|j^0 - F(\rho^0))\|_1 = \mathrm{O}(\varepsilon),$$

converges, and the piecewise-linear interpolants $\rho^{\Delta,\varepsilon}(x,t)$ *satisfy the convergence rate estimates*

$$\|\rho^{\Delta,\varepsilon}(\cdot,T) - \rho(\cdot,T)\|_{W^{s,p}} \leqslant C_T(\Delta x + \varepsilon)^{(1-sp)/2p} \quad \textit{for } -1 \leqslant s \leqslant \frac{1}{p}, \; 1 \leqslant p \leqslant \infty, \tag{5.17}$$

as well as

$$|\rho^{\Delta,\varepsilon}(x,T) - \rho(x,T)| \leqslant \mathrm{Const}_{x,T}(\Delta x + \varepsilon)^{1/3}, \tag{5.18}$$

$$\mathrm{Const}_{x,T} \sim 1 + |\rho_x(\cdot,T)|_{L^\infty(x-(\Delta x+\varepsilon)^{1/3}, x+(\Delta x+\varepsilon)^{1/3})}.$$

Remark. (1) When $(s,p) = (-1,1)$, the error estimate (5.17) turns into the Lip$'$ error estimate (5.16).

(2) When $(s,p) = (0,1)$, (5.17) yields L^1-convergence rate of order $\mathrm{O}(\sqrt{\Delta x + \varepsilon})$.

(3) Uniform convergence which corresponds to $(s,p) = (0,\infty)$ in (5.17) fails in this case, due to the possible presence of shock discontinuities in the entropy solution $\rho(\cdot,t)$. But we have pointwise convergence (5.18) away from the singular support of $\rho(\cdot,t)$.

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Appendix

In this appendix we present two technical lemmas that are needed for the estimate of the approximate solution. This first lemma gives us the positivity of the relaxation step.

Lemma A.1. *Let* (w_0, z_0) *be a nonnegative initial data for system* (2.5) *with corresponding solution* $(w(t), z(t))$. *Then the solution is nonnegative for any* $t \geqslant 0$, *i.e.*,

$$(w(t), z(t)) \geqslant 0.$$

Proof. Let $\Sigma = \{(w,z), w \geqslant 0, z \geqslant 0\}$. We have assumed $(w_0, z_0) \in \Sigma$. We claim that the trajectory of (2.5) originating from $(w_0, z_0) \in \Sigma$ will remain in Σ for all $t \in [0,T]$. To see this, we show that the vector field

$$V = \begin{pmatrix} -\frac{1}{\varepsilon}(\alpha w - \beta z + \gamma wz) \\ \frac{1}{\varepsilon}(\alpha w - \beta z + \gamma wz) \end{pmatrix}$$

points strictly into Σ on $\partial\Sigma = \{w = 0\} \cup \{z = 0\}$. This immediately follows from the fact

$$\begin{aligned} V_1|_{w=0} &= \frac{\beta}{\varepsilon} z \geqslant 0 \quad \text{on the axis } z \geqslant 0, \\ V_2|_{z=0} &= \frac{\alpha}{\varepsilon} w \geqslant 0 \quad \text{on the axis } w \geqslant 0. \end{aligned} \tag{A.1}$$

This proves that $(w(t), z(t)) \in \Sigma$ for any $t \in [0, T]$. □

This next lemma allows us to obtain the L^1 stability for the solutions to the splitting scheme.

Lemma A.2. *Let* $(w_0, z_0) \geqslant 0, (\tilde{w}_0, \tilde{z}_0) \geqslant 0$ *be two initial conditions for system* (2.5), *then the corresponding solutions* $(w(t), z(t))$ *and* $(\tilde{w}(t), \tilde{z}(t))$ *satisfy*

$$\begin{aligned} &|w(t) - \tilde{w}(t)| + |z(t) - \tilde{z}(t)| \\ &\quad \leqslant \exp\left(\frac{2\gamma}{\beta} \left(z_0 - \frac{\alpha}{\gamma} \right)_+ \left[1 - \exp\left(-\frac{\beta}{\varepsilon} t \right) \right] \right) [|w_0 - \tilde{w}_0| + |z_0 - \tilde{z}_0|]. \end{aligned}$$

Proof. Set $(\bar{w}, \bar{z})(t) = (w, z)(t)–(\tilde{w}, \tilde{z})(t)$, then $(\bar{w}(t), \bar{z}(t))$ satisfy the system

$$\begin{aligned} \frac{d\bar{w}}{dt} &= -\frac{1}{\varepsilon}\{\alpha\bar{w} - \beta\bar{z} - \gamma\bar{w}z - \gamma\tilde{w}\bar{z}\}, \\ \frac{d\bar{z}}{dt} &= \frac{1}{\varepsilon}\{\alpha\bar{w} - \beta\bar{z} - \gamma\bar{w}z - \gamma\tilde{w}\bar{z}\}. \end{aligned}$$

Multiply the first equation by $\operatorname{sgn}(\bar{w})$, and the second equation by $\operatorname{sgn}(\bar{z})$, this gives

$$\begin{aligned} \frac{d}{dt}|\bar{w}| &= -\frac{1}{\varepsilon}[\alpha|\bar{w}| - \beta\bar{z}\operatorname{sgn}(\bar{w}) - \gamma|\bar{w}|z - \gamma\tilde{w}\bar{z}\operatorname{sgn}(\bar{w})], \\ \frac{d}{dt}|\bar{z}| &= \frac{1}{\varepsilon}[\alpha\bar{w}\operatorname{sgn}(\bar{z}) - \beta|\bar{z}| - \gamma\bar{w}z\operatorname{sgn}(\bar{z}) - \gamma\tilde{w}|\bar{z}|]. \end{aligned}$$

Adding the two equations, we get

$$\begin{aligned} \frac{d}{dt}[|\bar{w}| + |\bar{z}|] = \frac{1}{\varepsilon}[&-\alpha|\bar{w}| + \alpha\bar{w}\operatorname{sgn}(\bar{z}) + \beta\bar{z}\operatorname{sgn}(\bar{w}) - \beta|\bar{z}| \\ &+\gamma|\bar{w}|z - \gamma\bar{w}z\operatorname{sgn}(\bar{z}) + \gamma\tilde{w}\bar{z}\operatorname{sgn}(\bar{w}) - \gamma\tilde{w}|\bar{z}|]. \end{aligned}$$

If $\operatorname{sgn}(\bar{w}) = \operatorname{sgn}(\bar{z})$, then

$$\frac{d}{dt}[|\bar{w}| + |\bar{z}|] = 0,$$

otherwise, $\operatorname{sgn}(\bar{w}) = -\operatorname{sgn}(\bar{z})$, then

$$\frac{d}{dt}[|\bar{w}| + |\bar{z}|] = \frac{2\gamma}{\varepsilon}|\bar{w}| \left(z - \frac{\alpha}{\gamma} \right) - \frac{2}{\varepsilon}(\beta + \gamma\tilde{w})|\bar{z}|.$$

We set $T(t)=|\bar{w}|+|\bar{z}|$ and note that $\tilde{w}\geqslant 0$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)\leqslant\frac{2\gamma}{\varepsilon}\left(z-\frac{\alpha}{\gamma}\right)_+|\bar{w}|\leqslant\frac{2\gamma}{\varepsilon}\left(z-\frac{\alpha}{\gamma}\right)_+T(t).$$

Thus,

$$T(t)\leqslant T(0)\exp\left(\frac{2\gamma}{\varepsilon}\int_0^t\left(z(\tau)-\frac{\alpha}{\gamma}\right)_+\mathrm{d}\tau\right)\quad\text{for } 0<t\leqslant T.$$

By using the technique in the proof of Theorem 3.2 we get

$$\left(z(t)-\frac{\alpha}{\gamma}\right)_+\leqslant\left(z_0-\frac{\alpha}{\gamma}\right)_+\exp\left(-\frac{\beta t}{\varepsilon}\right).$$

Thus,

$$T(t)\leqslant T(0)\exp\left(\frac{2\gamma}{\beta}\left(z_0-\frac{\alpha}{\gamma}\right)_+\left[1-\exp\left(-\frac{\beta t}{\varepsilon}\right)\right]\right).$$

Hence Lemma A.2 follows. □

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