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Asymptotic representations for hypergeometric-Bessel type function and fractional integrals

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Abstract

The paper is devoted to the study of asymptotic relations for the function

$$\lambda_{\gamma,\sigma}^{(\beta)}(z)=\frac{\beta}{\Gamma(\gamma+1-1/\beta)}\int_1^{\infty}(t^{\beta}-1)^{\gamma-1/\beta}t^{\sigma}\,e^{-zt}\,dt$$

generalising Tricomi confluent hypergeometric function and modified Bessel function of the third kind. The full asymptotic representations for $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ at zero and infinity are established. Applications are given to obtain full asymptotic expansions near zero and infinity for the Liouville fractional integral

$$(I_{-}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}\frac{f(t)\,dt}{(t-x)^{1-\alpha}}\quad(x>0;\ \alpha\in\mathbf{C},\ \mathrm{Re}(\alpha)>0)$$

and for the Erdelyi–Kober-type fractional integral

$$(I_{-;\beta,\eta}^{\alpha}f)(x)=\frac{\beta x^{\beta\eta}}{\Gamma(\alpha)}\int_x^{\infty}\frac{t^{\beta(1-\alpha-\eta)-1}f(t)\,dt}{(t^{\beta}-x^{\beta})^{1-\alpha}}\quad(x>0;\ \alpha\in\mathbf{C},\ (\mathrm{Re}(\alpha)>0)$$

with $\beta>0$ and $\eta\in\mathbf{C}$ of power-exponential function $f(t)$, and for three other fractional integrals.

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1. Introduction

The paper deals with a function $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ defined by

$$\lambda^{(\beta)}_{\gamma,\sigma}(z) = \frac{\beta}{\Gamma(\gamma+1-1/\beta)} \int_1^\infty (t^\beta - 1)^{\gamma-1/\beta} t^\sigma \, \mathrm{e}^{-zt} \, \mathrm{d}t \tag{1.1}$$

for

$$\beta > 0; \ \gamma \in \mathbf{C}, \ \mathrm{Re}(\gamma) > \tfrac{1}{\beta} - 1; \ \sigma \in \mathbf{R}; \ z \in \mathbf{C} \ (\mathrm{Re}(z) > 0), \tag{1.2}$$

C and **R** being the sets of complex and real numbers, respectively. This function is analytic with respect to z for $\mathrm{Re}(z) > 0$. When $\beta = 1$ and 2, $\sigma = 0$, then

$$\lambda^{(1)}_{\gamma,\sigma}(z) = \mathrm{e}^{-z} \Psi(\gamma, \gamma+\sigma+1; z) \tag{1.3}$$

and

$$\lambda^{(2)}_{\gamma,0}(z) = 2^{\gamma+1} \pi^{-1/2} z^{-\gamma} K_{-\gamma}(z), \tag{1.4}$$

where $\Psi(\gamma, \gamma+\sigma+1; z)$ is the Tricomi confluent hypergeometric function [5, 6.5(2)] and $K_{-\gamma}(z)$ is the modified Bessel function of the third kind known also as McDonald function [6, Section 7.2.2]. Therefore, we call (1.1) as a function of hypergeometric–Bessel type.

The function $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ was introduced by Glaeske, Kilbas and Saigo [9]. It is a generalisation of the function

$$\lambda^{(n)}_{\gamma}(z) = \frac{(2\pi)^{(n-1)/2}\sqrt{n}}{\Gamma(\gamma+1-1/n)} \left(\frac{z}{n}\right)^{\gamma n} \int_1^\infty (t^n - 1)^{\gamma-1/n} \, \mathrm{e}^{-zt} \, \mathrm{d}t \tag{1.5}$$

$$\left(n \in \mathbf{N} = \{1, 2, \cdots\}; \ \gamma \in \mathbf{C}, \ \mathrm{Re}(\gamma) > \tfrac{1}{n} - 1; \ z \in \mathbf{C} \ (\mathrm{Re}(z) > 0)\right),$$

introduced by Kratzel in [13] for natural parameter n, who in [13–16] investigated integral transforms with such function kernels and gave application to solution of some ordinary differential equations. It should be noted that function (1.5) is invariant with the accuracy of indices with respect to the usual differentiation [15,16], while (1.1) has the invariant property to within indices

$$(I^\alpha_- \lambda^{(\beta)}_{\gamma,\sigma})(x) = \lambda^{(\beta)}_{\gamma,\sigma-\alpha}(x) \tag{1.6}$$

with respect to the Liouville fractional integration [26, (5.2)].

$$(I^\alpha_- f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)\,\mathrm{d}t}{(t-x)^{1-\alpha}} \quad (x \in \mathbf{R}_+ = (0,\infty); \ \alpha \in \mathbf{C}, \ \mathrm{Re}(\alpha) > 0) \tag{1.7}$$

and the same property with respect to the corresponding Liouville fractional derivative $D^\alpha_- \varphi$ [26, 5.8]. The latter results being proved in [9], were applied in [4] to find the explicit solutions of certain types of integral and differential equations of fractional order in terms of function (1.1).

The first terms of the asymptotic behaviour of $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ at zero and infinity and its Mellin transform were also investigated in [9]. These results were applied in [9] and [3] to study the mapping properties of the integral transforms involving $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ in the kernels, in spaces of tested and generalised functions $\mathscr{F}_{p,\mu}$ and $\mathscr{F}'_{p,\mu}$ (see [19]) and in the weighted space of summable functions $\mathscr{L}_{v,r}$ (see, for example [25]), respectively.

The Liouville fractional integral (1.7) of power-exponential function can be also evaluated via the function $\lambda^{(\beta)}_{\gamma,\sigma}(z)$. Such a result was proved in [12] together with the similar representation for the Erdelyi–Kober-type fractional integral $I^{\alpha}_{-;\beta,\eta}f$ defined for $\alpha \in \mathbf{C}$ ($\mathrm{Re}(\alpha) > 0$), $\beta > 0$ and $\eta \in \mathbf{C}$ by [26, (18.7)].

$$(I^{\alpha}_{-;\beta,\eta}f)(x) = \frac{\beta x^{\beta\eta}}{\Gamma(\alpha)} \int_x^{\infty} \frac{t^{\beta(1-\alpha-\eta)-1} f(t)\,\mathrm{d}t}{(t^{\beta} - x^{\beta})^{1-\alpha}} \quad (x \in \mathbf{R}_+). \tag{1.8}$$

The present paper is devoted to investigate asymptotic properties of the function $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ and fractional integrals (1.7) and (1.8) of power-exponential functions. We establish full asymptotic expansions of these functions at zero and infinity. We also deduce the full asymptotic expansions for the so-called fractional integral of a function by a power function [26, (18.41)]

$$(I^{\alpha}_{-;\,x^{\beta}}f)(x) = \frac{\beta}{\Gamma(\alpha)} \int_x^{\infty} \frac{t^{\beta-1} f(t)\,\mathrm{d}t}{(t^{\beta} - x^{\beta})^{1-\alpha}} \quad (x \in \mathbf{R}_+;\ \alpha \in \mathbf{C},\ \mathrm{Re}(\alpha) > 0;\ \beta > 0) \tag{1.9}$$

for the Kober and Erdelyi–Kober fractional integrals defined for $\alpha \in \mathbf{C}$ ($\mathrm{Re}(\alpha) > 0$) and $\eta \in \mathbf{C}$ via (1.7) by [26, (18.6), (18.8)]

$$(K^{-}_{\eta,\alpha}f)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} \frac{t^{-\alpha-\eta} f(t)\,\mathrm{d}t}{(t - x)^{1-\alpha}} \quad (x \in \mathbf{R}_+) \tag{1.10}$$

and

$$(K_{\eta,\alpha}f)(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^{\infty} \frac{t^{1-2(\alpha+\eta)} f(t)\,\mathrm{d}t}{(t^2 - x^2)^{1-\alpha}} \quad (x \in \mathbf{R}_+), \tag{1.11}$$

respectively. Note that the fractional integrals (1.9)–(1.11) are deduced from the Erdelyi–Kober-type fractional integral (1.8). We present here the asymptotic results for these integrals because they as well as the Erdelyi–Kober-type fractional integrals are arisen in applications, in particular while solving dual integral equations and partial differential equations arisen in the potential theory—see, for example [26, Sections 38, 39, 41, 43].

It should be noted that one may find asymptotic representations at zero and infinity for some special functions in the handbooks of Erdelyi et al. [5–7] and monographs in [17,18,21–24,8]. Asymptotic estimates for the fractional integrals are studied less. In this connection we indicate that the asymptotic representations at infinity for the left-sided fractional integrals, corresponding to (1.7) and (1.8) in which the integration over (x, ∞) is replaced by the one taken over $(0, x)$, were proved in [20] and the first author [10,11] in the cases when $f(t)$ has simplest power and general power asymptotic expansions, respectively. Asymptotic estimates for such a fractional integral of the form (1.9), being taking over $(0, x)$, were proved in [2] in the case when $f(t)$ has power-exponential expansion. See the results and bibliography in [26, Sections 16 and 17].

The paper is organised as follows. Section 2 contains some preliminary assertions. Section 3 deals with full asymptotic representations of $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ at zero and infinity. Special cases of such asymptotic relations for $\beta = 1$, 2 and $1/\gamma$ are considered in Section 4. Section 5 and 6 are devoted to full asymptotic expansions at zero and infinity for the Liouville and Erdelyi–Kober-type fractional integrals of power-exponential functions, respectively. Asymptotic estimates for the fractional integrals (1.9), (1.10) and (1.11) are presented in Sections 7, 8 and 9, respectively.

2. Preliminaries

We consider a function

$$f(t) = [g_\beta(t)]^{\gamma-1/\beta}(t+1)^\sigma, \tag{2.1}$$

where

$$g_\beta(t) = \frac{(t+1)^\beta - 1}{\beta t}, \quad g_\beta(0) = 1. \tag{2.2}$$

Note that near $t=0$ the function $g_\beta(t)$ can be represented in the series form

$$g_\beta(t) = \frac{1}{\beta}\sum_{i=1}^{\infty}\frac{(-1)^i(-\beta)_i}{i!}t^{i-1} = \sum_{j=0}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\beta-j)}\frac{t^j}{(j+1)!}, \tag{2.3}$$

where $(a)_i$ is the Pochhammer symbol:

$$(a)_0 = 1, \quad (a)_i = a(a+1)\cdots(a+i-1) \quad (i = 1,2,\ldots). \tag{2.4}$$

Then for $i \in \mathbf{N}$

$$g_\beta^{(i)}(t) = \sum_{j=0}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\beta-j)}\frac{1}{(j+1)(j-i)!}t^{j-i}$$

and hence

$$g_\beta^{(i)}(0) = \frac{\Gamma(\beta)}{(i+1)\Gamma(\beta-i)}. \tag{2.5}$$

Further we need to know the value of the following limit

$$g_k \equiv g_k(\gamma,\beta) = \lim_{t\to 0} D^k([g_\beta(t)]^{\gamma-1/\beta}), \quad D = \frac{\mathrm{d}}{\mathrm{d}t}. \tag{2.6}$$

Lemma 2.1. *If $\beta > 0$, $\gamma \in \mathbf{C}$ and $g_\beta(t)$ is given by* (2.2), *then for any $k \in \mathbf{N}$ there hold the relations*

$$g_k = k!\sum_{m=0}^{k}\left(\lim_{t\to 0}[D^m([g_\beta(t)]^{\gamma-1/\beta})]\right)\sum\prod_{i=1}^{k}\frac{1}{P_i!}\left[\frac{\Gamma(\beta)}{(i+1)!\Gamma(\beta-i)}\right]^{P_i}, \tag{2.7}$$

where $\sum$ is taken over all combinations of nonnegative integer values of $P_1, P_2, \ldots, P_k$ such that

$$\sum_{i=1}^{k} iP_i = k, \sum_{i=1}^{k} P_i = m. \tag{2.8}$$

Proof. There is known the following Faa di Bruno formula (see [1, p. 823])

$$D^k h[g(t)] = k!\sum_{m=0}^{k} D^m h[g(t)]\sum\prod_{i=1}^{k}\frac{1}{P_i!}\left[\frac{(D^i g(t))}{i!}\right]^{P_i}, \tag{2.9}$$

where a summation $\sum$ is taken over all combinations of nonnegative integer values of $P_1, P_2, \ldots, P_k$ satisfying (2.8). Using this formula with $h(g) = g^{\gamma-1/\beta}$ and $g(t) \equiv g_\beta = [(t+1)^\beta - 1]/(\beta t)$ and taking into account (2.5), from (2.9) we obtain (2.7). □

Corollary 2.2. *The constants* g_1, g_2 *and* g_3 *are given by*

$$g_1 = \left(\gamma - \frac{1}{\beta}\right)\frac{\beta}{2}, \tag{2.10}$$

$$g_2 = \left(\gamma - \frac{1}{\beta}\right)\left(\gamma - \frac{1}{\beta} - 1\right)\left(\frac{\beta}{2}\right)^2 + \left(\gamma - \frac{1}{\beta}\right)\frac{\beta(\beta-1)}{3} \tag{2.11}$$

and

$$g_3 = \left(\gamma - \frac{1}{\beta}\right)\left(\gamma - \frac{1}{\beta} - 1\right)\left(\gamma - \frac{1}{\beta} - 2\right)\left(\frac{\beta}{2}\right)^3 + 3\left(\gamma - \frac{1}{\beta}\right)\left(\gamma - \frac{1}{\beta} - 1\right)\left(\frac{\beta}{2}\right)^2\frac{\beta(\beta-1)}{3} + \left(\gamma - \frac{1}{\beta}\right)\frac{\beta(\beta-1)(\beta-2)}{4}, \tag{2.12}$$

respectively.

3. Asymptotic representations for $\lambda_{\gamma,\sigma}^{(\beta)}(z)$

First we investigate asymptotic representation for function (1.1) at zero. There holds the following assertion.

Lemma 3.1. *Let* $\beta > 0$, $\gamma \in \mathbf{C}$, $\sigma \in \mathbf{R}$ *and* $N \in \mathbf{N} = \{1, 2, \ldots\}$ *be such that*

$$\frac{1}{\beta} - 1 < \mathrm{Re}(\gamma) < -\frac{\sigma + N}{\beta}. \tag{3.1}$$

Then for $z \in \mathbf{C}$, $\mathrm{Re}(z) > 0$, *the function* (1.1) *can be represented by*

$$\lambda_{\gamma,\sigma}^{(\beta)}(z) = \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\frac{\Gamma[-\gamma-(\sigma+n)/\beta]}{\Gamma[1-(\sigma+n+1)/\beta]}z^n + R_N(z), \tag{3.2}$$

where

$$R_N(z) = \mathrm{O}(z^N) \quad (z \to 0). \tag{3.3}$$

Proof. Let $n \in \mathbf{N}$. Using the Taylor formula

$$\mathrm{e}^{-zt} = \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}(zt)^n + r_N(zt), \quad r_N(u) = \mathrm{O}(u^N)\ (u \to 0), \tag{3.4}$$

we have

$$\lambda_{\gamma,\sigma}^{(\beta)}(z) = \frac{\beta}{\Gamma(\gamma + 1 - 1/\beta)}\sum_{n=0}^{N-1}\frac{(-z)^n}{n!}\int_1^\infty (t^\beta - 1)^{\gamma - 1/\beta}t^{\sigma+n}\,\mathrm{d}t + R_N(z), \tag{3.5}$$

where

$$R_N(z) = \frac{\beta}{\Gamma(\gamma + 1 - 1/\beta)}\int_1^\infty (t^\beta - 1)^{\gamma - 1/\beta}t^\sigma r_N(zt)\,\mathrm{d}t. \tag{3.6}$$

Making the change $t=s^{-1/\beta}$ and applying the known formulas for the Beta-function (see, for example [5, 1.5(1) and 1.5(5)]) we obtain for $n=0,1,\ldots,N-1$

$$\beta\int_1^\infty (t^\beta-1)^{\gamma-1/\beta}t^{\sigma+n}\,\mathrm{d}t=\int_0^1 s^{-\gamma-1-(\sigma+n)/\beta}(1-s)^{\gamma-1/\beta}\,\mathrm{d}s$$
$$=B\left(-\gamma-\frac{\sigma+n}{\beta},\gamma+1-\frac{1}{\beta}\right)=\frac{\Gamma[-\gamma-(\sigma+n)/\beta]}{\Gamma[1-(\sigma+k+1)/\beta]}.$$

Substitution of this relation into (3.5) yields (3.2). Estimate (3.3) for the remainder $R_N(z)$ follows from (3.6) and (3.4) if we take into account the condition (3.1). □

Using the definition of asymptotic expansion (see, for example [7,21]) from (3.2) we obtain the asymptotic expansion of $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ at zero in the form

$$\lambda_{\gamma,\sigma}^{(\beta)}(z)\sim\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma[-\gamma-(\sigma+n)/\beta]}{\Gamma[1-(\sigma+n+1)/\beta]}z^n\quad(z\to 0).\tag{3.7}$$

We note that relation (3.7) is true provided $\beta>0$, $\gamma\in\mathbf{C}$ and $\sigma\in\mathbf{R}$ satisfy the condition

$$\gamma+\frac{\sigma+n}{\beta}\neq 0,1,2,\ldots\ (n=0,1,2,\ldots).\tag{3.8}$$

To investigate the asymptotic behaviour of $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ at infinity we rewrite it in the form

$$\lambda_{\gamma,\sigma}^{(\beta)}(z)=\frac{\beta^{\gamma+1-1/\beta}}{\Gamma(\gamma+1-1/\beta)}\,\mathrm{e}^{-z}\int_0^\infty f(t)t^{\gamma-1/\beta}\,\mathrm{e}^{-zt}\,\mathrm{d}t,\tag{3.9}$$

where $f(t)$ is given by (2.1)–(2.2).

Theorem 3.2. *Let $\beta>0$ and $\gamma\in\mathbf{C}$ be such that* $\mathrm{Re}(\gamma)>-1+1/\beta$ *and let $\sigma\in\mathbf{R}$. Then there holds the asymptotic expansion for $z\to\infty$* $(\mathrm{Re}(z)>0)$

$$\lambda_{\gamma,\sigma}^{(\beta)}(z)\sim\frac{\beta^{\gamma+1-1/\beta}}{\Gamma(\gamma+1-1/\beta)}\,\mathrm{e}^{-z}z^{-(\gamma+1-1/\beta)}\sum_{n=0}^{\infty}\Gamma\left(\gamma-\frac{1}{\beta}+1+n\right)\frac{c_n}{n!}z^{-n},\tag{3.10}$$

where

$$c_n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-1)^{n-k}(-\sigma)_{n-k}g_k\quad(n=0,1,\ldots),\tag{3.11}$$

$g_0=1$ *and g_k being given by* (2.7) *for* $k=1,2,\ldots$.

Proof. First of all we note that

$$(t^\beta-1)^{\gamma-1/\beta}t^\sigma\sim t^{\beta\,\mathrm{Re}(\gamma)+\sigma-1}\quad(t\to\infty)$$

and hence there exist the numbers $A>0$ and $K>0$ such that

$$|(t^\beta-1)^{\gamma-1/\beta}t^\sigma|\leqslant Kt^{\beta\,\mathrm{Re}(\gamma)+\sigma-1}\quad(t\geqslant A).\tag{3.12}$$

Using the representation of $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ in the form (3.9) we divide this integral in two taking over $(0,A)$ and (A,∞):

$$\lambda_{\gamma,\sigma}^{(\beta)}(z)=\frac{\beta^{\gamma+1-1/\beta}}{\Gamma(\gamma+1-1/\beta)}\,\mathrm{e}^{-z}\int_0^A f(t)t^{\gamma-1/\beta}\mathrm{e}^{-zt}\,\mathrm{d}t$$
$$+\frac{\beta^{\gamma+1-1/\beta}}{\Gamma(\gamma+1-1/\beta)}\,\mathrm{e}^{-z}\int_A^\infty f(t)t^{\gamma-1/\beta}\mathrm{e}^{-zt}\,\mathrm{d}t=I_1(z)+I_2(z). \tag{3.13}$$

Using (3.12) we estimate $I_2(z)$ for $\mathrm{Re}(z)\geqslant 1$:

$$|I_2(z)|\leqslant\frac{\beta^{\mathrm{Re}(\gamma)+1-1/\beta}}{|\Gamma(\gamma+1-1/\beta)|}\int_{A+1}^\infty (t^\beta-1)^{\gamma-1/\beta}t^\sigma\,\mathrm{e}^{-\mathrm{Re}(z)t}\,\mathrm{d}t$$
$$\leqslant\frac{K\beta^{\mathrm{Re}(\gamma)+1-1/\beta}}{|\Gamma(\gamma+1-1/\beta)|}\int_{A+1}^\infty t^{\beta\,\mathrm{Re}(\gamma)+\sigma-1}\,\mathrm{e}^{-\mathrm{Re}(z)t}\,\mathrm{d}t$$
$$\leqslant\frac{K\beta^{\mathrm{Re}(\gamma)+1-1/\beta}}{|\Gamma(\gamma+1-1/\beta)|}\left(\frac{1}{\mathrm{Re}(z)}\right)^{(\beta\,\mathrm{Re}(\gamma)+\sigma)}\int_1^\infty u^{\beta\,\mathrm{Re}(\gamma)+\sigma-1}\,\mathrm{e}^{-u}\,\mathrm{d}u.$$

Hence

$$I_2(z)=\mathrm{O}(z^{-(\beta\gamma+\sigma)})\quad(z\to\infty) \tag{3.14}$$

and this estimate is asymptotically small in compare with any term of the series in (3.10).

To find the asymptotic expansion of the first integral in (3.13)

$$\int_0^A f(t)t^{\gamma-1/\beta}\,\mathrm{e}^{-zt}\,\mathrm{d}t, \tag{3.15}$$

$f(t)$ being given by (2.1)–(2.2), we apply the Watson lemma (for example, see [26, Lemma 16.3]). According to this lemma if $\alpha>0$, $\beta>0$, $f(t)$ is continuous for $0\leqslant t\leqslant A$ and infinitely differentiable in the neighbourhood of $t=0$, then the asymptotic equality

$$\int_0^A f(t)t^{\beta-1}\,\mathrm{e}^{-zt^\alpha}\,\mathrm{d}t\sim\frac{1}{\alpha}\sum_{n=0}^\infty\Gamma\left(\frac{n+\beta}{(\alpha)}\right)\frac{f^{(n)}(0)}{n!}z^{-(n+\beta)/\alpha} \tag{3.16}$$

is true as $z\to\infty$ ($\mathrm{Re}(z)>0$). Applying this Watson lemma with $\alpha=1$ and $\beta=\gamma+1-1/\beta$ to the integral in (3.15) we have

$$\int_0^A f(t)t^{\gamma-1/\beta}\,\mathrm{e}^{-zt}\,\mathrm{d}t\sim\sum_{n=0}^\infty\Gamma\left(\gamma-\frac{1}{\beta}+1+n\right)\frac{f^{(n)}(0)}{n!}z^{-(\gamma+1+n-1/\beta)}$$

as $z\to\infty$ ($\mathrm{Re}(z)>0$). Using the Leibnitz rule, the formula

$$D^m(1+t)^\sigma=(-1)^m(-\sigma)_m(1+t)^{\sigma-m}\quad(m\in\mathbf{N})$$

and relations (2.1), (2.2), (2.6), (2.7) we evaluate $f^{(n)}(0)$:

$$f^{(n)}(0)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-1)^{n-k}(-\sigma)_{n-k}g_k=c_n.$$

Thus

$$\int_0^A f(t)t^{\gamma-1/\beta}\,\mathrm{e}^{-zt}\,\mathrm{d}t\sim\sum_{n=0}^{\infty}\Gamma\left(\gamma-\frac{1}{\beta}+1+n\right)\frac{c_n}{n!}z^{-(\gamma+1+n-1/\beta)}$$

as $z\to\infty$ ($\mathrm{Re}(z)>0$). Substituting this estimate into (3.13) and taking (3.14) into account we arrive at estimate (3.10) which completes the proof of theorem. □

4. Asymptotic representations for $\lambda_{\gamma,\sigma}^{(b)}(z)$ in special cases

Setting $\beta=1$, $1/\gamma$ and 2 in (3.7) we deduce the asymptotic expansions of $\lambda_{\gamma,\sigma}^{(1)}(z)$, $\lambda_{\gamma,\sigma}^{(1/\gamma)}(z)$ and $\lambda_{\gamma,\sigma}^{(2)}(z)$ at zero in the forms

$$\lambda_{\gamma,\sigma}^{(1)}(z)\sim\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma(-\gamma-\sigma-n)}{[\Gamma(-\sigma-n)]}z^n\ (z\to 0) \tag{4.1}$$

$$(\gamma+\sigma+n\neq 0,1,2,\ldots;\ n=0,1,2,\ldots), \tag{4.2}$$

$$\lambda_{\gamma,\sigma}^{(1/\gamma)}(z)\sim\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma[-(1+\sigma+n)\gamma]}{\Gamma[1-(\sigma+n+1)\gamma]}z^n(z\to 0) \tag{4.3}$$

$$(1+\sigma+n\gamma\neq 0,1,2,\ldots;\ n=0,1,2,\ldots) \tag{4.4}$$

and

$$\lambda_{\gamma,\sigma}^{(2)}(z)\sim\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma[-\gamma-(\sigma+n)/2]}{\Gamma[1-(\sigma+n+1)/2]}z^n\ (z\to 0) \tag{4.5}$$

$$\left(\gamma+\frac{\sigma+n}{2}\neq 0,1,2,\ldots;\ n=0,1,2,\ldots\right), \tag{4.6}$$

respectively

In the cases $\beta=1$, $1/\gamma$ and 2 asymptotic expansions of $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ at infinity can be obtained without using relation (2.9).

Theorem 4.1. *There hold the following asymptotic expansions, as* $z\to\infty$ $(\mathrm{Re}(z)>0)$.

(a) *If* $\gamma\in\mathbf{C}\,(\mathrm{Re}(\gamma)>0)$ *and* $\sigma\in\mathbf{R}$, *then*

$$\lambda_{\gamma,\sigma}^{(1)}(z)\sim\ \mathrm{e}^{-z}z^{-\gamma}\sum_{n=0}^{\infty}(-1)^n\frac{(\gamma)_n(-\sigma)_n}{n!}z^{-n}. \tag{4.7}$$

(b) *If* $\gamma \in \mathbf{C}$ *and* $\sigma \in \mathbf{R}$, *then*

$$\lambda_{\gamma,\sigma}^{(1/\gamma)}(z) \sim \frac{1}{\gamma}\,\mathrm{e}^{-z} z^{-1} \sum_{n=0}^{\infty} (-1)^n (-\sigma)_n z^{-n}. \tag{4.8}$$

(c) *If* $\gamma \in \mathbf{C}\,(\mathrm{Re}(\gamma) > -1/2)$ *and* $\sigma \in \mathbf{R}$, *then*

$$\lambda_{\gamma,\sigma}^{(2)}(z) \sim \left(\frac{2}{z}\right)^{\gamma+1/2} \mathrm{e}^{-z} \sum_{n=0}^{\infty} (-1)^n \left(\gamma + \frac{1}{2}\right)_n c_n z^{-n}, \tag{4.9}$$

where

$$c_n = \sum_{k=0}^{n} \frac{2^k}{k!(n-k)!} \left(\frac{1}{2} - \gamma\right)_k (-\sigma)_{n-k}. \tag{4.10}$$

In particular,

$$\lambda_{\gamma,0}^{(2)}(z) \sim \left(\frac{2}{z}\right)^{\gamma+1/2} \mathrm{e}^{-z} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\gamma + \frac{1}{2}\right)_n \left(\frac{1}{2} - \gamma\right)_n (2z)^{-n}. \tag{4.11}$$

Proof. There hold the following representations in the form (3.9) for the functions in the left-hand sides of (4.7), (4.8) and (4.9):

$$\lambda_{\gamma,\sigma}^{(1)}(z) = \frac{1}{\Gamma(\gamma)}\,\mathrm{e}^{-z} \int_0^{\infty} t^{\gamma-1/\beta} (1+t)^{\sigma}\, \mathrm{e}^{-zt}\, \mathrm{d}t, \tag{4.12}$$

$$\lambda_{\gamma,\sigma}^{(1/\gamma)}(z) = \frac{1}{\gamma}\,\mathrm{e}^{-z} \int_0^{\infty} (t+1)^{\sigma}\, \mathrm{e}^{-zt}\, \mathrm{d}t \tag{4.13}$$

and

$$\lambda_{\gamma,\sigma}^{(2)}(z) = \frac{2}{\Gamma(\gamma+1/2)}\,\mathrm{e}^{-z} \int_0^{\infty} t^{\gamma-1/2} (t+2)^{\gamma-1/2} (1+t)^{\sigma} \mathrm{e}^{-zt}\, \mathrm{d}t \tag{4.14}$$

respectively. We use arguments similar to those in the proof of Theorem 3.2. Namely we divide the integrals in (4.12),(4.13) and (4.14) into two integrals, being taken over $(0,A)$ $(A > 0)$ and (A,∞), and show that the second integrals are asymptotically small in compare with any terms of series in the right-hand sides of (4.7), (4.8) and (4.9). Then we apply relation (3.16) to the first integrals with $\alpha = 1$, $\beta = \gamma$, $f(t) = (1+t)^{\sigma}$ for $\lambda_{\gamma,\sigma}^{(1/\gamma)}(z)$, with $\alpha = \beta = 1$, $f(t) = (1+t)^{\sigma}$ for $\lambda_{\gamma,\sigma}^{(1/\gamma)}(z)$ and $\alpha = 1$, $\beta = \gamma + 1/2$, $f(t) = (t+2)^{\gamma+1/2}(1+t)^{\sigma}$ for $\lambda_{\gamma,\sigma}^{(2)}(z)$. The direct calculations yield the asymptotic results in (4.7), (4.8) and (4.9). (4.9) with $\sigma = 0$ yields (4.11). □

Remark 4.2. Using (1.3) and (4.7) with $\gamma = a$ and $\sigma = c - a - 1$, we obtain the asymptotic expansion for the Tricomi confluent hypergeometric function $\Psi(a,c;z)$ at infinity

$$\Psi(a,c;z) \sim z^{-a} \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (a+1-c)_n}{n!} z^{-n} \quad (z \to \infty, \mathrm{Re}(z) > 0). \tag{4.15}$$

Such an expansion is well known—see, for example [5, 6.13(1)].

Remark 4.3. Using (1.4) and (4.11) we arrive at the asymptotic expansion for the McDonald function $K_\gamma(z)$, as $z \to \infty$ $(\mathrm{Re}(z) > 0)$,

$$K_\gamma(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} \mathrm{e}^{-z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2} - \gamma\right)_n \left(\frac{1}{2} + \gamma\right)_n (2z)^{-n}. \tag{4.16}$$

Since

$$(-1)^n \left(\frac{1}{2} - \gamma\right)_n \left(\frac{1}{2} + \gamma\right)_n = \frac{2^{-2n}}{n!}[4\gamma^2 - 1][4\gamma^2 - 3^2]\cdots[4\gamma^2 - (2n-1)^2]$$

relation (4.16) coincides with the known asymptotic estimate for $K_\gamma(z)$ at infinity—for example, see [6, 7.13(7)].

5. Asymptotic expansions for Liouville fractional integrals

To obtain asymptotic representations for the Liouville fractional integral (1.7) and for the Erdelyi–Kober-type fractional integral (1.8) of power-exponential function $f(t)$ we use the following result proved in [12, Theorem 5.1].

Lemma 5.1. *Let* $\alpha \in \mathbf{C}$, $\mathrm{Re}(\alpha) > 0$ *and let* $\mu \in \mathbf{C}$, $v > 0$ *and* $\delta > 0$. *Then the relation*

$$(I_-^\alpha[t^{\mu-1}\exp(-\delta t^v)])(x) = x^{\alpha+\mu-1}\lambda^{(1/v)}_{\alpha+v-1,-1+\mu/v}(\delta x^v) \tag{5.1}$$

holds for $x > 0$.

Applying the asymptotic estimate (3.7) to (5.1), we obtain the following asymptotic expansion near zero for the Liouville fractional integral:

$$(I_-^\alpha[t^{\mu-1}\exp(-\delta t^v)])(x) \sim x^{\alpha+\mu-1}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(1-\alpha-\mu-vn)}{\Gamma(1-\mu-vn)}(-\delta x^v)^n \quad (x \to +0) \tag{5.2}$$

$$(\mu \in \mathbf{C};\ v > 0;\ \delta > 0;\ \alpha - 1 + \mu + vn \neq 0, 1, 2, \ldots;\ n = 0, 1, 2, \ldots). \tag{5.3}$$

From Theorem 3.2 and relation (5.1) we deduce the full asymptotic expansion at infinity for the Liouville fractional integral of power-exponential function.

Theorem 5.2. *Let* $\alpha \in \mathbf{C}(\mathrm{Re}(\alpha) > 0)$, $\mu \in \mathbf{C}$, $v > 0$ *and* $\delta > 0$. *Then the following relation holds, as* $x \to +\infty$,

$$(I_-^\alpha[t^{\mu-1}\exp(-\delta t^v)])(x) \sim \frac{(v\delta)^{-\alpha}}{\Gamma(\alpha)}x^{\mu-1+\alpha(1-v)}\exp(-\delta x^v)\sum_{n=0}^{\infty}(\alpha)_n c_n(\delta x^v)^{-n}, \tag{5.4}$$

in particular,

$$(I_-^\alpha[t^{\mu-1}\exp(-t^v)])(x) \sim \frac{v^{-\alpha}}{\Gamma(\alpha)}x^{\mu-1+\alpha(1-v)}\exp(-x^v)\sum_{n=0}^{\infty}(\alpha)_n c_n x^{-vn}. \tag{5.5}$$

Here

$$c_n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(-1)^{n-k}\left(\frac{\mu}{v}-1\right)_{n-k} g_k \quad (n=0,1,\ldots), \tag{5.6}$$

$$g_0 = 1, g_k = k!\sum_{m=0}^{k}\left(\lim_{t\to 0}[D^m([g_v(t)]^{\alpha-1})]\right)\sum\prod_{i=1}^{k}\frac{1}{P_i!}\left[\frac{\Gamma(1/v)}{(i+1)!\Gamma([1/v]-i)}\right]^{P_i} \tag{5.7}$$

for $k=1,2,\ldots,n$;

$$g_v(t) = \sum_{j=0}^{\infty}\frac{\Gamma(1/v)}{\Gamma(1/v-j)}\frac{t^j}{(j+1)!} \tag{5.8}$$

and a summation $\sum$ *is taken over all combinations of nonnegative integer values of* $P_1, P_2, \ldots, P_k$ *satisfying* (2.8).

Corollary 5.3. *Let* $\alpha\in\mathbf{C}(\mathrm{Re}(\alpha)>0)$, $\mu\in\mathbf{C}$ *and* $\delta>0$. *Then, as* $x\to+\infty$,

$$(I_-^\alpha[t^{\mu-1}\,\mathrm{e}^{-\delta t}])(x) \sim \delta^{-\alpha}x^{\mu-1}\,\mathrm{e}^{-\delta x}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\alpha)_n(1-\mu)_n(\delta x)^{-n}. \tag{5.9}$$

Corollary 5.4. *Let* $\alpha\in\mathbf{C}(\mathrm{Re}(\alpha)>0)$, $\mu\in\mathbf{C}$ *and* $\delta>0$. *Then, as* $x\to+\infty$,

$$(I_-^\alpha[t^{\mu-1}\exp(-\delta t^{1/2})])(x) \sim \left(\frac{2}{\delta}\right)^{\alpha} x^{\alpha/2+\mu-1}\exp(-\delta x^{1/2})\sum_{n=0}^{\infty}(-1)^n(\alpha)_n c_n(\delta x^{1/2})^{-n}, \tag{5.10}$$

where

$$c_n = \sum_{k=0}^{n}\frac{2^k}{k!(n-k)!}(1-\alpha)_k(1-2\mu)_{n-k} \quad (n=0,1,2\ldots). \tag{5.11}$$

In particular,

$$(I_-^\alpha[t^{-1/2}\exp(-\delta t^{1/2})])(x) \sim \left(\frac{2}{\delta}\right)^{\alpha} x^{(\alpha-1)/2}\exp(-\delta x^{1/2})\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}(\alpha)_n(1-\alpha)_n(\delta x^{1/2})^{-n} \quad (x\to+\infty). \tag{5.12}$$

Corollaries 5.3 and 5.4 follow from (5.1) with $v=1$ and $v=1/2$, respectively, if we take into account the asymptotic estimate (4.7) for the former while the asymptotic relations (4.9)–(4.11) for the latter.

Remark 5.5. In particular, if $\mu=m+1(m\in\mathbf{N})$, the asymptotic expansion in (5.9) yields the exact expression. Namely, it was proved in [12, Corollary 5.2] that if $\alpha\in\mathbf{C}$ $(\mathrm{Re}(\alpha)>0)$, $m\in\mathbf{N}$ and $\delta>0$, then there holds the relation for $x>0$

$$(I_-^\alpha[t^m\,\mathrm{e}^{-\delta t}])(x) = \delta^{-\alpha}x^m\,\mathrm{e}^{-\delta x}\sum_{n=0}^{m}\frac{(\alpha)_n(-m)_n}{n!\delta^n}\frac{(-1)^n}{x^n}. \tag{5.13}$$

6. Asymptotic expansions for Erdelyi–Kober-type fractional integrals

To obtain asymptotic representations for the Erdelyi–Kober-type fractional integral (1.8) of power-exponential function $f(t)$ we use the following result proved in [12, Theorem 5.2].

Lemma 6.1. *Let* $\alpha \in \mathbf{C}$, $\mathrm{Re}(\alpha) > 0$ *and let* $\beta > 0$, $\eta \in \mathbf{C}$, $\mu \in \mathbf{C}$, $v > 0$ *and* $\delta > 0$. *Then the relation*

$$(I^{\alpha}_{-;\beta,\eta}[t^{\mu}\exp(-\delta t^{v})])(x) = x^{\mu}\lambda^{(\beta/v)}_{\alpha-1+v/\beta,[\beta(1-\alpha-\eta)+\mu]/v-1}(\delta x^{v}) \tag{6.1}$$

is valid for $x > 0$.

Applying the asymptotic estimate (3.7) to (6.1), we obtain the following asymptotic expansion near zero for the Erdelyi–Kober-type fractional integral:

$$(I^{\alpha}_{-;\beta,\eta}[t^{\mu}\exp(-\delta t^{v})])(x) \sim x^{\mu}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma[\eta-(\mu+vn)/\beta)]}{\Gamma[\alpha+\eta-(\mu+vn)/\beta)]}(-\delta x^{v})^{n} \quad (x\to+0) \tag{6.2}$$

with $\eta \in \mathbf{C}$, $\beta > 0$, $\mu \in \mathbf{C}$, $v > 0$ and $\delta > 0$ such that

$$\left(-\eta+\frac{\mu+vn}{\beta}\neq 0,1,2,\ldots;\ n=0,1,2,\ldots\right). \tag{6.3}$$

From Theorem 3.2 and relation (6.1) we deduce the full asymptotic expansion at infinity for the Erdelyi–Kober-type fractional integral of power-exponential function.

Theorem 6.2. *Let* $\alpha \in \mathbf{C}$ $(\mathrm{Re}(\alpha) > 0)$. $\beta > 0$, $\eta \in \mathbf{C}$, $\mu \in \mathbf{C}$, $v > 0$ *and* $\delta > 0$. *Then the following relation is valid, as* $x \to +\infty$:

$$(I^{\alpha}_{-;\beta,\eta}[t^{\mu}\exp(-\delta t^{v})])(x) \sim \frac{(\beta/v\delta)^{\alpha}}{\Gamma(\alpha)}x^{\mu-v\alpha}\exp(-\delta x^{v})\sum_{n=0}^{\infty}(\alpha)_{n}c_{n}(\delta x^{v})^{-n}, \tag{6.4}$$

in particular,

$$(I^{\alpha}_{-;\beta,\eta}[t^{\mu}\exp(-t^{v})])(x) \sim \frac{(\beta/v)^{\alpha}}{\Gamma(\alpha)}x^{\mu-v\alpha}\exp(-x^{v})\sum_{n=0}^{\infty}(\alpha)_{n}c_{n}x^{-vn}. \tag{6.5}$$

Here

$$c_{n} = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-1)^{n-k}\left(1-\frac{\beta(1-\alpha-\eta)+\mu}{v}\right)_{n-k} g_{k}, \tag{6.6}$$

$$g_{0}=1, g_{k} = k!\sum_{m=0}^{k}\left(\lim_{t\to 0}[D^{m}([g_{\beta,v}(t)]^{\alpha-1})]\right)\sum\prod_{i=1}^{k}\frac{1}{P_{i}!}\left[\frac{\Gamma(1/v)}{(i+1)!\Gamma([\beta/v]-i)}\right]^{P_{i}} \tag{6.7}$$

for $k = 1,2,\ldots$;

$$g_{\beta,v}(t) = \sum_{j=1}^{\infty}\frac{\Gamma(\beta/v)}{\Gamma(\beta/v-j)}\frac{t^{j}}{(j+1)!} \tag{6.8}$$

and a summation $\sum$ *is taken over all combinations of nonnegative integer values of* $P_{1}, P_{2},\ldots,P_{k}$ *satisfying* (2.8).

Corollary 6.3. *Let* $\alpha\in\mathbf{C}(\mathrm{Re}(\alpha)>0)$. $\beta>0$, $\eta\in\mathbf{C}$, $\mu\in\mathbf{C}$ *and* $\delta>0$. *Then, as* $x\to+\infty$,

$$(I^{\alpha}_{-;\beta,\eta}[t^{\mu}\exp(-\delta t^{\beta})])(x)$$
$$\sim \delta^{-\alpha}x^{\mu-\alpha\beta}\exp(-\delta x^{\beta})\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\alpha)_n\left(\alpha+\eta-\frac{\mu}{\beta}\right)_n(\delta x^{\beta})^{-n}. \tag{6.9}$$

Corollary 6.4. *Let* $\alpha\in\mathbf{C}$ $(\mathrm{Re}(\alpha)>0)$, $\beta>0$, $\eta\in\mathbf{C}$, $\mu\in\mathbf{C}$ *and* $\delta>0$. *Then*

$$(I^{\alpha}_{-;\beta,\eta}[t^{\mu}\exp(-\delta t^{\beta/2})])(x)$$
$$\sim\left(\frac{2}{\delta}\right)^{\alpha}x^{\mu-\alpha\beta/2}\exp(-\delta x^{\beta/2})\sum_{n=0}^{\infty}(-1)^n(\alpha)_n c_n(\delta x^{\beta/2})^{-n}\quad(x\to+\infty), \tag{6.10}$$

where

$$c_n=\sum_{k=0}^{n}\frac{2^k}{k!(n-k)!}(1-\alpha)_k\left(2\left[\alpha+\eta-\frac{\mu}{\beta}\right]-1\right)_{n-k}\quad(n=0,1,2\ldots). \tag{6.11}$$

In particular, as $x\to+\infty$,

$$(I^{\alpha}_{-;\beta,\eta}[t^{\beta(\alpha+\eta-1/2)}\exp(-\delta t^{\beta/2})])(x)$$
$$\sim\left(\frac{2}{\delta}\right)^{\alpha}x^{\beta[\eta+(\alpha-1)/2]}\exp(-\delta x^{\beta/2})\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}(\alpha)_n(1-\alpha)_n(\delta x^{\beta/2})^{-n}. \tag{6.12}$$

Corollaries 6.3 and 6.4 follow from (6.1) with $v=\beta$ and $v=\beta/2$, respectively, if we take into account the asymptotic estimate (4.7) for the former while the asymptotic relations (4.9)–(4.11) for the latter.

Remark 6.5. In particular, when $\mu=\beta(\alpha+\eta+m)$ $(m\in\mathbf{N})$, the asymptotic expansion in (6.9) yields the exact expression. Namely, it was proved in [12, Corollary 6.3] that if $\alpha\in\mathbf{C}(\mathrm{Re}(\alpha)>0)$, $\beta>0$, $\eta\in\mathbf{C}$, $m\in\mathbf{N}$, $\delta>0$, then there holds the relation for $x>0$

$$(I^{\alpha}_{-;\beta,\eta}[t^{\beta(\alpha+\eta+m)}\exp(-\delta t^{\beta})])(x)=\delta^{-\alpha}x^{\beta(\eta+m)}\exp(-\delta x^{\beta})\sum_{n=0}^{m}\frac{(\alpha)_n(-m)_n}{n!\delta^n}\frac{(-1)^n}{x^{\beta n}}. \tag{6.13}$$

7. Asymptotic expansions for fractional integrals of a function by a power function

It is known [26, (18.39)] the following connection between the fractional integral $I^{\alpha}_{-;x^{\beta}}f$ and the Erdelyi–Kober-type fractional integral $I^{\alpha}_{-;\beta,-\alpha}f$:

$$(I^{\alpha}_{-;x^{\beta}}f)(x)=x^{\beta\alpha}(I^{\alpha}_{-;\beta,0}f)(x)\quad(x\in\mathbf{R}_+;\alpha\in\mathbf{C},\mathrm{Re}(\alpha)>0;\ \beta>0). \tag{7.1}$$

Therefore, the asymptotic expansions for the fractional integral (1.9) follow from (7.1) and the results in Section 6 with $\eta=-\alpha$.

From (7.1) and (6.2) we deduce the asymptotic expansion of $(I^{\alpha}_{-;x^\beta}f)(x)$, as $x \to +0$,

$$(I^{\alpha}_{-;x^\beta}[t^\mu \exp(-\delta t^v)])(x) \sim x^{\mu+\beta\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma[-\alpha-(\mu+vn)/\beta)]}{\Gamma[-(\mu+vn)/\beta)]}(-\delta x^v)^n \tag{7.2}$$

$$\left(\beta>0;\mu\in\mathbf{C};v>0;\delta>0;\alpha+\frac{\mu+vn}{\beta}\neq 0,1,2,\ldots;n=0,1,2,\ldots\right). \tag{7.3}$$

Relation (7.1) and Theorem 6.2 yield the asymptotic expansion for the fractional integral (1.9) at infinity.

Theorem 7.1. *Let $\alpha\in\mathbf{C}(\mathrm{Re}(\alpha)>0)$, $\mu\in\mathbf{C}$, $\beta>0$ and $\delta>0$. Then there holds the following relation, as $x\to+\infty$,*

$$(I^{\alpha}_{-;x^\beta}[t^\mu \exp(-\delta t^v)])(x) \sim \frac{(\beta/v\delta)^\alpha}{\Gamma(\alpha)}x^{\mu+\alpha(\beta-v)}\exp(-\delta x^v)\sum_{n=0}^{\infty}(\alpha)_n c_n(\delta x^v)^{-n}, \tag{7.4}$$

in particular

$$(I^{\alpha}_{-;x^\beta}[t^\mu \exp(-t^v)])(x) \sim \frac{(\beta/v)^\alpha}{\Gamma(\alpha)}x^{\mu+\alpha(\beta-v)}\exp(-x^v)\sum_{n=0}^{\infty}(\alpha)_n c_n x^{-vn}. \tag{7.5}$$

Here

$$c_n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-1)^{n-k}\left(1-\frac{\beta+\mu}{v}\right)_{n-k}g_k, \tag{7.6}$$

where g_k $(k=0,1,2,\ldots)$ are given by (6.7)–(6.8).

Corollary 7.2. *Let $\alpha\in\mathbf{C}(\mathrm{Re}(\alpha)>0)$, $\beta>0$, $\mu\in\mathbf{C}$ and $\delta>0$. Then, as $x\to+\infty$,*

$$(I^{\alpha}_{-;x^\beta}[t^\mu\exp(-\delta t^\beta)])(x)$$
$$\sim \delta^{-\alpha}x^\mu\exp(-\delta x^\beta)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\alpha)_n\left(-\frac{\mu}{\beta}\right)_n(\delta x^\beta)^{-n}, \tag{7.7}$$

Corollary 7.3. *Let $\alpha\in\mathbf{C}(\mathrm{Re}(\alpha)>0)$, $\beta>0$, $\mu\in\mathbf{C}$ and $\delta>0$. Then*

$$(I^{\alpha}_{-;x^\beta}[t^\mu\exp(-\delta t^{\beta/2})])(x)$$
$$\sim\left(\frac{2}{\delta}\right)^\alpha x^{\mu+\alpha\beta/2}\exp(-\delta x^{\beta/2})\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\alpha)_n\left(\alpha-\frac{\mu}{\beta}\right)_n(\delta x^{\beta/2})^{-n}, \tag{7.8}$$

as $x\to+\infty$, where

$$c_n=\sum_{k=0}^{n}\frac{2^k}{k!(n-k)!}(1-\alpha)_k\left(-\frac{2\mu}{\beta}-1\right)_{n-k}\quad(n=0,1,2\ldots). \tag{7.9}$$

In particular, as $x \to +\infty$,

$$(I^{\alpha}_{-;\,x^\beta}[t^{\beta(\alpha+\eta-1/2)}\exp(-\delta t^\beta)])(x)$$
$$\sim \left(\frac{2}{\delta}\right)^{\alpha} x^{\beta[\eta+(\alpha+1)/2]}\exp(-\delta x^{\beta/2})\sum_{n=0}^{m}\frac{(-2)^n}{n!}(\alpha)_n(1-\alpha)_n(\delta x^{\beta/2})^{-n}. \tag{7.10}$$

Corollaries 7.2 and 7.3 follow from (7.1) and Corollaries 6.3 and 6.4 with $\eta=-\alpha$.

Remark 7.4. In particular, if $\mu=\beta m (m\in\mathbf{N})$, the asymptotic expansion in (7.7) yields the exact expression for $(I^{\alpha}_{-;\,x^\beta}[t^{\beta m}])(x)$. It is directly proved that if $\alpha\in\mathbf{C}$ $(\mathrm{Re}(\alpha)>0)$, $\beta>0$, $m\in\mathbf{N}$, $\delta>0$, then there holds the relation for $x>0$

$$(I^{\alpha}_{-;\,x^\beta}[t^{\beta m}\exp(-\delta t^\beta)])(x)$$
$$=\delta^{-\alpha}x^{\beta m}\exp(-\delta x^\beta)\sum_{n=0}^{m}\frac{(-1)^n}{n!}(\alpha)_n(-m)_n(\delta x^\beta)^{-n}. \tag{7.11}$$

Such a formula also can be deduced from (5.14) if we take into account the connection between the fractional derivative $I^{\alpha}_{-;\,x^\beta}f$ and the Liouville fractional derivative $I^{\alpha}_{-}f$ given by

$$(I^{\alpha}_{-;\,x^\beta}f(t))(x)=(I^{\alpha}_{-}f(t^{1/\beta}))(x^\beta). \tag{7.12}$$

8. Asymptotic expansions for Kober fractional integrals

The Kober fractional integral (1.10) is connected with the Erdelyi–Kober-type fractional integral (1.8) by the following formula:

$$(K^{-}_{\eta,\alpha}f)(x)=(I^{\alpha}_{-;1,\eta}f)(x)\ (x\in\mathbf{R}_+;\ \alpha\in\mathbf{C},\ \mathrm{Re}(\alpha)>0;\ \eta\in\mathbf{C}). \tag{8.1}$$

So (8.1) and the results in Section 6 with $\beta=1$ yield the asymptotic expansions for this integral. (8.1) and (6.2) yield the asymptotic relation for the Kober fractional integral near zero

$$(K^{-}_{\eta,\alpha}[t^\mu\exp(-\delta t^\nu)])(x)\sim x^\mu\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\eta-\mu-\nu n)}{\Gamma(\alpha+\eta-\mu-\nu n)}(-\delta x^\nu)^n\ (x\to+0) \tag{8.2}$$

$$(\eta\in\mathbf{C};\ \mu\in\mathbf{C};\ \nu>0;\ \delta>0;\ -\eta+\mu+\nu n\neq 0,1,2,\cdots;\ n=0,1,2,\cdots). \tag{8.3}$$

From (8.1) and Theorem 6.2 we deduce the asymptotic expansion for the fractional integral (1.10) at infinity.

Theorem 8.1. *Let* $\alpha\in\mathbf{C}(\mathrm{Re}(\alpha)>0)$, $\eta\in\mathbf{C}$, $\mu\in\mathbf{C}$, $\nu>0$ *and* $\delta>0$. *Then the following relation holds, as* $x\to+\infty$:

$$(K^{-}_{\eta,\alpha}[t^\mu\exp(-\delta t^\nu)])(x)\sim\frac{(\nu\delta)^{-\alpha}}{\Gamma(\alpha)}x^{\mu-\nu\alpha}\exp(-\delta x^\nu)\sum_{n=0}^{\infty}(\alpha)_n c_n(\delta x^\nu)^{-n}, \tag{8.4}$$

in particular,

$$(K^{-}_{\eta,\alpha}[t^{\mu}\exp(-t^{\nu})])(x) \sim \frac{(\nu)^{-\alpha}}{\Gamma(\alpha)} x^{\mu-\nu\alpha}\exp(-x^{\nu})\sum_{n=0}^{\infty}(\alpha)_n c_n x^{-\nu n}. \tag{8.5}$$

Here

$$c_n = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-1)^{n-k}\left(1-\frac{1-\alpha-\eta+\mu}{\nu}\right)_{n-k} g_k, \tag{8.6}$$

where g_k $(k=0,1,2,\cdots)$ *are given by* (6.7)–(6.8) *with* $\beta=1$.

Corollary 8.2. *Let* $\alpha\in\mathbf{C}$ $(\mathrm{Re}(\alpha)>0)$, $\eta\in\mathbf{C}$, $\mu\in\mathbf{C}$ *and* $\delta>0$. *Then, as* $x\to+\infty$,

$$(K^{-}_{\eta,\alpha}[t^{\mu}\,\mathrm{e}^{-\delta t}])(x)$$
$$\sim \delta^{-\alpha}x^{\mu-\alpha}\,\mathrm{e}^{-\delta x}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\alpha)_n(\alpha+\eta-\mu)_n(\delta x)^{-n}. \tag{8.7}$$

Corollary 8.3. *Let* $\alpha\in\mathbf{C}$ $(\mathrm{Re}(\alpha)>0)$, $\eta\in\mathbf{C}$, $\mu\in\mathbf{C}$ *and* $\delta>0$. *Then*

$$(K^{-}_{\eta,\alpha}[t^{\mu}\exp(-\delta t^{1/2})])(x)$$
$$\sim\left(\frac{2}{\delta}\right)^{\alpha}x^{\mu-\alpha/2}\exp\left(-\delta x^{\beta/2}\right)\sum_{n=0}^{\infty}(-1)^n(\alpha)_n c_n(\delta x^{\beta/2})^{-n}\quad (x\to+\infty), \tag{8.8}$$

where

$$c_n=\sum_{k=0}^{n}\frac{2^k}{k!(n-k)!}(1-\alpha)_k(2[\alpha+\eta-\mu]-1)_{n-k}\quad (n=0,1,2\cdots). \tag{8.9}$$

In particular, as $x\to+\infty$,

$$(K^{-}_{\eta,\alpha}[t^{\alpha+\eta-1/2}\exp(-\delta t^{1/2})])(x)$$
$$\sim\left(\frac{2}{\delta}\right)^{\alpha}x^{\eta+(\alpha-1)/2}\exp(-\delta x^{\beta/2})\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}(\alpha)_n(1-\alpha)_n(\delta x^{\beta/2})^{-n}. \tag{8.10}$$

Corollaries 8.2 and 8.3 follow from (8.1) and Corollaries 6.3 and 6.4 with $\beta=1$.

Remark 8.4. In particular, when $\mu=\alpha+\eta+m$ $(m\in\mathbf{N})$, the asymptotic expansion in (8.7) yields the exact expression. Namely, it was proved in [12, Corollary 7.5] that if $\alpha\in\mathbf{C}$ $(\mathrm{Re}(\alpha)>0)$, $\eta\in\mathbf{C}$, $m\in\mathbf{N}$ and $\delta>0$, then there holds the relation for $x>0$

$$(K^{-}_{\eta,\alpha}[t^{\alpha+\eta+\mu}\,\mathrm{e}^{-\delta t}])(x)=\delta^{-\alpha}x^{\eta+m)}\,\mathrm{e}^{-\delta x}\sum_{n=0}^{m}\frac{(\alpha)_n(-m)_n}{n!\delta^n}\frac{(-1)^n}{x^n}. \tag{8.11}$$

9. Asymptotic expansions for Erdelyi–Kober fractional integrals

The Erdelyi–Kober fractional integral (1.11) is connected with the Erdelyi–Kober-type fractional integral (1.8) by the following relation:

$$(K_{\eta,\alpha}f)(x) = (I^{\alpha}_{-;2,\eta}f)(x) \quad (x \in \mathbf{R}_+;\ \alpha \in \mathbf{C}, \mathrm{Re}(\alpha) > 0;\ \eta \in \mathbf{C}). \tag{9.1}$$

Thus the asymptotic expansions for this integral follow from (9.1) and the results in Section 6 with $\beta = 2$.

From (9.1) and (6.2) we obtain the asymptotic relation for the Erdelyi–Kober integral near zero

$$(K_{\eta,\alpha}[t^{\mu}\exp(-\delta t^{v})])(x) \sim x^{\mu}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma[\eta-(\mu+vn)/2]}{\Gamma[\alpha+\eta-(\mu+vn)/2]}(-\delta x^{v})^{n} \quad (x \to +0) \tag{9.2}$$

$$\left(\eta \in \mathbf{C};\ \mu \in \mathbf{C};\ v > 0;\ \delta > 0;\ -\eta + \frac{\mu+vn}{2} \neq 0, 1, 2, \ldots;\ n = 0, 1, 2, \ldots\right). \tag{9.3}$$

Relation (9.1) and Theorem 6.2 yield the asymptotic expansion for the fractional integral (1.11) at infinity.

Theorem 9.1. *Let $\alpha \in \mathbf{C}$ ($\mathrm{Re}(\alpha) > 0$), $\eta \in \mathbf{C}$, $\mu \in \mathbf{C}$, $v > 0$ and $\delta > 0$. Then there holds the asymptotic relation, as $x \to +\infty$,*

$$(K_{\eta,\alpha}[t^{\mu}\exp(-\delta t^{v})])(x) \sim \frac{(2/v\delta)^{\alpha}}{\Gamma(\alpha)}x^{\mu-v\alpha}\exp(-\delta x^{v})\sum_{n=0}^{\infty}(\alpha)_n c_n(\delta x^{v})^{-n}, \tag{9.4}$$

in particular,

$$(K_{\eta,\alpha}[t^{\mu}\exp(-t^{v})])(x) \sim \frac{(2/v)^{\alpha}}{\Gamma(\alpha)}x^{\mu-v\alpha}\exp(-x^{v})\sum_{n=0}^{\infty}(\alpha)_n c_n x^{-vn}. \tag{9.5}$$

Here

$$c_n = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-1)^{n-k}\left(1-\frac{2(1-\alpha-\eta)+\mu}{v}\right)_{n-k}g_k, \tag{9.6}$$

where g_k ($k = 0, 1, 2, \ldots$) are given by (6.7)–(6.8) with $\beta = 2$.

Corollary 9.2. *Let $\alpha \in \mathbf{C}$ ($\mathrm{Re}(\alpha) > 0$), $\eta \in \mathbf{C}$, $\mu \in \mathbf{C}$ and $\delta > 0$. Then, as $x \to +\infty$,*

$$(K_{\eta,\alpha}[t^{\mu}\exp(-\delta t^{2})])(x)$$
$$\sim \delta^{-\alpha}x^{\mu-2\alpha}\exp(-\delta x^{2})\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}(\alpha)_n\left(\alpha+\eta-\frac{\mu}{2}\right)_n(\delta x^{2})^{-n}. \tag{9.7}$$

Corollary 9.3. *Let* $\alpha \in \mathbf{C}$ $(\mathrm{Re}(\alpha) > 0)$, $\eta \in \mathbf{C}$, $\mu \in \mathbf{C}$ *and* $\delta > 0$. *Then*

$$(K_{\eta,\alpha}[t^{\mu}\,\mathrm{e}^{-\delta t}])(x) \sim \left(\frac{2}{\delta}\right)^{\alpha} x^{\mu-\alpha/2}\,\mathrm{e}^{-\delta x}\sum_{n=0}^{\infty}(-1)^n(\alpha)_n c_n(\delta x)^{-n} \quad (x \to +\infty), \tag{9.8}$$

where

$$c_n = \sum_{k=0}^{n}\frac{2^k}{k!(n-k)!}(1-\alpha)_k\left(2\left[\alpha+\eta-\frac{\mu}{2}\right]-1\right)_{n-k} \quad (n = 0,1,2\cdots). \tag{9.9}$$

In particular, as $x \to +\infty$,

$$(K_{\eta,\alpha}[t^{2(\alpha+\eta)-1}\,\mathrm{e}^{-\delta t}])(x) \sim \left(\frac{2}{\delta}\right)^{\alpha} x^{2\eta+\alpha-1}\,\mathrm{e}^{-\delta x}\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}(\alpha)_n(1-\alpha)_n(\delta x)^{-n}. \tag{9.10}$$

Corollaries 9.2 and 9.3 follow from (9.1) and Corollaries 6.3 and 6.4 with $\beta = 2$.

Remark 9.4. In particular, if $\mu = 2(\alpha+\eta+m)$ $(m \in \mathbf{N})$, the asymptotic expansion in (9.7) yields the exact expression. Namely, it was proved in [12, Corollary 7.2] that if $\alpha \in \mathbf{C}$ $(\mathrm{Re}(\alpha) > 0)$, $\eta \in \mathbf{C}$, $m \in \mathbf{N}$ and $\delta > 0$, then there holds the relation for $x > 0$

$$(K_{\eta,\alpha}[t^{2(\alpha+\eta+\mu)}\exp(-\delta t^2)])(x) = \delta^{-\alpha}x^{2(\eta+m)}\exp(-\delta x^2)\sum_{n=0}^{m}\frac{(\alpha)_n(-m)_n}{n!\,\delta^n}\frac{(-1)^n}{x^{2n}}. \tag{9.11}$$

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