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# On the mod-$p$ cohomology of $Out(F_{2(p-1)})$

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ABSTRACT

We study the mod-$p$ cohomology of the group $Out(F_n)$ of outer automorphisms of the free group $F_n$ in the case $n = 2(p-1)$ which is the smallest $n$ for which the $p$-rank of this group is 2. For $p = 3$ we give a complete computation, at least above the virtual cohomological dimension of $Out(F_4)$ (which is 5). More precisely, we calculate the equivariant cohomology of the $p$-singular part of outer space for $p = 3$. For a general prime $p > 3$ we give a recursive description in terms of the mod-$p$ cohomology of $Aut(F_k)$ for $k \leq p-1$. In this case we use the $Out(F_{2(p-1)})$-equivariant cohomology of the poset of elementary abelian $p$-subgroups of $Out(F_n)$.



## 1. Introduction: Background and results

Let $F_n$ denote the free group on $n$ generators and let $Out(F_n)$denote its group of outer automorphisms. We are interested in the cohomology ring $H^*(Out(F_n);\mathbb{F}_p)$ with coefficients in the prime field $\mathbb{F}_p$. The case $n = 2$ is well understood because of the isomorphism $Out(F_2) \cong GL_2(\mathbb{Z})$ and the stable cohomology of $Out(F_n)$ has been shown to agree with that of symmetric groups [1]. Apart from these results the only other complete calculation is that of the integral cohomology ring of $Out(F_3)$ by Brady [2]. There has been a fair amount of work on the Farell cohomology of $Out(F_n)$, but always in cases where the $p$-rank of $Out(F_n)$ is one [3–5]. (We recall that the $p$-rank of a group $G$ is the maximal $k$ for which $(\mathbb{Z}/p)^k$ embeds into $G$.) The $p$-rank of $Out(F_n)$ is known to be $[\frac{n}{p-1}]$. In this paper we consider the case $n = 2(p-1)$ for $p$ odd, i.e. the first case of $p$-rank two and compute the mod-$p$ cohomology ring, at least above dimension $2n-3$, the virtual cohomological dimension of $Out(F_n)$. For $p = 3$, $n = 4$ our result is completely explicit and can be neatly described in terms of the cohomology of certain finite subgroups of $Out(F_4)$. For $p > 3$ our result has a recursive nature, i.e. we express the result in terms of automorphism groups of free groups of lower rank. We remark that the cohomology of the closely related group of automorphisms $Aut(F_{2(p-1)})$ has been studied by Jensen [6]. In fact, for $p > 3$ we use essentially the same method and we take advantage of some of the work done in [6].

We will now describe our results. We start by exhibiting certain finite subgroups of $Out(F_n)$. For this we identify $F_n$ with the fundamental group $\pi_1(R_n)$ of the wedge of $n$ circles which we consider as a graph with one vertex and $n$ edges. As such it is also called the rose with $n$ leaves and is denoted $R_n$. If $\Gamma$ is a finite graph and $\alpha : R_n \to \Gamma$ is an (unpointed) homotopy equivalence then $\alpha$ induces an isomorphism $\alpha_* : Out(F_n) = Out(\pi_1(R_n)) \cong Out(\pi_1(\Gamma))$. The group $Aut(\Gamma)$ of graph automorphisms of $\Gamma$ is a finite group which for $n > 1$ embeds naturally into $Out(\pi_1(\Gamma))$ [7]. Therefore $\alpha_*^{-1}(Aut(\Gamma))$ is a finite subgroup of $Out(F_n)$. Note that the choice of (the homotopy class) of $\alpha$ (which is also called a marking of $\Gamma$) is important for getting an actual subgroup, and that running through the different choices for $\alpha$ amounts to running through the different representatives in the same conjugacy class of subgroups.

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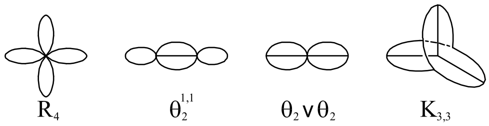



**Fig. 1.** The graphs whose automorphism groups determine $H^*(Out(F_4); \mathbb{F}_3)$ for $* > 5$.

Now let $p = 3$, $n = 4$ and let $R_4$ be the rose with four leaves (cf. Fig. 1). Its automorphism group can be identified with the wreath product $\mathbb{Z}/2 \wr \Sigma_4$. Choosing $\alpha = id : R_4 \to R_4$ gives us a subgroup of $Out(F_n)$ which we denote by $G_R$.

Next let $\Theta_2$ be the connected graph with two vertices and 3 edges between these two vertices, and let $\Theta_2^{1,1}$ be the wedge of $\Theta_2$ with a rose with one leaf attached to each of the two vertices of $\Theta_2$ (cf. Fig. 1). Its automorphism group is $\Sigma_3 \times D_8$ where $D_8$ is the dihedral group of order 8. Choosing any homotopy equivalence between $R$ and $\Theta_2^{1,1}$ gives us a subgroup of $Out(F_4)$ which we denote by $G_{1,1}$.

The automorphism group of the wedge $\Theta_2 \vee \Theta_2$ (cf. Fig. 1) is $\Sigma_3 \wr \mathbb{Z}/2$. After choosing a homotopy equivalence $R_4 \to \Theta_2 \vee \Theta_2$ we get a subgroup of $Out(F_4)$ which we denote by $G_2$.

Let $K_{3,3}$ be the Kuratowski graph with two blocks of 3 vertices and 9 edges which join all vertices from the first block to all vertices of the second block (cf. Fig. 1). Its automorphism group is again $\Sigma_3 \wr \mathbb{Z}/2$ and after choosing a homotopy equivalence $R_4 \to K_{3,3}$ we obtain a subgroup $G_K$ of $Out(F_4)$.

We will see below (cf. Section 3.1) that the subgroup $\Sigma_3 \times \mathbb{Z}/2$ of $G_K$ which is given by permuting the three vertices in the first block and independently two of the three vertices in the second block is conjugate in $Out(F_4)$ to the "diagonal" subgroup $\Delta\Sigma_3 \times \mathbb{Z}/2$ of $G_2 \cong \Sigma_3 \wr \mathbb{Z}/2$. By choosing appropriate markings of $\Theta_2 \vee \Theta_2$ and $K_{3,3}$ we can assume that this subgroup is the same. We denote it by $H$. Let $G_2 *_H G_K$ be the amalgamated product.

**Theorem 1.1.** *The inclusions of the finite subgroups $G_R$, $G_{1,1}$, $G_2$ and $G_K$ into $Out(F_4)$ induce a homomorphism of groups*

$$G_R * G_{1,1} * (G_2 *_H G_K) \to Out(F_4)$$

*and the induced map in mod 3-cohomology*

$$H^*(Out(F_4); \mathbb{F}_3) \to H^*(G_R; \mathbb{F}_3) \times H^*(G_{1,1}; \mathbb{F}_3) \times H^*(G_2 *_H G_K; \mathbb{F}_3)$$

*is an isomorphism above dimension 5.*

The mod-3 cohomology of the first two factors is the same as that of $\Sigma_3$, i.e. it is isomorphic to the tensor product $\mathbb{F}_3[a_4] \otimes \Lambda(b_3)$ of a polynomial algebra generated by a class $a_4$ of dimension 4 and an exterior algebra generated by a class $b_3$ of dimension 3. The cohomology of the amalgamated product can also be easily computed and we obtain the following explicit result.

**Corollary 1.2.** *In dimensions bigger than 5 we have*

$$H^*(Out(F_4); \mathbb{F}_3) \cong \prod_{i=1}^{2} \mathbb{F}_3[a_4^{(i)}] \otimes \Lambda(b_3^{(i)}) \times \mathbb{F}_3[r_4, r_8] \otimes \Lambda(s_3)\{1, t_7, \tilde{t}_7, t_8\}.$$

*Here the lower indices indicate the dimensions of the cohomology classes and the last factor is described as a free module of rank 4 over $\mathbb{F}_3[r_4, r_8] \otimes \Lambda(s_3)$ on the indicated classes. (The full multiplicative structure is given in Proposition 3.3.)*

We derive these results by analyzing the Borel construction $EOut(F_n) \times_{Out(F_n)} K_n$ with respect to the action of $Out(F_n)$ on the "spine $K_n$ of outer space" [8]. We recall that $K_n$ is a contractible simplicial complex of dimension $2n-3$ on which $Out(F_n)$ acts simplicially with finite isotropy groups, in particular the Borel construction is a classifying space for $Out(F_n)$. However, $K_4$ is already very difficult to analyze. We replace it therefore by its smaller and more accessible 3-singular locus $(K_4)_s$ so that our result really describes the mod-3 cohomology of $EOut(F_4) \times_{Out(F_4)} (K_4)_s$ which agrees with that of $Out(F_4)$ above degree 5.

However, for $p > 5$ and $n = 2(p-1)$, even the $p$-singular locus of $K_n$ becomes too difficult to analyze directly. An alternative approach towards the mod-$p$ cohomology of $Out(F_n)$ uses the normalizer spectral sequence [9] which is associated to the action of $Out(F_n)$ on the poset $\mathcal{A}$ of its elementary abelian $p$-subgroups and which also calculates the mod-$p$ cohomology of $Out(F_n)$ above its finite virtual cohomological dimension $2n-3$. This approach has also been used by Jensen [6] in the case of $Aut(F_{2(p-1)})$. Our analysis of the normalizers of elementary abelian subgroups of $Out(F_{2(p-1)})$ is quite similar to that of the normalizers in $Aut(F_{2(p-1)})$ which was carried out in [6] using results of Krstic [10].

In order to describe our result for $Out(F_{2(p-1)})$ we need to describe certain elementary abelian $p$- subgroups of $Out(F_{2(p-1)})$. For this we consider the following graphs. Let $R_{2(p-1)}$ denote the rose with $2(p-1)$ leaves, $\Theta_{p-1}$ denote the graph with two vertices with $p$ edges between them, $\Theta_{p-1}^{s,t}$ with $s+t = p-1$ denote the graph which is obtained from $\Theta_{p-1}$ by attaching a rose with $s$ leaves at one vertex and one with $t$ leaves at the other vertex of $\Theta_{p-1}$. Furthermore let $\Theta_{p-1} \vee \Theta_{p-1}$ denote the wedge of two $\Theta_{p-1}$ at a common vertex, see Fig. 2.

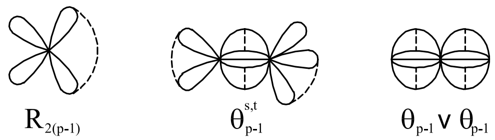



**Fig. 2.** The graphs which determine $H^*(Out(F_{2p-2}); \mathbb{F}_p)$ for $p > 3$ and $* > 4p - 7$.

After choosing appropriate markings these graphs determine as before subgroups

$$G_R \cong \mathbb{Z}/2 \wr \Sigma_{2(p-1)}$$
$$G_{s,p-1-s} \cong \begin{cases} (\mathbb{Z}/2 \wr \Sigma_s) \times \Sigma_p \times (\mathbb{Z}/2 \wr \Sigma_{p-1-s}) & \text{if } s \neq \frac{p-1}{2} \\ \big((\mathbb{Z}/2 \wr \Sigma_s) \times \Sigma_p \times (\mathbb{Z}/2 \wr \Sigma_s)\big) \rtimes \mathbb{Z}/2 & \text{if } s = \frac{p-1}{2} \end{cases}$$
$$G_2 \cong \Sigma_p \wr \mathbb{Z}/2.$$

(In the third line the action of $\mathbb{Z}/2$ is trivial on the middle factor $\Sigma_p$ while it interchanges the other two factors.) The $p$-Sylow subgroups in all these groups are elementary abelian. We choose $p$-Sylow subgroups and denote them by $E_R \cong \mathbb{Z}/p$ resp. $E_{s,p-1-s} \cong \mathbb{Z}/p$ resp. $E_2 \cong \mathbb{Z}/p \times \mathbb{Z}/p$. With a suitable choice of markings and $p$-Sylow subgroups we can assume that $E_{0,p-1}$ is one of the two factors of $E_2 \cong \mathbb{Z}/p \times \mathbb{Z}/p$. The structure of the normalizers of these elementary abelian subgroups and their relevant intersections is summarized in the following result.

In this result we will abbreviate the normalizer $N_{Out(F_{2(p-1)})}(E_R)$ of $E_R$ in $Out(F_{2(p-1)})$ by $N_R$, and likewise $N_{Out(F_{2(p-1)})}(E_{s,p-1-s})$ by $N_{s,p-1-s}$ and $N_{Out(F_{2(p-1)})}(E_2)$ by $N_2$; the normalizer of the diagonal subgroup $\Delta(E_2)$ of $E_2$ is abbreviated by $N_\Delta$. Furthermore $N_{\Sigma_p}(\mathbb{Z}/p)$ denotes the normalizer of $\mathbb{Z}/p$ in $\Sigma_p$ and $Aut(F_n)$ denotes the group of automorphisms of $F_n$.

**Proposition 1.3.** (a) *The groups $E_R$, $E_{s,p-1-s}$ for $0 \le s \le \frac{p-1}{2}$, $E_2$ and the diagonal $\Delta(E_2)$ of $E_2$ are pairwise non-conjugate, and any elementary abelian p-subgroup of $Out(F_{2(p-1)})$ is conjugate to one of them.*

(b) *$E_{0,p-1}$ is subconjugate to $E_2$, and neither $E_R$ nor any of the $E_{s,p-1-s}$ with $1 \le s \le \frac{p-1}{2}$ is subconjugate to $E_2$.*

(c) *There are canonical isomorphisms*

$$N_R \cong N_{\Sigma_p}(\mathbb{Z}/p) \times \big((F_{p-2} \rtimes Aut(F_{p-2})) \rtimes \mathbb{Z}/2\big)$$
$$N_{s,p-1-s} \cong N_{\Sigma_p}(\mathbb{Z}/p) \times Aut(F_s) \times Aut(F_{p-1-s}) \quad \textit{if } 0 \le s < \frac{p-1}{2}$$
$$N_{s,s} \cong N_{\Sigma_p}(\mathbb{Z}/p) \times (Aut(F_s) \wr \mathbb{Z}/2) \quad \textit{if } s = \frac{p-1}{2}$$
$$N_\Delta \cong \big((\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes (Aut(\mathbb{Z}/p) \times \mathbb{Z}/2)\big) *_{N_{\Sigma_p}(\mathbb{Z}/p) \times \mathbb{Z}/2} (N_{\Sigma_p}(\mathbb{Z}/p) \times \Sigma_3)$$
$$N_2 \cong N_{\Sigma_p}(\mathbb{Z}/p) \wr \mathbb{Z}/2$$

*where in the semidirect product $(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes (Aut(\mathbb{Z}/p) \times \mathbb{Z}/2)$ the group $Aut(\mathbb{Z}/p)$ acts diagonally on $\mathbb{Z}/p \times \mathbb{Z}/p$ and $\mathbb{Z}/2$ acts by interchanging the two factors. (We refer to Proposition 5.4. for a precise definition of the amalgamated product in the second to last isomorphism.)*

(d) *There are canonical isomorphisms*

$$N_{0,p-1} \cap N_2 \cong N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p).$$
$$N_\Delta \cap N_2 \cong (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes (Aut(\mathbb{Z}/p) \times \mathbb{Z}/2)$$

*where in the semidirect product in the second line $Aut(\mathbb{Z}/p)$ acts again diagonally on $\mathbb{Z}/p \times \mathbb{Z}/p$ and $\mathbb{Z}/2$ acts by interchanging the two factors.*

The evaluation of the normalizer spectral sequence yields the following result.

**Theorem 1.4.** *Let $p > 3$ be a prime and $n = 2(p - 1)$.*

(a) *The inclusions of the subgroups $N_R$, $N_{s,p-s}$ and $N_2$ into $Out(F_n)$ induce a homomorphism of groups*

$$N_R * N_{1,p-2} * N_{2,p-3} \ldots * N_{\frac{p-1}{2},\frac{p-1}{2}} * (N_{0,p-1} *_{N_{0,p-1} \cap N_2} N_2) \to Out(F_n)$$

*and the induced map*

$$H^*(Out(F_n); \mathbb{F}_p) \to H^*(N_R; \mathbb{F}_p) \times \prod_{s=1}^{\frac{p-1}{2}} H^*(N_{s,p-1-s}; \mathbb{F}_p) \times H^*(N_{0,p-1} *_{N_{0,p-1} \cap N_2} N_2; \mathbb{F}_p)$$

*is an isomorphism above dimension $2n - 3$.*

(b) *There is an epimorphism of $\mathbb{F}_p$-algebras*

$$H^*(N_{0,p-1} *_{N_{0,p-1}\cap N_2} N_2; \mathbb{F}_p) \to H^*(N_2; \mathbb{F}_p)$$

*whose kernel is isomorphic to the ideal $H^*(N_{\Sigma_p}(\mathbb{Z}/p); \mathbb{F}_p) \otimes K_{p-1}$ where $K_{p-1}$ is the kernel of the restriction map $H^*(Aut(F_{p-1}); \mathbb{F}_p) \to H^*(\Sigma_p; \mathbb{F}_p)$.*

In view of Proposition 1.3 this result reduces the explicit calculation of the mod-$p$ cohomology of $Out(F_{2(p-1)})$ above the finite virtual cohomological dimension (v.c.d. in what follows) to the calculation of the cohomology of $Aut(F_s)$ with $s \le p-1$, i.e. to calculations in which the $p$-rank is one. If $s < p - 1$ these cohomologies are finite, and if $s = p - 1$ the cohomology has been calculated above the v.c.d. in [3].

We remark that the result for $p = 3$ can also be derived by the normalizer method but requires special considerations which are caused by the existence of the graph $K_{3,3}$ and the corresponding additional elementary abelian 3-subgroup $E_K \cong \mathbb{Z}/3 \times \mathbb{Z}/3$. In addition in this approach a partial analysis of outer space $K_4$ still seems needed in order to determine the structure of some of the normalizers. We find it therefore preferable to present the case $p = 3$ via the isotropy spectral sequence for the Borel construction of $(K_4)_s$.

The remainder of this paper is organized as follows. In Section 2 we recall the definition of the spine $K_n$ of outer space associated with $Out(F_n)$ and analyze the 3-singular part of $K_4$, $(K_4)_s$ (cf. [4]). In Section 3 we evaluate the isotropy spectral sequence of $(K_4)_s$, thus proving Theorem 1.1 and Corollary 1.2. In Section 4 we discuss the poset of elementary abelian $p$-subgroups of $Out(F_{2(p-1)})$ for $p > 3$ and prove part (a) and (b) of Proposition 1.3. In Section 5 we study the normalizers of these elementary abelian $p$-subgroups and prove the remaining parts of the same proposition. Finally in Section 6 we derive the cohomological consequences and prove Theorem 1.4.

## 2. The spine of outer space and 3-singular graphs in the rank 4 case

### *2.1. The spine of outer space*

We recall that the spine $K_n$ of outer space is defined as the geometric realization of the poset of equivalence classes of marked admissible finite graphs or rank $n$ (i.e. they have the homotopy type of the rose $R_n$), where the poset relation is generated by collapsing trees [8].

In more detail, for our purposes a finite graph $\Gamma$ is a quadruple $(V(\Gamma), E(\Gamma), \sigma, t)$ where $V(\Gamma)$ and $E(\Gamma)$ are finite sets, $\sigma$ is a fixed point free involution of $E(\Gamma)$ and $t$ is a map from $E(\Gamma)$ to $V(\Gamma)$. To such a graph one can associate canonically a one-dimensional $CW$-complex with 0-skeleton $V(\Gamma)$ and with 1-cells in bijection with the $\sigma$-orbits of $E(\Gamma)$. The attaching map of a 1-cell $e$ is given by $t(e)$, the terminal vertex of $e$, and by $t(\sigma(e))$, the initial vertex of $e$. A graph $\Gamma$ is called admissible if it (i.e. the associated $CW$-complex which will also be denoted by $\Gamma$) is connected, all vertices have valency at least 3, and it does not contain any separating edges. A marking of a graph $\Gamma$ is a choice of a homotopy equivalence $\alpha : R_n \to \Gamma$. Two markings $R_n \xrightarrow{\alpha_i} \Gamma_i$, $i = 1, 2$, are equivalent if there exists a homeomorphism $\varphi : \Gamma_1 \to \Gamma_2$ such that $\alpha_2$ and $\varphi\alpha_1$ are freely homotopic.

The poset relation is defined as follows: $R_n \xrightarrow{\alpha_2} \Gamma_2$ is bigger than $R_n \xrightarrow{\alpha_1} \Gamma_1$ if there exists a forest in $\Gamma_2$ such that $\Gamma_1$ is obtained from $\Gamma_2$ by collapsing each tree in this forest to a point, and $\alpha_1$ is freely homotopic to the composite of $\alpha_2$ followed by the collapse map. It is clear that this induces a poset structure on equivalence classes of marked graphs.

The group $Out(F_n)$ can be identified with the group of free homotopy equivalences of $R_n$ to itself, and with this identification we obtain a right action of $Out(F_n)$ on $K_n$, given by precomposing the marking with an unbased homotopy equivalence of $R_n$ to itself.

The isotropy group of (the equivalence class of) a marked graph $(\Gamma, \alpha)$ (with marking $\alpha$) can be identified via $\alpha_*^{-1}$ with the automorphism group $Aut(\Gamma)$ of the underlying graph $\Gamma$. (Note that graph automorphisms have to be taken in the sense of the definition of a graph given above, in particular they are allowed to reverse edges!)

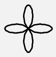
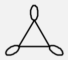
If $(\Gamma_i, \alpha_i)$, $0 \le i < k$, is obtained from $(\Gamma_k, \alpha_k)$ by collapsing a sequence of forests $\tau_i$ with $\tau_i \supset \tau_{i+1}$ then the isotropy group of the $k$-simplex with vertices $(\Gamma_0, \alpha_0), \ldots, (\Gamma_k, \alpha_k)$ can be identified with the subgroup of $Aut(\Gamma_k)$ which leaves each of the forests $\tau_i$ invariant, cf. [7].

It is easy to check that an admissible graph of rank $n$ has at most $3n - 3$ and at least $n$ edges, and the complex $K_n$ has dimension $2n - 3$.

### *2.2. The 3-singular graphs of rank 4*

With these preparations we can now start discussing the 3-singular locus $(K_4)_s$ of $K_4$, i.e. the subspace of $K_4$ of points whose isotropy groups contain non-trivial elements of order 3. For this we first need to determine the graphs with a non-trivial element of order 3 in its automorphism group (we call them 3-singular graphs) and then the invariant chains of forests in them which are preserved by a non-trivial element of order 3. The necessary analysis is analogous to that of the $p$-singular locus of $K_{p+1}$ for $p > 3$ [4]. Our notation partially follows that of [4], however we have chosen different notation for the graphs $\Theta_r^{s,t}$, $K_{3,3}$ (which was labelled $S_1$ in [4]) and $\Theta_2 : \Theta_1$ (which was labelled $\Theta_2 * \Theta_2$ in [4]).

**Table 1**
3-singular admissible graphs of rank 4.

| Graph | name | ♯ Edges | Stabilizer |
|---|---|---|---|
|  | $R_4$ | 4 | $Z/2 \wr \Sigma_4$ |
|  | $\theta_4$ | 5 | $Z/2 \times \Sigma_5$ |
|  | $\theta_3^{0,1}$ | 5 | $\Sigma_4 \times Z/2$ |
|  | $\theta_2^{0,2}$ | 5 | $\Sigma_3 \times D_8$ |
|  | $\theta_2^{1,1}$ | 5 | $\Sigma_3 \times D_8$ |
|  | $\theta_2 \vee \theta_2$ | 6 | $\Sigma_3 \wr Z/2$ |
|  | $\theta_2 \vee \theta_1 \vee R_1$ | 6 | $\Sigma_3 \times (Z/2)^2$ |
|  | $\theta_3 * R_1$ | 6 | $\Sigma_3 \times (Z/2)^2$ |
|  | $\theta_2 \Diamond Y$ | 6 | $\Sigma_3 \times Z/2$ |
|  | $T_1$ | 6 | $Z/2 \wr \Sigma_3$ |
|  | $T_0$ | 6 | $Z/2 \wr \Sigma_3$ |
|  | $\theta_2 : \theta_1$ | 7 | $\Sigma_3 \times (Z/2)^2$ |
|  | $W_3 \vee R_1$ | 7 | $\Sigma_3 \times (Z/2)$ |
|  | $\theta_2 * *\theta_1$ | 7 | $\Sigma_3 \times (Z/2)^2$ |
|  | $P_1$ | 9 | $\Sigma_3 \times Z/2$ |
|  | $S_0$ | 9 | $Z/2 \wr \Sigma_3$ |
|  | $K_{3,3}$ | 9 | $\Sigma_3 \wr Z/2$ |

In this section we give a brief outline how one arrives at the list of 3-singular graphs described in Table 1. We already know that these graphs will have at least 4 and at most 9 edges.

*2.2.1*

With 4 edges we only find the rose $R_4$ which is clearly 3-singular. Its automorphism group is clearly the wreath product $\mathbb{Z}/2 \wr \Sigma_4$.

*2.2.2*

With 5 edges we need 2 vertices which are necessarily fixed with respect to any graph automorphism of order 3. In order to make the graph 3-singular and admissible, we need to have at least 3 edges connecting the two vertices. The resulting graphs and corresponding isomorphism groups are

$$\begin{array}{ll} \Theta_4 & \mathbb{Z}/2 \times \Sigma_5 \\ \Theta_3^{0,1} & \Sigma_4 \times \mathbb{Z}/2 \\ \Theta_2^{1,1} & \Sigma_3 \times D_8 \\ \Theta_2^{0,2} & \Sigma_3 \times D_8. \end{array}$$

*2.2.3*

With 6 edges we need 3 vertices. First we consider the case that a 3-Sylow subgroup $P$ of $Aut(\Gamma)$ fixes these vertices. If the $P$-orbit of each edge is non-trivial then we must have 2 orbits of length 3. The resulting graph and its automorphism group are

$$\Theta_2 \vee \Theta_2 \qquad \Sigma_3 \wr \mathbb{Z}/2.$$

If there is an edge which is fixed by $P$ then there are three fixed edges and there is one orbit of edges of length 3. In this situation we have one of the following cases with corresponding automorphism groups

$\Theta_2 \vee \Theta_1 \vee R_1 \qquad \Sigma_3 \times (\mathbb{Z}/2)^2$

$\Theta_3 * R_1 \qquad \Sigma_3 \times (\mathbb{Z}/2)^2$

$\Theta_2 \diamond Y \qquad \Sigma_3 \times \mathbb{Z}/2.$

Now assume that $P$ permutes all three vertices. Then the number of edges between any two vertices is either 1 or 2 and we have one of the following cases

$T_0 \qquad \mathbb{Z}/2 \wr \Sigma_3$

$T_1 \qquad \mathbb{Z}/2 \wr \Sigma_3.$

*2.2.4*

Next we consider the case of 7 edges and 4 vertices. First we assume again that a 3-Sylow subgroup $P$ of $Aut(\Gamma)$ fixes all vertices. Then we can have only one non-trivial $P$ orbit of edges (otherwise the graph would not be admissible) and we are in one of the following cases

$\Theta_2 : \Theta_1 \qquad \Sigma_3 \times (\mathbb{Z}/2)^2$

$\Theta_2 ** \Theta_1 \qquad \Sigma_3 \times (\mathbb{Z}/2)^2.$

If $P$ acts non-trivially on the vertices then we have an orbit of length 3 and one orbit of length 1. Admissibility forces that there are two orbits of edges of length 3 and the graph and its automorphism group are

$W_3 \vee R_1 \qquad \Sigma_3 \times \mathbb{Z}/2.$

*2.2.5*

Now consider the case of 8 edges and 5 vertices. If a 3-Sylow subgroup $P$ acts trivially on the vertices then $\Gamma$ contains $\Theta_2$ as subgraph (because it is supposed to be 3-singular and admissible) and the valency 3 condition together with the requirement that there are no separating edges in $\Gamma$ imply that one would need more than 8 edges and hence there is no such graph. If $P$ acts non-trivially on the vertices then the set of edges which have one of the moving vertices as an endpoint must form two orbits each of length 3. Then the two fixed edges must join the two fixed vertices and because these vertices must have valency at least 3 they must also be endpoints of moving edges. However, this violates the valency condition for the moving vertices and the fact there are only 8 edges in $\Gamma$.

*2.2.6*

Finally we consider the case of 9 edges and 6 vertices. Then all vertices have valency 3 and connectivity forces that $\Gamma$ cannot contain $\Theta_2$ as a subgraph. This implies that a non-trivial graph automorphism of order 3 cannot fix all vertices. Furthermore, if such an automorphism does have fixed points then it has precisely three, and the valency condition implies that we are in the case

$K_{3,3} \qquad \Sigma_3 \wr \mathbb{Z}/2.$

If such an automorphism has no fixed points we consider the two orbits of vertices both of length 3. If there are two vertices which have more than one edge joining them then admissibility of the graph requires these vertices to be in different orbits and thus there are 3 pairs of vertices with precisely two edges between them. In this case the graph and its automorphism group is

$S_0 \qquad \mathbb{Z}/2 \wr \Sigma_3.$

In the remaining case there are either no edges between vertices in the same orbit and we obtain again $K_{3,3}$, or among the three orbits of edges there are two each of which forms a triangle joining the vertices in the same orbit and the third orbit of edges connects both triangles. The resulting graph and its automorphism group are

$P_1 \qquad \Sigma_3 \times \mathbb{Z}/2.$

## 3. The isotropy spectral sequence in case $p = 3$

We recall that for a discrete group $G$ and a $G$-CW-complex $X$ there is an "isotropy spectral sequence" [9] converging to the cohomology $H^*_G(X; \mathbb{F}_p)$ of the Borel construction $EG \times_G X$. It takes the form

$$E_1^{p,q} \cong \prod_{\overline{\sigma} \in C_p(X)} H^q(Stab(\sigma); \mathbb{F}_p) \Rightarrow H_G^{p+q}(X; \mathbb{F}_p)$$

where $\overline{\sigma}$ runs through the set of $G$-orbits of $p$-cells of $X$ and $Stab(\sigma)$ is the stabilizer of a representative $\sigma$ of this orbit.

**Table 2**
1-cells in $(K_4)_s/Out(F_4)$.

| Vertices of the cell | Invariant forest | Isotropy |
|---|---|---|
| $\Theta_2 **\Theta_1, \Theta_3 * R_1$ | One of the top horizontal edges | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 **\Theta_1, \Theta_2 \diamond Y$ | One of the vertical edges | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 **\Theta_1, \Theta_3^{0,1}$ | A top horizontal edge and a vertical edge | $\Sigma_3$ |
| $\Theta_2 **\Theta_1, \Theta_4$ | Both vertical edges | $\Sigma_3 \times (\mathbb{Z}/2)^2$ |
| $\Theta_2 **\Theta_1, R_4$ | Both vertical edges and a top horizontal edge | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_3 * R_1, \Theta_3^{0,1}$ | An edge joining the base of $R_1$ to another vertex | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_3 * R_1, R_4$ | Both edges joining the base of $R_1$ to the other vertices | $\Sigma_3 \times (\mathbb{Z}/2)^2$ |
| $\Theta_2 \diamond Y, \Theta_3^{0,1}$ | One of the vertical right hand edges | $\Sigma_3$ |
| $\Theta_2 \diamond Y, \Theta_4$ | Vertical left hand edges | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 \diamond Y, R_4$ | Vertical left hand edge and a vertical right hand edge | $\Sigma_3$ |
| $\Theta_3^{0,1}, R_4$ | One of the edges of $\Theta_3$ | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_4, R_4$ | One of the edges of $\Theta_4$ | $\Sigma_4 \times \mathbb{Z}/2$ |
| $W_3 \vee R_1, R_4$ | Symmetric tree around the center | $\Sigma_3 \times \mathbb{Z}/2$ |
| $K_{3,3}, \Theta_2 \vee \Theta_2$ | $K_{3,1}$ | $\Sigma_3 \times \mathbb{Z}/2$ |
| $K_{3,3}, T_1$ | Invariant forest made of 3 disjoint edges | $\Sigma_3 \times \mathbb{Z}/2$ |
| $P_1, T_1$ | The three edges joining the two triangles | $\Sigma_3 \times \mathbb{Z}/2$ |
| $S_0, T_1$ | Invariant forest of an orbit of a “single edge” | $\mathbb{Z}/2 \wr \Sigma_3$ |
| $S_0, T_0$ | Invariant forest of an orbit of a “double edge” | $\Sigma_3$ |
| $\Theta_2 : \Theta_1, \Theta_2 \vee \Theta_2$ | One of the two edges joining $\Theta_2$ to the vertical $\Theta_1$ | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 : \Theta_1, \Theta_2 \vee \Theta_1 \vee R_1$ | One of the two edges of the vertical $\Theta_1$ | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 : \Theta_1, \Theta_2^{0,2}$ | The union of the two edges in the previous two cases | $\Sigma_3$ |
| $\Theta_2 : \Theta_1, \Theta_2^{0,2}$ | Both edges joining $\Theta_2$ to the vertical $\Theta_1$ | $\Sigma_3 \times (\mathbb{Z}/2)^2$ |
| $\Theta_2 \vee \Theta_1 \vee R_1, \Theta_2^{0,2}$ | One of the edges of $\Theta_1$ | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 \vee \Theta_2, \Theta_2^{0,2}$ | Any edge | $\Sigma_3 \times \mathbb{Z}/2$ |

In this section we prove Theorem 1.1 and Corollary 1.2 by evaluating the isotropy spectral sequence for the action of $Out(F_4)$ on the singular locus $(K_4)_s$ and we compute thus the equivariant mod-3 cohomology $H^*_{Out(F_4)}((K_4)_s; \mathbb{F}_3)$.

We note that $H^*_{Out(F_4)}(K_4, (K_4)_s; \mathbb{F}_3)$ is isomorphic to the cohomology of the quotient $H^*(Out(F_4) \setminus (K_4, (K_4)_s); \mathbb{F}_3)$. In particular it vanishes in dimensions bigger than 5 and hence the result of our computation agrees with $H^*(Out(F_4); \mathbb{F}_3)$ in dimensions bigger than 5.

### 3.1. The cell structure of the quotient complex $(K_4)_s/Out(F_4)$

The 0-cells of this quotient are in one-to-one correspondence with the 3-singular graphs given in Table 1.

The 1-cells are in bijection with pairs given by a 3-singular graph $\Gamma$ together with an orbit of an invariant forest with respect to the action of $Aut(\Gamma)$. By going through the list of 3-singular graphs we get the list of 1-cells given in Table 2. The first column in this table gives the vertices of each cell. (It turns out that there are no loops and that with the exception of two edges all edges are determined by their vertices.) The second column describes the invariant forest in the graph which appears first in the name of the 1-cell; the tree $K_{3,1}$ in this column is given its standard name as the complete bipartite graph on one block of three vertices and one block of one vertex. The last column gives the abstract structure of the isotropy group of a representative of the edge in $(K_4)_s$; this abstract group is always to be regarded as a subgroup of the automorphism group of the first graph (with a fixed marking), namely as the subgroup which leaves the given forest invariant.

Similarly, Table 3 resp. Table 4 give the lists of 2-cells resp. 3-cells. Again the first column gives the vertices, the second column the chain of forests to be collapsed and the last column the automorphism group of a representing 2-simplex resp. 3-simplex in $(K_4)_s$. (The maximal trees in the chain of forests describing the 3-cells are supposed to be outside the subgraph $\Theta_2$.)

We immediately see that the quotient complex $(K_4)_s/Out(F_4)$ has 3 connected components which we will call the rose component resp. the $\Theta_2^{1,1}$ component resp. the $K_{3,3}$ component. We let $K_A$ resp. $K_B$ resp. $K_C$ denote the preimages of these components in $(K_4)_s$. Then we have a canonical isomorphism

$$H^*_{Out(F_4)}((K_4)_s; \mathbb{F}_3) \cong H^*_{Out(F_4)}(K_A; \mathbb{F}_3) \oplus H^*_{Out(F_4)}(K_B; \mathbb{F}_3) \oplus H^*_{Out(F_4)}(K_C; \mathbb{F}_3)$$

The analysis of the first two summands is quite straightforward, while the analysis of the last one is more delicate.

### 3.2. The rose component

This component turns out to be the realization of a poset which is described in Fig. 3. As a simplicial complex it is a one point union of the 1-cell with vertices $W_3 \vee R_1$ and $R_4$, and the cone (with cone point $\Theta_2 **\Theta_1$) over the pentagon formed by the adjacent 2-simplices with vertices $\Theta_3 * R_1$, $\Theta_3^{0,1}$, $R_4$ resp. $\Theta_2 \diamond Y$, $\Theta_3^{0,1}$, $R_4$ resp. $\Theta_2 \diamond Y$, $\Theta_4$, $R_4$. In particular, the rose component is contractible. Furthermore in the $E_1$-term of the isotropy spectral sequence for $K_A$ the contribution of each

**Table 3**
2-cells in $(K_4)_s/Out(F_4)$.

| Cell | Chain of invariant forests | Isotropy |
|---|---|---|
| $\Theta_2 * *\Theta_1, \Theta_3 * R_1, \Theta_3^{0,1}$ | One of the top horizontal edges, then a vertical edge | $\Sigma_3$ |
| $\Theta_2 * *\Theta_1, \Theta_3 * R_1, R_4$ | One of the top horizontal edges, then maximal tree outside of $\Theta_2$ | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 * *\Theta_1, \Theta_2 \diamond Y, \Theta_3^{0,1}$ | A vertical edge, then one of the top horizontal edges | $\Sigma_3$ |
| $\Theta_2 * *\Theta_1, \Theta_2 \diamond Y, \Theta_4$ | A vertical edge, then the other vertical edge | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 * *\Theta_1, \Theta_2 \diamond Y, R_4$ | A vertical edge, then maximal tree outside of $\Theta_2$ | $\Sigma_3$ |
| $\Theta_2 * *\Theta_1, \Theta_3^{0,1}, R_4$ | Top horizontal and vertical edge, then maximal tree outside of $\Theta_2$ | $\Sigma_3$ |
| $\Theta_2 * *\Theta_1, \Theta_4, R_4$ | Both vertical edges, then maximal tree outside of $\Theta_2$ | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_3 * R_1, \Theta_3^{0,1}, R_4$ | One edge, then both edges joining $\Theta_3$ with $R_1$ | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 \diamond Y, \Theta_3^{0,1}, R_4$ | One right hand vertical edge, then maximal tree outside of $\Theta_2$ | $\Sigma_3$ |
| $\Theta_2 \diamond Y, \Theta_4, R_4$ | Left hand vertical edge, then maximal tree outside of $\Theta_2$ | $\Sigma_3$ |
| $\Theta_2 : \Theta_1, \Theta_2 \vee \Theta_1 \vee R_1, \Theta_2^{0,2}$ | One edge of $\Theta_1$, then add edge between $\Theta_2$ and $\Theta_1$ | $\Sigma_3$ |
| $\Theta_2 : \Theta_1, \Theta_2 \vee \Theta_2, \Theta_2^{0,2}$ | One edge between $\Theta_2$ and $\Theta_1$, then both edges | $\Sigma_3 \times \mathbb{Z}/2$ |
| $\Theta_2 : \Theta_1, \Theta_2 \vee \Theta_2, \Theta_2^{0,2}$ | One edge between $\Theta_2$ and $\Theta_1$, then add edge of $\Theta_1$ | $\Sigma_3$ |

**Table 4**
3-cells in $(K_4)_s/Out(F_4)$.

| Cell | Chain of invariant forests | Isotropy |
|---|---|---|
| $\Theta_2 * *\Theta_1, \Theta_3 * R_1, \Theta_3^{0,1}, R_4$ | Top horizontal edge, then add a vertical edge, then maximal tree | $\Sigma_3$ |
| $\Theta_2 * *\Theta_1, \Theta_2 \diamond Y, \Theta_3^{0,1}, R_4$ | One vertical edge, then top horizontal edge, then maximal tree | $\Sigma_3$ |
| $\Theta_2 * *\Theta_1, \Theta_2 \diamond Y, \Theta_4, R_4$ | One vertical edge, then both vertical edges, then maximal tree | $\Sigma_3$ |

cell is isomorphic to $H^*(\Sigma_3; \mathbb{F}_3)$ and all faces induce the identity via this identification (cf. Lemma 4.1 of [4]). Therefore the equivariant cohomology turns out to be

$$H^*_{Out(F_4)}(K_A; \mathbb{F}_3) \cong H^*(\Sigma_3; \mathbb{F}_3) \cong H^*(G_R; \mathbb{F}_3).$$

### 3.3. The $\Theta_2^{1,1}$ component

This component is even simpler; it consists of a single point and therefore we get

$$H^*_{Out(F_4)}(K_B; \mathbb{F}_3) \cong H^*(\Sigma_3; \mathbb{F}_3) \cong H^*(G_{1,1}; \mathbb{F}_3).$$

### 3.4. The $K_{3,3}$ component

The geometry of the remaining component is as follows: there is a "critical edge" joining $K_{3,3}$ to $\Theta_2 \vee \Theta_2$, there is a "free part" (which in Fig. 4 is to the right of the critical edge and in which the 3-Sylow subgroups of the automorphism groups of the graphs act freely on the graph), and there is the "fixed part" which is attached to $\Theta_2 \vee \Theta_2$ (on which the 3-Sylow subgroups of the automorphism groups of the graphs fix at least one vertex of the graph), see Fig. 4.

We note that in this figure there are two triangles both of whose vertices are $\Theta_2 : \Theta_1, \Theta_2 \vee \Theta_2$ and $\Theta_2^{0,2}$.

In the $E_1$-term of the isotropy spectral sequence for $K_C$ the contribution of each cell except those of the 0-cells on the "critical edge" between $K_{3,3}$ and $\Theta_2 \vee \Theta_2$ is isomorphic to $H^*(\Sigma_3; \mathbb{F}_3)$, and all faces except those involving the critical edge induce the identity via this identification (cf. Lemma 4.1 of [4] again). This together with the observation that the critical edge is a deformation retract of the quotient $Out(F_4) \setminus K_C$ implies that the inclusion of the preimage $K_4^e$ of the critical edge into $K_C$ induces an isomorphism in equivariant cohomology

$$H^*_{Out(F_4)}(K_C; \mathbb{F}_3) \to H^*_{Out(F_4)}(K_4^e; \mathbb{F}_3).$$

So it remains to calculate $H^*_{Out(F_4)}(K_4^e; \mathbb{F}_3)$. Because $K_4^e$ is a one-dimensional $Out(F_4)$-complex with quotient an edge the following lemma finishes off the proof of Theorem 1.1.

**Lemma 3.1.** *Let $G$ be a discrete group and $X$ a one-dimensional $G$-CW-complex with fundamental domain a segment. Let $G_1$ and $G_2$ be the isotropy groups of the vertices of the segment and $H$ be the isotropy group of the segment itself and let $A$ be any abelian group. Then there is a canonical isomorphism*

$$H^*_G(X; A) \cong H^*(G_1 *_H G_2; A).$$

**Proof.** The inclusions of $G_1$, $G_2$ and $H$ into $G$ determine a homomorphism from $G_1 *_H G_2$ to $G$. Furthermore the tree $T$ associated to the amalgamated product admits a $G_1 *_H G_2$-equivariant map to $X$ such that the induced map on spectral sequences converging to $H^*_G(X; A)$ resp. to $H^*_{G_1 *_H G_2}(T; A) \cong H^*(G_1 *_H G_2; A)$ is an isomorphism on $E_1$-terms. □

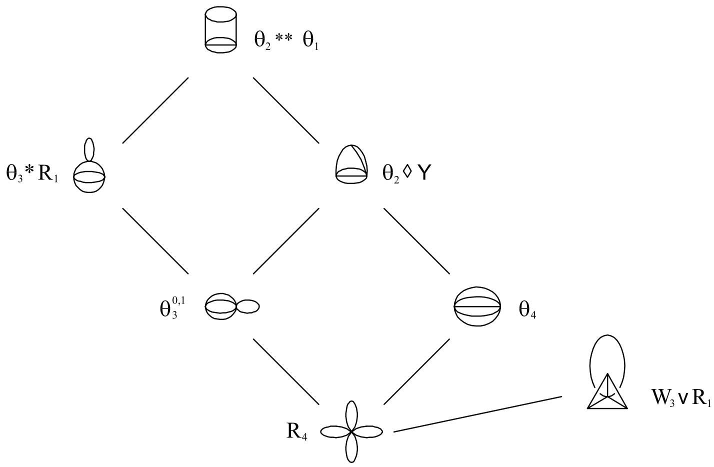



**Fig. 3.** The rose component of $Out(F_4) \setminus (K_4)_s$.

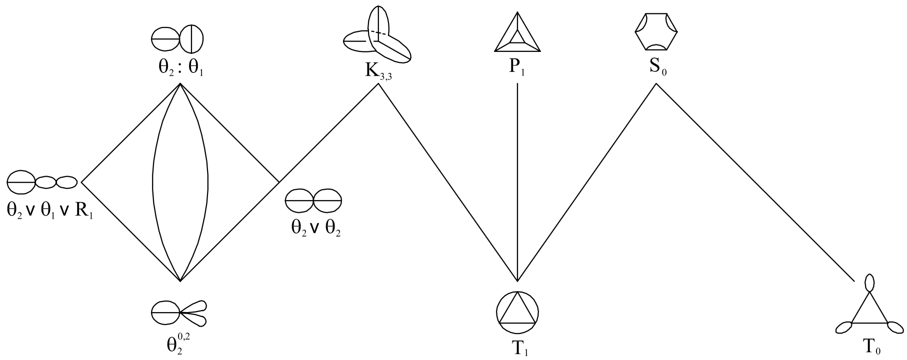



**Fig. 4.** The $K_{3,3}$ component of $Out(F_4) \setminus (K_4)_s$.

*3.5. The critical edge and the proof of* Corollary 1.2

The isotropy groups along this edge are as follows: for $K_{3,3}$ and for $\Theta_2 \vee \Theta_2$ we get both times $\Sigma_3 \wr \mathbb{Z}/2$ while for the edge we get $\Sigma_3 \times \mathbb{Z}/2$.

To understand the two inclusions of $\Sigma_3 \times \mathbb{Z}/2$ into $\Sigma_3 \wr \mathbb{Z}/2$ we observe that the edge is obtained by collapsing an invariant $K_{3,1}$ tree, hence the isotropy group of the edge embeds into that of $K_{3,3}$ via the embedding of $\Sigma_3 \times \mathbb{Z}/2$ into $\Sigma_3 \times \Sigma_3$. On the other hand it embeds into the isotropy group of $\Theta_2 \vee \Theta_2$ via the "diagonal embedding".

The cohomology of the isotropy groups considered as abstract groups is given in the following proposition whose proof is straightforward and left to the reader.

**Proposition 3.2.** (a)

$$H^*(\Sigma_3 \times \mathbb{Z}/2; \mathbb{F}_3) \cong H^*(\Sigma_3; \mathbb{F}_3) \cong \mathbb{F}_3[a_4] \otimes \Lambda(b_3)$$

*where the indices give the dimensions of the elements.*

(b)

$$H^*(\Sigma_3 \wr \mathbb{Z}/2; \mathbb{F}_3) \cong H^*(\Sigma_3 \times \Sigma_3; \mathbb{F}_3)^{\mathbb{Z}/2} \cong (\mathbb{F}_3[c_{4,1}, c_{4,2}] \otimes \Lambda(d_{3,1}, d_{3,2}))^{\mathbb{Z}/2}$$

*where the first index gives the dimension and the second index refers to the first resp. second copy of* $\Sigma_3$. *Consequently*

$$H^*(\Sigma_3 \wr \mathbb{Z}/2; \mathbb{F}_3) \cong \mathbb{F}_3[c_4, c_8] \otimes \Lambda(d_3, d_7)$$

*where*

$$c_4 = c_{4,1} + c_{4,2} \qquad c_8 = (c_{4,1} - c_{4,2})^2$$
$$d_3 = d_{3,1} + d_{3,2} \qquad d_7 = (c_{4,1} - c_{4,2})(d_{3,1} - d_{3,2}). \quad \square$$

Next we turn towards the description of the restriction maps. Let $\alpha$ denote the map in mod-3 cohomology induced by the inclusion of $\Sigma_3 \times \mathbb{Z}/2$ into the isotropy group $G_K$ and $\beta$ be the map induced by the inclusion of $\Sigma_3 \times \mathbb{Z}/2$ into $G_2$.

We change notation and write

$$H^*(G_K;\mathbb{F}_3)\cong \mathbb{F}_3[x_4,x_8]\otimes\Lambda(u_3,u_7)$$
$$H^*(G_2;\mathbb{F}_3)\cong \mathbb{F}_3[y_4,y_8]\otimes\Lambda(v_3,v_7)$$

and

$$H^*(\Sigma_3\times\mathbb{Z}/2;\mathbb{F}_3)\cong \mathbb{F}_3[z_4]\otimes\Lambda(w_3).$$

Then the effect of the two restriction maps is as follows:

$$\alpha(x_4)=z_4,\qquad \alpha(x_8)=z_4^2,\qquad \alpha(u_3)=w_3,\qquad \alpha(u_7)=z_4w_3$$

and

$$\beta(y_4)=2z_4,\qquad \beta(y_8)=0,\qquad \beta(v_3)=2w_3,\qquad \beta(v_7)=0.$$

**Proposition 3.3.** (a) *The inclusions of $K_4^e$ into the $Out(F_4)$-orbits of $K_{3,3}$ and $\Theta_2\vee\Theta_2$ induce an isomorphism*

$$H^*_{Out(F_4)}(K_4^e;\mathbb{F}_3)\cong Eq(\alpha,\beta)$$

*between $H^*_{Out(F_4)}(K_4^e;\mathbb{F}_3)$ and the subalgebra of $\mathbb{F}_3[x_4,x_8]\otimes\Lambda(u_3,u_7)\times\mathbb{F}_3[y_4,y_8]\otimes\Lambda(v_3,v_7)$ equalized by the maps $\alpha$ and $\beta$.*

(b) *This equalizer contains the tensor product of the polynomial subalgebra generated by the elements $r_4=(x_4,2y_4)$ and $r_8=(x_8,y_4^2+y_8)$ with the exterior algebra generated by the element $s_3=(u_3,2v_3)$, and as a module over this tensor product it is free on generators $1,t_7,\widetilde{t_7},t_8$, i.e. we can write*

$$H^*_{Out(F_4)}(X_4^e;\mathbb{F}_3)\cong \mathbb{F}_3[r_4,r_8]\otimes\Lambda(s_3)\{1,t_7,\widetilde{t_7},t_8\}$$

*with $1=(1,1)$, $t_7=(0,v_7)$, $\widetilde{t_7}=(u_7,y_4v_3)$, $t_8=(0,y_8)$.*

(c) *The additional multiplicative relations in this algebra are given as*

$$t_7^2=\widetilde{t_7}^2=0,\qquad t_8^2=(r_8-r_4^2)t_8,\qquad \widetilde{t_7}t_7=r_4s_3t_7,\qquad t_8t_7=(r_8-r_4^2)t_7,\qquad t_8\widetilde{t_7}=r_4s_3t_8.$$

**Proof.** (a) It is clear that $\alpha$ is onto and this implies that $H^*_{Out(F_4)}(K_4^e;\mathbb{F}_3)$ is given as the equalizer of $\alpha$ and $\beta$.

(b) It is straightforward to check that all of the elements $r_4,r_8,s_3,1,t_7,\widetilde{t_7}$ and $t_8$ are contained in the equalizer and that the subalgebra generated by $r_4,r_8$ and $s_3$ has structure as claimed. It is also easy to check that the elements $1,t_7,\widetilde{t_7}$ and $t_8$ are linearly independent over this subalgebra. Furthermore we have the following equation for the Euler Poincaré series $\chi$ of the equalizer:

$$\chi+\frac{1+t^3}{1-t^4}=2\frac{(1+t^3)(1+t^7)}{(1-t^4)(1-t^8)}.$$

From this we get

$$\chi=\frac{(1+t^3)(1+2t^7+t^8)}{(1-t^4)(1-t^8)}$$

and hence the result.

(c) The additional multiplicative relations in the algebra $H^*_{Out(F_4)}(X_4^e;\mathbb{F}_3)$ can be easily determined by considering it as subalgebra of $\mathbb{F}_3[x_4,x_8]\otimes\Lambda(u_3,u_7)\times\mathbb{F}_3[y_4,y_8]\otimes\Lambda(v_3,v_7)$. □

## 4. Elementary abelian p-subgroups in $Out(F_{2(p-1)})$ for $p>3$

In this section we determine the conjugacy classes of elementary abelian $p$-subgroups of the group $Out(F_{2(p-1)})$. The strategy is as follows. If $G$ is a finite subgroup then by results of Culler [11] and Zimmermann [12] there exists a finite graph $\Gamma$ with $\pi_1(\Gamma)\cong F_n$ and a subgroup $G'$ of $Aut(\Gamma)$ such that $G'$ gets identified with $G$ via the induced outer action on $\pi_1(\Gamma)\cong F_n$. In what follows, we will call such a graph a $G$-graph. Changing the marking of a $G$-graph amounts to changing $G$ within its conjugacy class.

### *4.1. Reduced $\mathbb{Z}/p$-graphs of rank $n=2(p-1)$*

We say that $\Gamma$ is $G$-reduced, if $\Gamma$ does not contain any $G$-invariant forest. By collapsing tress in invariant forests to a point we may assume that the graph $\Gamma$ in the result of Culler and Zimmermann is reduced. This gives an upper bound for the conjugacy classes in terms of isomorphism classes of reduced graphs of rank $n$ with a $\mathbb{Z}/p$-symmetry.

We thus proceed to classify $\mathbb{Z}/p$-reduced graphs of rank $n = 2(p-1)$.

(a) If there is a vertex $v_1$ which is fixed by $\mathbb{Z}/p$ and if $e$ is any edge joining this vertex to any distinct vertex $v_2$ then $v_2$ has to be also fixed (otherwise the orbit of this edge gives an invariant forest). Therefore if there is one vertex which is fixed then all vertices will be fixed.
(b) If there is only one fixed vertex then the graph has to be the rose $R_{2(p-1)}$ and $\mathbb{Z}/p$ acts transitively on $p$ of the edges of the rose and fixes the others. We choose a marking so that we obtain an isomorphic subgroup in $Out(F_n)$ which we denote $E_R$.
(c) If we have two fixed vertices then any edge between them cannot be fixed because otherwise it would define an invariant forest. Therefore $\Gamma$ contains $\Theta_{p-1}$ and there cannot be any other edge between the two vertices because they would have to be fixed and thus there would be an invariant forest. Consequently the graph has to be isomorphic to $\Theta_{p-1}^{s,t}$ with $s+t = p-1$ and $0 \le s \le \frac{p-1}{2}$. After having chosen a marking, we get a subgroup $\mathbb{Z}/p \cong E_{s,p-1-s}$ of $Out(F_n)$.
(d) If there are three fixed vertices then $\Gamma$ has to be isomorphic to $\Theta_{p-1} \vee \Theta_{p-1}$ and $\mathbb{Z}/p$ acts non-trivially on both $\Theta_{p-1}$'s. In this case the action of $\mathbb{Z}/p$ can be extended to an action of $\mathbb{Z}/p \times \mathbb{Z}/p$ with the left resp. right hand factor $\mathbb{Z}/p$ acting on the left resp. right hand copy of $\Theta_{p-1}$. After having chosen a marking, we get a subgroup $\mathbb{Z}/p \times \mathbb{Z}/p \cong E_2$ of $Out(F_n)$. Its diagonal will be denoted $\Delta(E_2)$.
(e) Clearly there are no reduced graphs of rank $2(p-1)$ which have more than 3 fixed vertices.
(f) If $\mathbb{Z}/p$ acts without fixed vertex then there are also no fixed edges and we get for the Euler characteristic

$$1 - 2(p-1) = \chi(\Gamma) \equiv 0 \bmod (p)$$

and hence $p = 3$.

We have thus proved the following result (cf. Proposition 4.2 of [6]).

**Proposition 4.1.** *Let $p > 3$ be a prime and $\Gamma$ be a reduced $\mathbb{Z}/p$-graph of rank $n = 2(p-1)$. Then $\Gamma$ is isomorphic to one of the graphs $R_n$, $\Theta_{p-1}^{s,t}$ with $s+t = n$ and $0 \le s \le \frac{p-1}{2}$, or $\Theta_{p-1} \vee \Theta_{p-1}$ (with diagonal action of $\mathbb{Z}/p$).* □

*4.2. Nielsen transformations of G-graphs and conjugacy classes of finite subgroups of $Out(F_n)$*

In order to distinguish conjugacy classes of elementary abelian $p$-subgroups we make use of Krstic's theory of equivariant Nielsen transformations [10] which we will also use to determine most of the centralizers and normalizers of these elementary abelian subgroups.

**Definition 4.2.** Let $G$ be a finite subgroup of $Out(F_n)$ and $\Gamma$ be a reduced $G$-graph of rank $n$ with vertex set $V$ and edge set $E$. Suppose

- $e_1$ and $e_2$ are edges such that $e_2$ is neither in the orbit of $e_1$ nor of its opposite $\sigma(e_1)$,
- $e_1$ and $e_2$ have the same terminal points, $t(e_1) = t(e_2)$,
- the stabilizer of $e_1$ in $G$ is contained in that of $e_2$.

Then there is a unique graph $\Gamma'$ with $V' = V$, $E' = E$, the same $G$-action as on $\Gamma$, $\sigma' = \sigma$, $t'(e) = e$ if $e$ is not in the orbit of $e_1$, and $t'(ge_1) = t(\sigma(ge_2))$ for every $g \in G$. This graph is denoted $\langle e_1, e_2\rangle\Gamma$ and the assignment $\Gamma \mapsto \langle e_1, e_2\rangle\Gamma$ is called a Nielsen transformation.

In the situation of this definition there is a $G$-equivariant isomorphism, also called a $G$-equivariant Nielsen isomorphism, $\langle e_1, e_2\rangle : \Pi(\Gamma) \to \Pi(\Gamma')$ of fundamental groupoids; it is determined by its map on edges by $\langle e_1, e_2\rangle(e) = e$ if $e$ is neither in the $G$-orbit of $e_1$ or $\sigma(e_1)$, and $\langle e_1, e_2\rangle(ge_1) = g(e_1e_2)$ for every $g \in G$.

**Proposition 4.3.** *Let $p > 3$ be a prime.*

(a) *The groups $E_R$, $E_{s,p-1-s}$ for $0 \le s \le \frac{p-1}{2}$, $E_2$ and the diagonal $\Delta(E_2)$ of $E_2$ are pairwise non-conjugate, and any elementary abelian p-subgroup of $Out(F_{2(p-1)})$ is conjugate to one of them.*
(b) *$E_{0,p-1}$ is conjugate to the subgroup $\mathbb{Z}/p$ of $E_2$ which acts only on the left hand $\Theta_{p-1}$ in $\Theta_{p-1} \vee \Theta_{p-1}$.*
(c) *Neither $E_R$ nor any of the $E_{s,p-1-s}$ with $1 \le s \le \frac{p-1}{2}$ is conjugate to a subgroup of $E_2$.*

**Proof.** Proposition 4.1 and the realization result of Culler and Zimmermann show that every elementary abelian $p$-subgroup of $Out(F_{2(p-1)})$ is conjugate to one in the given list. (Here we have implicitly used that the action of the group $Aut(\Theta_{p-1} \vee \Theta_{p-1})$ on its subgroups of order $p$ has just two orbits, that of $\Delta(E_2)$ and that of the subgroup fixing the right hand $\Theta_{p-1}$. The latter one can be omitted from our list because the corresponding graph is not reduced.)

Furthermore, if $G$ is a finite subgroup of $Out(F_{2(p-1)})$ which is realized by reduced $G$-graphs $\Gamma_1$ and $\Gamma_2$ then by Theorem 2 of [10] there is a sequence of $G$-equivariant Nielsen transformations from $\Gamma_1$ to a graph which is $G$-equivariantly isomorphic to $\Gamma_2$.

(a) Inspection shows that any $E_R$- (resp. $\Delta(E_2)$- resp. $E_{s,p-1-s}$-) equivariant Nielsen transformation starting in $R_n$ (resp. $\Theta_{p-1} \vee \Theta_{p-1}$ resp. $\Theta_{p-1}^{s,t}$) ends up in a graph which is isomorphic to the initial graph. This shows, in particular, that none of the subgroups $E_R$, $E_{s,p-1-s}$ and $\Delta(E_2)$ can be conjugate.

(b) The only subgroups of $E_2$ for which the graph $\Theta_{p-1} \vee \Theta_{p-1}$ is not reduced are those which act non-trivially only on one of the $\Theta_{p-1}$'s in $\Theta_{p-1} \vee \Theta_{p-1}$. By collapsing one of the fixed edges in the other $\Theta_{p-1}$ we pass to $\Theta_{p-1}^{0,p-1}$ and this implies that $E_{0,p-1}$ is conjugate to one of these subgroups of $E_2$.

(c) For all other subgroups of $E_2$ the graph $\Theta_{p-1} \vee \Theta_{p-1}$ is reduced and by using Theorem 2 of [10] once more we see that neither $E_R$ nor $E_{s,p-1-s}$ for $1 \le s \le \frac{p-1}{2}$ is conjugate to a subgroup of $E_2$. □

## 5. Normalizers of elementary abelian *p*-subgroups

Equivariant Nielsen transformations of graphs will also be used in the analysis of the normalizers of the elementary abelian subgroups of $Out(F_n)$. We will thus start this section by recalling more results of [10].

Given a reduced $G$-graph $\Gamma$ of rank $n$ Krstic establishes in Corollary 2 together with Proposition 2 an exact sequence of groups

$$1 \to Inn_G(\Pi(\Gamma)) \to Aut_G(\Pi(\Gamma)) \to C_{Out(F_n)}(G) \to 1.$$

Here $Aut_G(\Pi(\Gamma))$ resp. $Inn_G(\Pi(\Gamma))$ denote the $G$-equivariant automorphisms resp. inner automorphisms of the fundamental groupoid of $\Gamma$. We recall that an endomorphism $J$ of a fundamental groupoid $\Pi$ is called inner if there is a collection of paths $\lambda_v$, $v \in V$, such that $t(\sigma(\lambda_v)) = v$ for every $v \in V$ and $J(\alpha) = \sigma(\lambda_{t(\sigma(\alpha))})\alpha\lambda_{t\alpha}$ for every path $\alpha \in \Pi$. For a general $G$-graph an inner endomorphism need not be an automorphism. However, if $\Gamma$ is $G$-reduced, then any inner endomorphism is an isomorphism (cf. Proposition 3 of [10]).

Furthermore, the analysis of $Aut_G(\Pi(\Gamma))$ is made possible by Proposition 4 in Krstic [10] which says that each $G$-equivariant automorphisms of $\Pi(\Gamma)$ can be written as a composition of $G$-equivariant Nielsen isomorphisms followed by a $G$-equivariant graph isomorphism. We will use this in order to determine the normalizers of all elementary abelian $p$-subgroups of $Out(F_{2(p-1)})$, $p > 3$, except that of $\Delta(E_2)$. Our discussion is very close to that of Section 5 in [6] where the same ideas are used to determine normalizers of elementary abelian subgroups of $Aut(F_{2(p-1)})$. Therefore our discussion will be fairly brief and the reader who wants to see more details may want to have a look at [6].

### 5.1. $\Gamma = R_{2(p-1)}$

We begin by constructing homomorphisms (cf. [6])

$$\begin{aligned} &F_{p-2} \times F_{p-2} \to Aut_{E_R}(\Pi(R_{2(p-1)})), \qquad (v, w) \mapsto (y_i \mapsto y_i, x_i \mapsto v^{-1}x_iw) \\ &Aut(F_{p-2}) \to Aut_{E_R}(\Pi(R_{2(p-1)})), \qquad \alpha \mapsto (y_i \mapsto \alpha(y_i), x_i \mapsto x_i) \\ &\mathbb{Z}/p \to Aut_{E_R}(\Pi(R_{2(p-1)})), \qquad \sigma \mapsto (y_i \mapsto y_i, x_i \mapsto x_{i+1}) \\ &\mathbb{Z}/2 \to Aut_{E_R}(\Pi R_{2(p-1)}), \qquad \tau \mapsto (y_i \mapsto y_i, x_i \mapsto x_i^{-1}). \end{aligned}$$

Here the $x_i$ are the edges of $R_n$ which get cyclically moved by $\mathbb{Z}/p \cong E_R$, the $y_i$ are the fixed edges of $R_n$, $v$ and $w$ are words in the $y_i$ and their inverses, and $\sigma$ is a suitable generator of $E_R$. These maps determine a homomorphism

$$\psi_R : \mathbb{Z}/p \times ((F_{p-2} \times F_{p-2}) \rtimes (Aut(F_{p-2}) \times \mathbb{Z}/2)) \to C_{Aut(F_{2(p-1)})}(E_R)$$

in which the action of $Aut(F_{p-2})$ on $F_{p-2} \times F_{p-2}$ is the canonical diagonal action, while $\tau$ acts via $\tau(v, w)\tau^{-1} = (w, v)$. This homomorphism is surjective by Proposition 4 of [10] and arguing with reduced words in free groups shows that it is also injective.

The elements

$$(v^{-1}, v^{-1}, c_v, 1) \in (F_{p-2} \times F_{p-2}) \rtimes Aut(F_{p-2}) \times \mathbb{Z}/2$$

(where $c_v$ denotes conjugation $y_i \mapsto vy_iv^{-1}$) correspond to the inner automorphisms $z \mapsto vzv^{-1}$. Furthermore, we have the following identity

$$\tau(v, 1, \alpha, 1)\tau^{-1} = (1, v, \alpha, 1) = (v, v, c_{v^{-1}}, 1)(v^{-1}, 1, c_v\alpha, 1)$$

in $(F_{p-2} \times F_{p-2}) \rtimes (Aut(F_{p-2}) \times \mathbb{Z}/2)$. This implies that there is an induced isomorphism

$$\varphi_R : \mathbb{Z}/p \times (F_{p-2} \rtimes Aut(F_{p-2})) \rtimes \mathbb{Z}/2\mathbb{Z} \to C_{Out(F_{2(p-1)})}(E_R)$$

where the action of $\mathbb{Z}/2$ on $F_{p-2} \rtimes Aut(F_{p-2})$ is given by $\tau(x, \alpha) = (x^{-1}, c_x\alpha)$. So we have already proved the first part of the following result.

**Proposition 5.1.** (a) *There is an isomorphism*

$$\varphi_R : \mathbb{Z}/p \times (F_{p-2} \rtimes Aut(F_{p-2})) \rtimes \mathbb{Z}/2\mathbb{Z} \to C_{Out(F_{2(p-1)})}(E_R).$$

(b) $\varphi_R$ *extends to an isomorphism*

$$\widetilde{\varphi}_R : N_\Sigma(\mathbb{Z}/p) \times (F_{p-2} \rtimes Aut(F_{p-2})) \rtimes \mathbb{Z}/2\mathbb{Z} \to N_{Out(F_{2(p-1)})}(E_R).$$

**Proof.** Part (a) has already been proved. For (b) we note that there is an exact sequence

$$1 \to C_{Out(F_2(p-1))}(E_R) \to N_{Out(F_2(p-1))}(E_R) \to Aut(E_R) \to 1.$$

The normalizer does indeed surject to $Aut(E_R)$ because graph automorphisms induce a splitting of this sequence. In fact, the homomorphism $\psi_R$ can be extended to a homomorphism

$$\widetilde{\psi}_R : N_\Sigma(\mathbb{Z}/p) \times (F_{p-2} \times F_{p-2}) \rtimes (Aut(F_{p-2}) \times \mathbb{Z}/2) \to N_{Aut(F_{2(p-1)})}(E_R)$$

which can easily be seen to be an isomorphism and which induces the isomorphism $\widetilde{\varphi}_R$. $\square$

### 5.2. $\Gamma = \Theta_{p-1}^{s,p-1-s}$

To simplify notation we let $t = p - 1 - s$. Again we begin by constructing homomorphisms (cf. [6])

$$F_s \times F_t \to Aut_{E_{s,t}}(\Pi(\Theta_{p-1}^{s,t})), \qquad (v, w) \mapsto (a_i \mapsto a_i, b_i \mapsto v^{-1}b_iw, c_i \mapsto c_i)$$

$$Aut(F_s) \times Aut(F_t) \to Aut_{E_{s,t}}(\Pi(\Theta_{p-1}^{s,t})), \qquad (\alpha, \beta) \mapsto (a_i \mapsto \alpha(a_i), b_i \mapsto b_i, c_i \mapsto \beta(c_i))$$

$$\mathbb{Z}/p \to Aut_{E_{s,t}}(\Pi(\Theta_{p-1}^{s,t})), \qquad \sigma \mapsto (a_i \mapsto a_i, b_i \mapsto b_{i+1}, c_i \mapsto c_i).$$

Here the $a_i$ denote the fixed edges attached to the left hand vertex of $\Theta_{p-1}$, the $b_i$ are the edges of $\Theta_{p-1}$ which get cyclically moved by $E_{s,t}$ and have the right hand vertex as their terminal point, the $c_i$ are the fixed edges attached to the right hand vertex of $\Theta_{p-1}$, $v$ resp. $w$ are words in the $a_i$ resp. $c_i$ and their inverses, and $\sigma$ is a suitable generator of $\mathbb{Z}/p \cong E_{s,t}$.

These homomorphisms determine a homomorphism

$$\psi_{s,t} : \mathbb{Z}/p \times (F_s \rtimes Aut(F_s)) \times (F_t \rtimes Aut(F_t)) \to Aut^*_{E_{s,t}}(\Pi(\Theta_{p-1}^{s,t}))$$

where $Aut^*_{E_{s,t}}(\Theta_{p-1}^{s,t})$ denotes the equivariant automorphisms of $\Theta_{p-1}^{s,t}$ which fix both vertices. In fact, this homomorphism is surjective by Proposition 4 of [10], and arguing with reduced words in free groups shows that it is also injective.

If $s \neq t = p - 1 - s$ we find $Aut_{E_{s,t}}(\Pi(\Theta_{p-1}^{s,t})) = Aut^*_{E_{s,t}}(\Pi(\Theta_{p-1}^{s,t}))$, and if $s = t$ the group $Aut^*_{E_{s,t}}(\Pi(\Theta_{p-1}^{s,t}))$ is of index 2 in $Aut^*_{E_{s,t}}(\Pi(\Theta_{p-1}^{s,t}))$, and there is an obvious extension of $\psi_{s,s}$ to an isomorphism

$$\psi'_{s,s} : \mathbb{Z}/p \times \big(F_s \rtimes Aut(F_s)\big) \wr \mathbb{Z}/2 \to Aut_{E_{s,s}}(\Pi(\Theta_{p-1}^{s,s})).$$

In order to get $C_{Out_{F_{s,t}}(E_{s,p-1-s})}$ we need to quotient out the group of equivariant inner automorphisms of $\Pi(\Theta_{p-1}^{s,t})$. Any inner automorphism is given by two paths $\lambda_1$ resp. $\lambda_2$ terminating in the two vertices $v_1$ resp. $v_2$ of $\Theta_{p-1}^{s,t}$. Equivariance requires these paths to be fixed under the action of $E_{s,t}$. Therefore $\lambda_1$ can be identified with a word $v$ in the $a_i$ and their inverses and $\lambda_2$ can be identified with a word $w$ in the $c_i$ and their inverses, and it follows that the inner automorphism determined by $\lambda_1$ and $\lambda_2$ corresponds to the tuple $(1, v^{-1}, c_v, w^{-1}, c_w)$ in $\mathbb{Z}/p \times (F_s \rtimes Aut(F_s)) \times (F_{p-1-s} \rtimes Aut(F_{p-1-s}))$. Passing to the quotient by the inner automorphism gives the first half of the following result. The second half is proved as before in the case of the rose.

**Proposition 5.2.** (a) *For* $s \neq \frac{p-1}{2}$ *the isomorphism* $\psi_{s,p-1-s}$ *induces an isomorphism*

$$\varphi_{s,p-1-s} : \mathbb{Z}/p \times Aut(F_s) \times Aut(F_{p-1-s}) \to C_{Out(F_{2(p-1)})}(E_{s,p-1-s})$$

*while for* $s = \frac{p-1}{2}$ *the isomorphism* $\psi'_{s,s}$ *induces an isomorphism*

$$\varphi'_{s,s} : \mathbb{Z}/p \times (Aut(F_s) \wr \mathbb{Z}/2) \to C_{Out(F_{2(p-1)})}(E_{s,s}).$$

(b) *For* $s \neq \frac{p-1}{2}$ *the isomorphism* $\varphi_{s,p-1-s}$ *extends to an isomorphism*

$$\widetilde{\varphi}_{s,p-1-s} : N_\Sigma(\mathbb{Z}/p) \times Aut(F_s) \times Aut(F_{p-1-s}) \to N_{Out(F_{2(p-1)})}(E_{s,p-1-s})$$

*while for* $s = \frac{p-1}{2}$ *the isomorphism* $\varphi_{s,s}$ *extends to an isomorphism*

$$\widetilde{\varphi}'_{s,s} : N_{\Sigma_p}(\mathbb{Z}/p) \times (Aut(F_s) \wr \mathbb{Z}/2) \to C_{Out(F_{2(p-1)})}(E_{s,s}). \quad \square$$

### 5.3. $\Gamma = \Theta_{p-1} \vee \Theta_{p-1}$

In this case we have to look at $Aut_{E_2}(\Pi(\Theta_{p-1} \vee \Theta_{p-1}))$. There are no non-trivial Nielsen transformations in this case and therefore the centralizer resp. normalizer is given completely in terms of graph automorphisms and therefore has the following form (cf. [6]).

**Proposition 5.3.**

$$C_{Out(F_{2(p-1)})}(E_2) \cong \mathbb{Z}/p \times \mathbb{Z}/p$$
$$N_{Out(F_{2(p-1)})}(E_2) \cong N_\Sigma(\mathbb{Z}/p) \wr \mathbb{Z}/2. \quad \square$$

It remains to determine the structure of the normalizer of $\Delta(E_2)$, which we will abbreviate as in the introduction by $N_\Delta$ in order to simplify notation. In this case we prefer to use information on the fixed point space $(K_{2(p-1)})_s^{\Delta(E_2)}$ rather than working with Krstic's method. We immediately note that this fixed point space is equipped with an action of $N_\Delta$ and it contains the graph $\Theta_{p-1} \vee \Theta_{p-1}$ as well as the bipartite graph $K_{p,3}$ with markings which are compatible with respect to the collapse of the invariant tree $K_{p,1}$ inside $K_{p,3}$. In the following result we fix such a marking.

**Proposition 5.4.** *Let $p \geq 3$ be a prime.*

(a) *The fixed point space $(K_{2(p-1)})_s^{\Delta(E_2)}$ is a tree and the edge e determined by the collapse of $K_{p,1}$ in $K_{p,3}$ is a fundamental domain for the action of $N_\Delta$ on it.*

(b) *There is an isomorphism*

$$N_\Delta \cong Stab_{N_\Delta}(\Theta_{p-1} \vee \Theta_{p-1}) *_{Stab_{N_\Delta}(e)} Stab_{N_\Delta}(K_{p,3})$$
$$\cong ((\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes (Aut(\mathbb{Z}/p) \times \mathbb{Z}/2)) *_{N_{\Sigma_p}(\mathbb{Z}/p)\times\mathbb{Z}/2} (N_{\Sigma_p}(\mathbb{Z}/p) \times \Sigma_3)$$

*where $Stab_{N_\Delta}(\Gamma)$ denotes the stabilizer of $\Gamma$ with respect to the action of $N_\Delta$ and the action of $Aut(\mathbb{Z}/p)$ on $\mathbb{Z}/p \times \mathbb{Z}/p$ is given by the canonical diagonal action and that by $\mathbb{Z}/2$ by permuting the factors.*

**Proof.** Part (b) is an immediate consequence of part (a).

For (a) we note that $K_{2(p-1)}^{\Delta(E_2)}$ is contractible [13]. By the general theory of groups acting on trees it is therefore enough to show that this space is a one-dimensional complex and the edge $e$ is a fundamental domain for the action of $\Delta(E_2)$.

This follows immediately as soon as we have shown that up to isomorphism there is only one admissible $\Delta(E_2)$-graph containing a non-trivial $\Delta(E_2)$-invariant forest so that after collapsing this forest we get $\Theta_{p-1} \vee \Theta_{p-1}$ with the given action of $\Delta(E_2)$. In fact, if $\Gamma$ is a minimal such graph, then the invariant forest either consists of a non-trivial orbit of $p$ edges or of a single fixed edge.

In the first case, $\Gamma$ has $3p$ edges and $p+3$ vertices, and because the quotient $\Gamma/\Delta(E_2)$ collapses to $(\Theta_{p-1} \vee \Theta_{p-1})/\Delta(E_2)$, there are 3 non-trivial orbits of edges and 4 orbits of vertices, 3 of which are trivial and one with $p$ vertices. Because all edges are moved, the valency of each fixed vertex must be at least $p$ and because each of the moving vertices has valency at least 3, we see that, in fact, the valency of the fixed vertices is exactly $p$ and that of the others is 3. It is then easy to check that admissibility implies that $\Gamma$ must be $K_{p,3}$.

In the second case $\Gamma$ would have $2p + 1$ edges in one trivial and two non-trivial orbits and 4 trivial orbits of vertices. It is easy to see that such a $\Gamma$ cannot be admissible.

Finally, we claim that there cannot be any admissible $\Delta(E_2)$-graph $\Gamma$ with a $\Delta(E_2)$-invariant forest which collapses to $K_{p,3}$. In fact, such a minimal graph would either have $4p$ edges and $2p+3$ vertices, or $3p+1$ vertices and $p+4$ vertices. Again $\Gamma/\Delta(E_2)$ would have to collapse to $K_{p,3}/\Delta(E_2)$. Therefore, in the first case we would have 4 non-trivial orbits of edges, 2 non-trivial orbits of vertices and three trivial orbits of vertices. The valency of each fixed vertex would be at least $p$ and that of the others at least 3, so that the sum of the valencies would be at least $3p + 6p = 9p$ which is in contradiction to having only $4p$ edges. In the second case we would have 3 non-trivial and one trivial orbit of edges and 1 non-trivial orbit and 4 trivial orbits of vertices. The valency of at least 3 of the fixed vertices would have to be at least $p$ and then the total valency would be at least $3p + 3 + 3p = 6p + 3$ which contradicts having only $3p + 1$ edges. $\square$

Finally, we will need the intersection of the normalizers of $\Delta(E_2)$ and of $E_2$ resp. of $E_{0,p-1}$ and of $E_2$. This is now straightforward to deduce just from the structure of $N_{Out(F_{2(p-1)})}(E_2)$ given in Proposition 5.3. With notation as in the introduction we get the following result.

**Proposition 5.5.** *There are isomorphisms*

$$N_{0,p-1} \cap N_2 \cong N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p)$$
$$N_\Delta \cap N_2 \cong (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes (Aut(\mathbb{Z}/p) \times \mathbb{Z}/2).$$

## 6. Evaluation of the normalizer spectral sequence for $p > 3$

By Section 4 the poset $\mathcal{A}$ of elementary abelian $p$-subgroups of $Out(F_{2(p-1)})$, $p > 3$, consists of the orbits of $E_R$, of $E_{0,p-1-s}$ for $0 \le s \le \frac{p-1}{2}$, of $\Delta(E_2)$, of $E_2$ and the orbits of the two edges formed by $\Delta(E_2)$ and $E_2$ resp. by $E_{0,p-1}$ and $E_2$. Therefore the quotient of $\mathcal{A}$ by the action of $Out(F_{2(p-1)})$ consists of singletons corresponding to the orbits of $E_R$ resp. of $E_{s,p-1-s}$ for $1 \le s \le \frac{p-1}{2}$, and of one component of dimension 1. This latter component has three vertices formed by the orbits of $E_{0,p-1}$, of $\Delta(E_2)$ and of $E_2$, and two edges formed by the orbits of the edge between $E_{0,p-1}$ and $E_2$ resp. by the edge between $\Delta(E_2)$ and $E_2$. We will denote the preimage of these components in $\mathcal{A}$ by $\mathcal{A}_R$ resp. by $\mathcal{A}_{s,p-1-s}$ for $1 \le s \le \frac{p-1}{2}$ resp. by $\mathcal{A}_2$.

The contributions of $\mathcal{A}_R$ and of $\mathcal{A}_{s,p-1-s}$ to $H^*_{Out(F_{2(p-1)})}(\mathcal{A}; \mathbb{F}_p)$ are simply given by the cohomology of the corresponding normalizers. The more interesting part is given by the component $\mathcal{A}_2$ whose contribution is described in the following result. Together with Propositions 5.1 and 5.2 this finishes the proof of Theorem 1.4.

**Proposition 6.1.** (a) *There is a canonical isomorphism*

$$H^*_{Out(F_{2(p-1)})}(\mathcal{A}_2; \mathbb{F}_p) \cong H^*(N_{0,p-1} *_{N_{0,p-1} \cap N_2} N_2; \mathbb{F}_p).$$

(b) *The restriction map to the orbit of $E_2$ induces an epimorphism of rings*

$$H^*_{Out(F_{2(p-1)})}(\mathcal{A}_2; \mathbb{F}_p) \to H^*(N_2; \mathbb{F}_3) \cong H^*(N_\Sigma(\mathbb{Z}/p) \wr \mathbb{Z}/2; \mathbb{F}_p)$$

*whose kernel is isomorphic to the ideal $H^*(N_{\Sigma_p}(\mathbb{Z}/p); \mathbb{F}_p) \otimes K_{p-1}$ where $K_{p-1}$ is the kernel of the restriction map $H^*(Aut(F_{p-1}); \mathbb{F}_p) \to H^*(N_{\Sigma_p}(\mathbb{Z}/p); \mathbb{F}_p)$.*

**Proof.** We consider the isotropy spectral sequence for the map

$$EOut(F_{2(p-1)}) \times_{Out(F_{2(p-1)})} \mathcal{A}_2 \to \mathcal{A}_2/Out(F_{2(p-1)})$$

By Propositions 5.4 and 5.5 the edge between $\Delta(E_2)$ and $E_2$ gives the same contribution as the vertex $\Delta(E_2)$ (because $p > 3$) and the inclusion of $N_\Delta \cap N_2$ into $N_\Delta$ induces an isomorphism in mod-$p$ cohomology so that this edge may be ignored. Then (a) follows from Lemma 3.1.

Next we claim that the restriction map from $H^*(N_{0,p-1}; \mathbb{F}_p)$ to $H^*(N_{0,p-1} \cap N_2; \mathbb{F}_p)$ is onto; in fact by Propositions 5.2 and 5.5 it is enough to show that the restriction map

$$H^*(Aut(F_{p-1}); \mathbb{F}_p) \to H^*(N_\Sigma(\mathbb{Z}/p); \mathbb{F}_p)$$

is onto. This in turn follows from [3] where it is shown that the restriction map is an isomorphism in Farrell cohomology, together with the observation that the virtual cohomological dimension is $2p - 5$ and the cohomology of $N_{\Sigma_p}(\mathbb{Z}/p)$ is trivial below dimension $2p - 3$.

From the spectral sequence we see now that $H^*_{Out(F_{2(p-1)})}(\mathcal{A}_2; \mathbb{F}_p)$ is the equalizer of the two maps

$$H^*(N_{0,p-1}; \mathbb{F}_3) \times H^*(N_2; \mathbb{F}_3) \to H^*(N_{0,p-1} \cap N_2; \mathbb{F}_3)$$

and the result follows by using once more Propositions 5.2 and 5.5. □

**Remark 6.2.** For $p = 3$ the situation is somewhat more complicated due to the symmetry of the graph $K_{3,3}$ and the resulting additional elementary abelian subgroups of $Out(F_4)$. However, with a bit of effort one can carry out the same analysis for $p = 3$ and thus get a second proof of Theorem 1.4. We leave it to the interested reader to work out the details.

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