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Oscillation of second order self-conjugate differential equation with impulses

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Abstract

In this paper, we investigate the oscillation of second-order self-conjugate differential equation with impulses

$$(a(t)(x(t)+p(t)x(t-\tau))')'+q(t)x(t-\sigma)=0, \quad t \neq t_k, \ t \geqslant t_0, \tag{1}$$

$$x(t_k^+)=(1+b_k)x(t_k), \quad k=1,2,\ldots, \tag{2}$$

$$x'(t_k^+)=(1+b_k)x'(t_k), \quad k=1,2,\ldots, \tag{3}$$

where a, p, q are continuous functions in $[t_0,+\infty)$, $q(t)\geqslant 0$, $a(t)>0$, $\int_{t_0}^{\infty}(1/a(s))\,ds=\infty$, $\tau>0$, $\sigma>0$, $b_k>-1$, $0<t_0<t_1<t_2<\cdots<t_k<\cdots$ and $\lim_{k\to\infty}t_k=\infty$. We get some sufficient conditions for the oscillation of solutions of Eqs. (1)–(3).

MSC: 34c10; 34k11

Keywords: Self-conjugate differential equation; Impulses; Oscillation

1. Introduction

Consider the impulsive differential equation

$$(a(t)(x(t)+p(t)x(t-\tau))')'+q(t)x(t-\sigma)=0, \quad t \neq t_k, \ k=1,2,\ldots, \ t \geqslant t_0, \tag{1}$$

$$x(t_k^+)=(1+b_k)x(t_k), \quad k=1,2,\ldots, \tag{2}$$

$$x'(t_k^+)=(1+b_k)x'(t_k), \quad k=1,2,\ldots, \tag{3}$$

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where a, p, q are continuous functions in $[t_0, +\infty)$, $q(t) \geqslant 0$, $a(t) > 0$, $\int_{t_0}^{\infty}(1/a(s))\,ds = \infty$, $\tau > 0$, $\sigma > 0$, $b_k > -1$, $0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. Suppose that

$$x'(t_k) = x'(t_k^-) = \lim_{h\to 0-} \frac{x(t_k+h)-x(t_k)}{h}, \quad x'(t_k^+) = \lim_{h\to 0+} \frac{x(t_k+h)-x(t_k^+)}{h}.$$

It is well known that ordinary differential equations with impulses and delay differential equations have been considered by many authors. The theory of impulsive differential equations is emerging as an important area of investigation, since it is much richer than the corresponding theory of differential equations without impulsive effects. Moreover, such equations may exhibit several real world phenomena, such as rhythmical beating, merging of solutions, and noncontinuity of solutions.

In the recent years, there is increasing interest on the oscillation/nonoscillation of impulsive delay differential equations, and numerous papers have been published on this class of equations and good results have been obtained (see [1,2,4–8] etc. and the references therein). For example, in [4], Luo researched the equation

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + f(t, x(t)) = 0, \quad t \geqslant t_0,\ t \neq t_k,\ k = 1, 2, \ldots,$$

$$x(t_k^+) = g_k(x(t_k)), \quad x'(t_k^+) = h_k(x'(t_k)),\ k = 1, 2, \ldots,$$

$$x(t_0^+) = x_0, \quad x'(t_0^+) = x'(t_0).$$

He obtained sufficient conditions for oscillation of all solutions of the equation.

In [6], Wu et al. discussed the equation

$$[r(t)x'(t)]' + p(t)x'(t) + Q(t, x(t)) = 0, \quad t \geqslant t_0,\ t \neq t_k,\ k = 1, 2, \ldots,$$

$$x(t_k^+) = g_k(x(t_k)), \quad x'(t_k^+) = h_k(x'(t_k)),$$

$$x(t_0^+) = x_0, \quad x'(t_0^+) = x'_0.$$

They also investigated the oscillation of the above equation.

Though there have been many papers about them, fewer papers are on neutral differential equations with impulses. Hence we study Eqs. (1)–(3) and we get some sufficient conditions for the oscillation of solutions of Eqs. (1)–(3).

Definition 1. For $\phi \in C([t_0-\gamma, t_0], R)$, $\gamma = \max\{\tau, \sigma\}$, a function $x : [t_0-\gamma, \infty) \to R$ is called a solution of Eqs. (1)–(3) satisfying the initial value condition

$$x(t) = \phi(t), \quad t \in [t_0-\gamma, t_0]$$

if the following conditions are satisfied:

(i) $x(t) = \phi(t)$ for $t \in [t_0-\gamma, t_0]$, $x(t)$ is continuous for $t \in [t_0, \infty)/\{t = t_k, k = 1, 2, \ldots\}$;
(ii) $x(t) + p(t)x(t-\tau)$ is continuously differentiable for $t \geqslant t_0$, $t \neq t_k$, $t \neq t_k+\tau$, $t \neq t_k+\sigma$, $k = 1, 2, \ldots$ and $x(t)$ satisfies Eq. (1);
(iii) $x(t_k^+)$, $x(t_k^-)$, $x'(t_k^+)$, $x'(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $x'(t_k^-) = x'(t_k)$, $k = 1, 2, \ldots$ and Eqs. (2), (3) are satisfied.

Definition 2. A solution of Eqs. (1)–(3) is said to be nonoscillatory if the solution is eventually positive or eventually negative; otherwise, the solution is said to be oscillatory.

2. Main results

Throughout the paper, let $u(t) = x(t) + p(t)x(t-\tau)$, we obtain some conclusions as follows:

Theorem 1. *Assume* $0 \leqslant \prod_{t-\tau\leqslant t_k<t}(1+b_k)^{-1}p(t) \leqslant 1$, *there exists a differentiable function* $\psi(t) > 0$ *such that*

$$\int_{t_0}^{\infty}\left(\psi(s)\prod_{s-\sigma\leqslant t_k<s}(1+b_k)^{-1}q(s)\left(1-\prod_{s-\sigma-\tau\leqslant t_k<s-\sigma}(1+b_k)^{-1}p(s-\sigma)\right) - \frac{a(s-\sigma)(\psi'(s))^2}{4\psi(s)}\right)\mathrm{d}s = \infty, \tag{4}$$

then every solution of Eqs. (1)–(3) *is oscillatory.*

Theorem 2. *Assume* $p(t) \equiv p,\ 0 < p < 1,\ \gamma = \max\{\tau, \sigma\},\ b_k = b > 0,\ t_{k+1} - t_k = \tau,$

$$\int_{t_0}^{\infty}\prod_{t_0<t_{0,n}<s}\frac{1}{t_{k,\sigma}}(1-p)q(s)\,\mathrm{d}s = \infty, \tag{5}$$

where

$$t_{k,\sigma} = \begin{cases} \dfrac{1}{1+b}, & t_{0,n} = t_k + \sigma \neq t_m\ (m > k), \\ 1+b, & t_{0,n} = t_k, \\ 1, & t_{0,n} = t_k + \sigma = t_m, \end{cases}$$

and $t_{0,n} = t_k$ *or* $t_{0,n} = t_k + \sigma$ $(t_1 = t_{0,1} < t_{0,2} < \cdots)$, *then every solution of Eqs.* (1)–(3) *is oscillatory.*

Corollary 1. *If*

$$\int_{t_0}^{\infty}(1-p)q(s)\,\mathrm{d}s = \infty \tag{6}$$

replaces (5), *and* $2\sigma > \tau > \sigma$, *other conditions in Theorem* 2 *hold, then every solution of Eqs.* (1)–(3) *is oscillatory.*

Corollary 2. *If* $\sigma = k_0\tau,\ k_0$ *is a positive integer*, (5) *is replaced by that there exists a constant* $\alpha > 0$, *such that*

$$\frac{1}{1+b} \geqslant \left(1+\frac{\tau}{t_1}\right)^{\alpha}, \tag{7}$$

$$\int_{t_0}^{\infty}s^{\alpha}q(s)\,\mathrm{d}s = \infty, \tag{8}$$

other conditions in Theorem 2 *hold, then every solution of Eqs.* (1)–(3) *is oscillatory.*

Theorem 3. *Assume that*

$$-1 \leqslant \prod_{t-\tau\leqslant t_k<t}(1+b_k)^{-1}p(t) < 0, \quad 0 < c_1 \leqslant \prod_{k=1}^{\infty}(1+b_k) \leqslant c_2,\ \tau > \sigma,$$

$$\liminf_{t\to\infty}\int_{t}^{t+\tau-\sigma}\prod_{s-\sigma\leqslant t_k<s}(1+b_k)^{-1}q(s)\left(1-\prod_{s-\sigma-\tau\leqslant t_k<s-\sigma}(1+b_k)^{-1}p(s-\sigma)\right)\mathrm{d}s > 0, \tag{9}$$

then every unbounded solution of Eqs. (1)–(3) *is oscillatory.*

Theorem 4. *Assume that* $a(t) \equiv 1$, $p(t)$ *is oscillatory*, $\lim_{t\to\infty} p(t) = 0$, $0 < c_1 \leqslant \prod_{k=1}^{\infty} (1 + b_k) \leqslant c_2$, *there exists a differential function* $\psi(t) > 0$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \psi(s)Q(s)\,\mathrm{d}s = \infty, \quad \limsup_{t\to\infty} \int_{t_0}^{t} \frac{(\psi'(s))^2}{\psi(s)}\,\mathrm{d}s < \infty, \tag{10}$$

where $Q(t) = \prod_{t-\sigma \leqslant t_k < t} (1 + b_k)^{-1} q(t)$, *then every bounded solution of Eqs.* (1)–(3) *is oscillatory.*

In order to prove these theorems, we need the following lemmas.

We introduce the notation as follows: $PC^1[R^+, R] = \{x : R^+ \to R$; $x(t)$ is continuous and continuously differentiable everywhere except at some t_k where $x(t_k^+)$, $x(t_k^-)$, $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $x'(t_k^-) = x'(t_k)\}$.

Lemma 1 (*Chan and Ke [1]*). *Assume that*

(A_0) $v \in PC^1[R^+, R]$ *and* $v(t)$ *is left-continuous at* t_k, $k = 1, 2, \ldots$

(A_1) *for* $k = 1, 2, \ldots, t \geqslant t_0$,

$$v'(t) \leqslant p(t)v(t) + q(t), \quad t \neq t_k,$$
$$v(t_k^+) \leqslant d_k v(t_k) + b_k,$$

where $v'(t) = (\mathrm{d}/\mathrm{d}t)v(t)$, $p, q \in C(R^+, R)$, $d_k \geqslant 0$ *and* b_k *are real constants. Then*

$$v(t) \leqslant v(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t} p(s)\,\mathrm{d}s\right) + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^{t} p(s)\,\mathrm{d}s\right)\right) b_k$$
$$+ \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_{s}^{t} p(\sigma)\,\mathrm{d}\sigma\right) q(s)\,\mathrm{d}s, \quad t \geqslant t_0.$$

Lemma 2. *Assume that* $p(t) \equiv p \geqslant 0$, $\gamma = \max\{\tau, \sigma\}$, $b_k = b$, $t_{k+1} - t_k = \tau$. *Let* $x(t)$ *be a solution of Eqs.* (1)–(3) *and there exists* $T \geqslant t_0$ *such that* $x(t) > 0$, $t \geqslant T - \gamma$. *Then* $u'(t_k^+) \geqslant 0$, *and* $u'(t) \geqslant 0$ *for* $t \in (t_k, t_{k+1}]$, *where* $t_k \geqslant T$, $k = 1, 2, \ldots$.

Proof. Since $x(t) > 0$ for $t \geqslant T - \gamma$, then $x(t - \sigma) > 0$, $x(t - \tau) > 0$, $u(t) > 0$, $t \geqslant T$.

$$u(t_k^+) = x(t_k^+) + px(t_{k-1}^+) = (1 + b)x(t_k) + p(1 + b)x(t_{k-1})$$
$$= (1 + b)[x(t_k) + px(t_{k-1})] = (1 + b)u(t_k),$$
$$u'(t_k^+) = x'(t_k^+) + px'(t_{k-1}^+) = (1 + b)x'(t_k) + p(1 + b)x'(t_{k-1})$$
$$= (1 + b)u'(t_k).$$

We first prove $u'(t_k) \geqslant 0$ for $t_k \geqslant T$. If not, there exists $j \in N$ such that $u'(t_j) < 0$ for $t_j \geqslant T$, and $u'(t_j^+) = (1+b)u'(t_j) < 0$. From (1), for $t \in (t_{j+i-1}, t_{j+i}]$, $i = 1, 2, \ldots$, we get $(a(t)u'(t))' = -q(t)x(t - \sigma) \leqslant 0$. So

$$a(t_{j+1})u'(t_{j+1}) \leqslant a(t_j^+)u'(t_j^+) = a(t_j)(1 + b)u'(t_j),$$
$$a(t_{j+2})u'(t_{j+2}) \leqslant a(t_{j+1}^+)u'(t_{j+1}^+) = a(t_{j+1})(1 + b)u'(t_{j+1})$$
$$\leqslant (1 + b)^2 a(t_j)u'(t_j) < 0.$$

By induction, we know $a(t)u'(t) \leqslant (1 + b)^n a(t_j)u'(t_j) \triangleq (1 + b)^n(-\beta) < 0$ for $t \in (t_{j+n-1}, t_{j+n}]$. So $u'(t) \leqslant -(\beta/a(t))\prod_{t_j \leqslant t_k < t} (1 + b)$. By Lemma 1, we get

$$u(t) \leqslant u(t_j^+) \prod_{t_j < t_k < t} (1 + b) - \beta(1 + b) \int_{t_j}^{t} \prod_{s < t_k < t} (1 + b) \prod_{t_j < t_l < s} (1 + b) \frac{1}{a(s)}\,\mathrm{d}s, \quad t > t_j.$$

Since

$$\int_{t_j}^{t}\prod_{t_j<t_k<t}(1+b)\frac{1}{a(s)}\,\mathrm{d}s=\int_{t_j}^{t}\prod_{t_j<t_l<s}(1+b)\prod_{s<t_k<t}(1+b)\frac{1}{a(s)}\,\mathrm{d}s,$$

we get

$$u(t)\leqslant\left(u(t_j^+)-\beta(1+b)\int_{t_j}^{t}\frac{\mathrm{d}s}{a(s)}\right)\prod_{t_j<t_k<t}(1+b),\quad t>t_j. \tag{11}$$

From $u(t)>0$, we see (11) contradicts to $\int_{t_0}^{\infty}(1/a(s))\,\mathrm{d}s=\infty$. So $u'(t_k)\geqslant 0$ for $t_k\geqslant T$, and $u'(t_k^+)=(1+b)u'(t_k)\geqslant 0$. Since $(a(t)u'(t))'\leqslant 0$, i.e. $a(t)u'(t)$ is not increasing in $(t_{j+i-1},t_{j+i}]$, we know $a(t)u'(t)\geqslant 0$, for $t\in(t_{j+i-1},t_{j+i}]$, i.e. $u'(t)\geqslant 0$. This completes Lemma 2. □

Lemma 3. *Let* $R(t)=\prod_{t-\tau\leqslant t_k<t}(1+b_k)^{-1}p(t)$, $Q(t)=\prod_{t-\sigma\leqslant t_k<t}(1+b_k)^{-1}q(t)$, *then all solutions of Eqs.* (1)–(3) *are oscillatory if and only if all solutions of*

$$(a(t)(y(t)+R(t)y(t-\tau))')'+Q(t)y(t-\sigma)=0\quad a.e.\ t\geqslant\gamma \tag{12}$$

are oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eqs. (1)–(3), i.e. $x(t)\neq 0$ holds for $t\geqslant T\geqslant t_0$. Define $y(t)=\prod_{T\leqslant t_k<t}(1+b_k)^{-1}x(t)$. Since $b_k>-1$, we get $y(t)\neq 0$. From $x(t_k^+)=(1+b_k)x(t_k)$, $x'(t_k^+)=(1+b_k)x'(t_k)$, we get

$$y(t_k^+)=\prod_{T\leqslant t_j\leqslant t_k}(1+b_j)^{-1}x(t_k^+)=\prod_{T\leqslant t_j<t_k}(1+b_j)^{-1}x(t_k)=y(t_k),$$

$$y(t_k^-)=\prod_{T\leqslant t_j<t_k^-}(1+b_j)^{-1}x(t_k^-)=\prod_{T\leqslant t_j<t_k}(1+b_j)^{-1}x(t_k)=y(t_k).$$

So $y(t)$ is continuous in $[T,\infty)$.

$$y'(t_k^+)=\prod_{T\leqslant t_j\leqslant t_k}(1+b_j)^{-1}x'(t_k^+)=\prod_{T\leqslant t_j<t_k}(1+b_j)^{-1}x'(t_k)=y'(t_k),$$

$$y'(t_k^-)=\prod_{T\leqslant t_j<t_k^-}(1+b_j)^{-1}x'(t_k^-)=\prod_{T\leqslant t_j<t_k}(1+b_j)^{-1}x'(t_k)=y'(t_k),$$

and

$$\prod_{T\leqslant t_k<t}(1+b_k)^{-1}(a(t)(x(t)+p(t)x(t-\tau))')'+\prod_{T\leqslant t_k<t}(1+b_k)^{-1}q(t)x(t-\sigma)=0,\quad t\neq t_k.$$

So

$$(a(t)(y(t)+R(t)y(t-\tau))')'+Q(t)y(t-\sigma)=0\quad \text{a.e.}$$

Conversely, suppose $y(t)$ is a nonoscillatory solution of Eq. (12), i.e. $y(t) \neq 0$ holds for $t \geqslant T \geqslant t_0$. Let $x(t) = \prod_{T \leqslant t_k < t}(1+b_k)y(t)$, from (12), we get

$$\prod_{T \leqslant t_k < t}(1+b_k)(a(t)(y(t)+R(t)y(t-\tau))')' + \prod_{T \leqslant t_k < t}(1+b_k)Q(t)y(t-\sigma) = 0 \quad \text{a.e.},$$

$$\left(a(t)\left(\prod_{T \leqslant t_k < t}(1+b_k)y(t) + \prod_{T \leqslant t_k < t}(1+b_k)R(t)y(t-\tau)\right)'\right)'$$

$$+\, q(t)\prod_{T \leqslant t_k < t-\sigma}(1+b_k)y(t-\sigma) = 0 \quad \text{a.e.},$$

$$(a(t)(x(t)+p(t)x(t-\tau))')' + q(t)x(t-\sigma) = 0 \quad \text{a.e.}$$

and

$$x(t_k^+) = \lim_{t \to t_k^+}\prod_{T \leqslant t_j < t}(1+b_j)y(t) = \prod_{T \leqslant t_j \leqslant t_k}(1+b_j)y(t_k), \quad x(t_k) = \prod_{T \leqslant t_j < t_k}(1+b_j)y(t_k),$$

then $x(t_k^+) = (1+b_k)x(t_k)$.

$$x'(t_k^+) = \lim_{t \to t_k^+}\prod_{T \leqslant t_j < t}(1+b_j)y'(t) = \prod_{T \leqslant t_j \leqslant t_k}(1+b_j)y'(t_k), \quad x'(t_k) = \prod_{T \leqslant t_j < t_k}(1+b_j)y'(t_k),$$

then $x'(t_k^+) = (1+b_k)x'(t_k)$. So $x(t)$ is a nonoscillatory solution of Eqs. (1)–(3). □

Corollary 3. *Suppose that the conditions in Lemma* 3 *hold*:

(i) *If $x(t)$ is a solution of Eqs*. (1)–(3) *in* $[t_0, \infty)$, *then* $y(t) = \prod_{t_0 \leqslant t_k < t}(1+b_k)^{-1}x(t)$ *is a solution of Eq*. (12).
(ii) *If $y(t)$ is a solution of Eq*. (12) *in* $[t_0, \infty)$, *then* $x(t) = \prod_{t_0 \leqslant t_k < t}(1+b_k)y(t)$ *is a solution of Eqs*. (1)–(3).

Now, we begin to prove our theorems.

Proof of Theorem 1. Suppose $x(t)$ is an eventually positive solution of Eqs. (1)–(3). By Corollary 3, Eq. (12) has an eventually positive solution $y(t)$. That is, $y(t) > 0,\ y(t-\sigma) > 0,\ y(t-\tau) > 0$ for $t \geqslant T \geqslant t_0$. Let $z(t) = y(t) + R(t)y(t-\tau)$, then $z(t) > 0$ for $t \geqslant T$. From Eq. (12), we have $(a(t)z'(t))' = -Q(t)y(t-\sigma) \leqslant 0$, a.e. So $a(t)z'(t)$ is not increasing. We can prove $z'(t) \geqslant 0$ holds eventually.

If not, for any $t > T$, there exists $t_1 > t$ such that $z'(t_1) < 0$.

$$a(t)z'(t) \leqslant a(t_1)z'(t_1) = -\beta < 0, \quad t \geqslant t_1, \text{ a.e.},$$

$$z'(t) \leqslant \frac{-\beta}{a(t)} \quad \text{a.e.}$$

Integrating it from t_1 to ∞, from conditions, we know $\lim_{t\to\infty} z(t) = -\infty$, this contradicts with $z(t) > 0$. So $z'(t) \geqslant 0$ for some $t_2 \geqslant T$. By Eq. (12),

$$(a(t)z'(t))' + Q(t)(z(t-\sigma) - R(t-\sigma)y(t-\sigma-\tau)) = 0 \quad \text{a.e.},$$

$$(a(t)z'(t))' + Q(t)(1-R(t-\sigma))z(t-\sigma) \leqslant 0, \ t \geqslant t_2, \text{ a.e.}$$

Let

$$w(t) = \frac{\psi(t)a(t)z'(t)}{z(t-\sigma)} > 0, \quad t \geqslant t_2,$$

then

$$w'(t) \leqslant -\psi(t)Q(t)(1-R(t-\sigma)) + \frac{a(t)z'(t)\psi'(t)}{z(t-\sigma)} - \frac{a(t)\psi(t)z'(t-\sigma)z'(t)}{z^2(t-\sigma)}.$$

Notice that

$$\frac{a(t)\psi(t)z'(t-\sigma)z'(t)}{z^2(t-\sigma)} \geqslant \frac{a^2(t)\psi(t)(z'(t))^2}{z^2(t-\sigma)a(t-\sigma)}.$$

We have

$$w'(t) \leqslant -\psi(t)Q(t)(1-R(t-\sigma)) + \frac{a(t-\sigma)(\psi'(t))^2}{4\psi(t)} - \left(a(t)\sqrt{\frac{\psi(t)}{a(t-\sigma)}}\frac{z'(t)}{z(t-\sigma)} - \frac{\psi'(t)}{2\sqrt{\psi(t)/a(t-\sigma)}}\right)^2,$$

i.e. $w'(t) \leqslant -\psi(t)Q(t)(1-R(t-\sigma)) + a(t-\sigma)(\psi'(t))^2/4\psi(t)$ holds for $t \geqslant t_2$. Integrating it from t_2 to t, we get

$$w(t) \leqslant w(t_2) - \int_{t_2}^{t}\left(\psi(s)Q(s)(1-R(s-\sigma)) - \frac{a(s-\sigma)(\psi'(s))^2}{4\psi(s)}\right)\mathrm{d}s.$$

We get a contradiction as $t \to \infty$. □

Proof of Theorem 2. Suppose that $x(t)$ is a nonoscillatory solution of Eqs. (1)–(3), without loss of generality; we assume that $x(t) > 0$ for $t \geqslant T \geqslant t_0$. From the proof of Lemma 2, we know Eqs. (1)–(3) can be written as

$$\begin{aligned} &(a(t)u'(t))' + q(t)x(t-\sigma) = 0, \quad t \neq t_k, \ k = 1, 2, \ldots, \ t \geqslant t_0, \\ &u(t_k^+) = (1+b)u(t_k), \\ &u'(t_k^+) = (1+b)u'(t_k). \end{aligned} \tag{13}$$

From Lemma 2, $u'(t_k^+) \geqslant 0$, $u'(t) \geqslant 0$ hold for $t \in (t_k, t_{k+1}]$ and $t_k \geqslant T$. Eq. (13) is

$$\begin{aligned} &(a(t)u'(t))' + q(t)(u(t-\sigma) - px(t-\sigma-\tau)) = 0, \quad t \neq t_k, \\ &(a(t)u'(t))' + q(t)(1-p)u(t-\sigma) \leqslant 0, \quad t \neq t_k, \ t_k \geqslant T. \end{aligned}$$

Let $w(t) = a(t)u'(t)/u(t-\sigma)$, $t \geqslant T$. By Lemma 2, we have $w(t_k^+) \geqslant 0$, $w(t) \geqslant 0$ hold for $t \in (t_k, t_{k+1}]$ and $t_k \geqslant T$, $k = 1, 2, \ldots$.

$$w'(t) = \frac{(a(t)u'(t))'}{u(t-\sigma)} - \frac{a(t)u'(t)u'(t-\sigma)}{u^2(t-\sigma)} \leqslant -q(t)(1-p), \quad t \neq t_{0,n}, \tag{14}$$

where $t_{0,n} = t_k$ or $t_k + \sigma$. Notice that $a(t)$ is continuous, $1 + b > 1$, we have

$$\begin{aligned} w(t_k^+) &= \frac{a(t_k^+)u'(t_k^+)}{u(t_k^+ - \sigma)} \\ &\leqslant \begin{cases} \dfrac{a(t_k)(1+b)u'(t_k)}{u(t_k-\sigma)} = (1+b)w(t_k), & t_k - \sigma \neq t_m, \ 0 < m < k, \\ \dfrac{a(t_k)(1+b)u'(t_k)}{(1+b)u(t_k-\sigma)} = w(t_k), & t_k - \sigma = t_m, \ 0 < m < k, \end{cases} \end{aligned} \tag{15}$$

$$\begin{aligned} w(t_k^+ + \sigma) &= \frac{a(t_k^+ + \sigma)u'(t_k^+ + \sigma)}{u(t_k^+)} \\ &\leqslant \begin{cases} \dfrac{a(t_k+\sigma)u'(t_k+\sigma)}{(1+b)u(t_k)} = \dfrac{1}{(1+b)}w(t_k+\sigma), & t_k + \sigma \neq t_m, \ m > k, \\ \dfrac{a(t_k+\sigma)(1+b)u'(t_k+\sigma)}{(1+b)u(t_k)} = w(t_k+\sigma), & t_k + \sigma = t_m, \ m > k. \end{cases} \end{aligned} \tag{16}$$

From Eqs. (14)–(16), there are $w'(t) \leqslant (p-1)q(t)$, $t \neq t_{0,n}$, $w(t_{0,n}^{+}) \leqslant t_{k,\sigma} w(t_{0,n})$. By Lemma 1, there is

$$w(t) \leqslant \prod_{t_0 < t_{0,n} < t} t_{k,\sigma} \left(w(t_0) - \int_{t_0}^{t} \prod_{t_0 < t_{0,n} < s} \frac{1}{t_{k,\sigma}} (1-p) q(s)\, \mathrm{d}s \right). \tag{17}$$

In view of (5) and (17), we get a contradiction as $t \to \infty$. This completes the proof. □

Proof of Corollary 1. For $t \in (t_n, t_{n+1}]$, we have

$$\begin{aligned}
&\int_{t_0}^{t} \prod_{t_0 < t_{0,n} < s} \frac{1}{t_{k,\sigma}} (1-p) q(s)\, \mathrm{d}s \\
&\quad = (1+b) \int_{t_0}^{t_1} (1-p) q(s)\, \mathrm{d}s + (1+b) \int_{t_1}^{t_2} (1-p) q(s)\, \mathrm{d}s + \cdots + (1+b) \int_{t_n}^{t} (1-p) q(s)\, \mathrm{d}s \\
&\quad = (1+b) \int_{t_0}^{t} (1-p) q(s)\, \mathrm{d}s.
\end{aligned}$$

From (6), we get

$$\int_{t_0}^{\infty} \prod_{t_0 < t_{0,n} < s} \frac{1}{t_{k,\sigma}} (1-p) q(s)\, \mathrm{d}s = \infty.$$

By Theorem 2, every solution of Eqs. (1)–(3) is oscillatory. □

Proof of Corollary 2. For $t \in (t_{k_0}, t_{k_0+1}]$, we have

$$\begin{aligned}
&\int_{t_0}^{t} \prod_{t_0 < t_{0,n} < s} \frac{1}{t_{k,\sigma}} (1-p) q(s)\, \mathrm{d}s \\
&\quad \geqslant \int_{t_0}^{t_1} (1-p) q(s)\, \mathrm{d}s + \frac{1}{(1+b)} \int_{t_1}^{t_2} (1-p) q(s)\, \mathrm{d}s + \cdots + \frac{1}{(1+b)^{k_0}} \int_{t_{k_0}}^{t} (1-p) q(s)\, \mathrm{d}s.
\end{aligned} \tag{18}$$

From (7), we know

$$\begin{aligned}
&\frac{1}{(1+b)} \geqslant \left(\frac{t_2}{t_1} \right)^{\alpha}, \\
&\frac{1}{(1+b)} \geqslant \left(\frac{t_3}{t_2} \right)^{\alpha}, \\
&\frac{1}{(1+b)} \geqslant \left(\frac{t_{k+1}}{t_k} \right)^{\alpha}, \\
&\qquad \vdots \\
&\frac{1}{(1+b)^n} \geqslant \left(\frac{t_2}{t_1} \frac{t_3}{t_2} \cdots \frac{t_{n+1}}{t_n} \right)^{\alpha} = \left(\frac{t_{n+1}}{t_1} \right)^{\alpha}.
\end{aligned}$$

From (18), for $t \in (t_n, t_{n+1}]$,

$$\begin{aligned}
&\int_{t_0}^{t} \prod_{t_0<t_{0,n}<s} \frac{1}{t_{k,\sigma}}(1-p)q(s)\,\mathrm{d}s \\
&\geqslant \int_{t_0}^{t_1}(1-p)q(s)\,\mathrm{d}s + \frac{1}{(1+b)}\int_{t_1}^{t_2}(1-p)q(s)\,\mathrm{d}s + \cdots + \frac{1}{(1+b)^n}\int_{t_n}^{t}(1-p)q(s)\,\mathrm{d}s \\
&\geqslant \frac{1}{t_1^{\alpha}}\int_{t_0}^{t_1} t_1^{\alpha}(1-p)q(s)\,\mathrm{d}s + \frac{1}{t_1^{\alpha}}\int_{t_1}^{t_2} t_2^{\alpha}(1-p)q(s)\,\mathrm{d}s + \cdots + \frac{1}{t_1^{\alpha}}\int_{t_n}^{t} t_{n+1}^{\alpha}(1-p)q(s)\,\mathrm{d}s \\
&\geqslant \frac{1}{t_1^{\alpha}}\left(\int_{t_0}^{t_1} s^{\alpha}(1-p)q(s)\,\mathrm{d}s + \int_{t_1}^{t_2} s^{\alpha}(1-p)q(s)\,\mathrm{d}s + \cdots + \int_{t_n}^{t} s^{\alpha}(1-p)q(s)\,\mathrm{d}s\right) \\
&= \frac{1}{t_1^{\alpha}}\int_{t_0}^{t} s^{\alpha}(1-p)q(s)\,\mathrm{d}s.
\end{aligned}$$

From (8), we know (5) holds as $t \to \infty$. By Theorem 2, the conclusion is proved. □

Proof of Theorem 3. Assume that $x(t)$ is an unbounded nonoscillatory solution of Eqs. (1)–(3), without loss of generality, we suppose $x(t) > 0$ for $t \geqslant T_1 \geqslant t_0$. From Corollary 3, we can see Eq. (12) has an unbounded nonoscillatory solution $y(t) > 0,\ t \geqslant T_1$. Define $z(t) = y(t) + R(t)y(t-\tau)$, we will prove $z(t) > 0$.

If not, $z(t) \leqslant 0$, then $y(t) \leqslant -R(t)y(t-\tau) \leqslant y(t-\tau)$, i.e. $y(t)$ is bounded, it is a contradiction. So $z(t) > 0$. From Eq. (12), $(a(t)z'(t))' = -Q(t)y(t-\sigma) \leqslant 0$, a.e. Similar to the proof of Theorem 1, we know that there exists $T_2 > T_1$ such that $z'(t) \geqslant 0$ for $t \geqslant T_2$, and

$$(a(t)z'(t))' + Q(t)(1 - R(t-\sigma))z(t-\sigma) \leqslant 0, \quad t \geqslant T_2, \text{ a.e.}$$

Let

$$w(t) = \frac{a(t)z'(t)}{z(t-\sigma)} \geqslant 0, \quad t \geqslant T_2$$

$$\begin{aligned}
w'(t) &= \frac{(a(t)z'(t))'}{z(t-\sigma)} - \frac{a(t)z'(t)z'(t-\sigma)}{z^2(t-\sigma)} \\
&\leqslant -Q(t)(1-R(t-\sigma)) - \frac{a^2(t)(z'(t))^2}{a(t-\sigma)z^2(t-\sigma)} \\
&\leqslant -Q(t)(1-R(t-\sigma)).
\end{aligned} \tag{19}$$

From (9) we have

$$\int_{T_2}^{\infty} Q(t)(1-R(t-\sigma))\,\mathrm{d}t = \infty.$$

Integrating (19) from T_2 to t,

$$w(t) \leqslant w(T_2) - \int_{T_2}^{t} Q(s)(1-R(s-\sigma))\,\mathrm{d}s.$$

So $\lim_{t\to\infty} w(t) = -\infty$. This is a contradiction. We completed the proof. □

Proof of Theorem 4. Assume that $x(t)$ is a bounded nonoscillatory solution of Eqs. (1)–(3), then Eq. (12) has a bounded nonoscillatory solution $y(t)$. Without loss of generality, we suppose $y(t) > 0,\ y(t-\sigma) > 0$ for $t \geqslant T$. Let $z(t) = y(t) + R(t)y(t-\tau)$, then

$$z''(t) = -Q(t)y(t-\sigma) \leqslant 0, \quad t \geqslant T.$$

So $z'(t)$ and $z(t)$ are constant sign eventually. Since $p(t)$ is oscillatory, we get $R(t)$ is oscillatory and there exists $T_1 > T$, such that $z(t) > 0$ holds for $t > T_1$. So $z'(t) \geqslant 0$. Since $y(t)$ is bounded, $\lim_{t\to\infty} p(t)=0$, we get $\lim_{t\to\infty} R(t)y(t-\tau)=0$. Hence $y(t) = z(t) - R(t)y(t-\tau) > \frac{1}{2}z(t)$ for $t > T_2 \geqslant T_1$.

Let

$$h(t) = \frac{z'(t)}{z((t-\sigma)/2)} \geqslant 0,$$

then

$$\begin{aligned} h'(t) &= \frac{z''(t)}{z((t-\sigma)/2)} - \frac{h(t)z'((t-\sigma)/2)}{2z((t-\sigma)/2)} \\ &\leqslant -\frac{Q(t)z(t-\sigma)}{2z((t-\sigma)/2)} - \frac{1}{2}h^2(t) \\ &\leqslant \frac{-Q(t)}{2} - \frac{h^2(t)}{2} \end{aligned}$$

holds for $t > T_3 \geqslant T_2$, i.e.

$$\frac{Q(t)}{2} \leqslant -h'(t) - \frac{h^2(t)}{2}.$$

Multiplying $\psi(t)$ on both sides, and integrating it from T_3 to t, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{T_3}^{t} \psi(s)Q(s)\,\mathrm{d}s \\ &\quad\leqslant -\int_{T_3}^{t} \psi(s)h'(s)\,\mathrm{d}s - \frac{1}{2}\int_{T_3}^{t} \psi(s)h^2(s)\,\mathrm{d}s \\ &\quad= -\psi(t)h(t) + \psi(T_3)h(T_3) + \int_{T_3}^{t} \psi'(s)h(s)\,\mathrm{d}s - \frac{1}{2}\int_{T_3}^{t} \psi(s)h^2(s)\,\mathrm{d}s \\ &\quad\leqslant \psi(T_3)h(T_3) - \frac{1}{2}\int_{T_3}^{t} \psi(s)\left(h(s) - \frac{\psi'(s)}{\psi(s)}\right)^2 \mathrm{d}s + \frac{1}{2}\int_{T_3}^{t} \frac{(\psi'(s))^2}{\psi(s)}\,\mathrm{d}s \\ &\quad\leqslant \psi(T_3)h(T_3) + \frac{1}{2}\int_{T_3}^{t} \frac{(\psi'(s))^2}{\psi(s)}\,\mathrm{d}s. \end{aligned}$$

From the conditions, we get

$$\begin{aligned} \infty &= \frac{1}{2}\limsup_{t\to\infty}\int_{T_3}^{t} \psi(s)Q(s)\,\mathrm{d}s \\ &\leqslant \psi(T_3)h(T_3) + \frac{1}{2}\limsup_{t\to\infty}\int_{T_3}^{t} \frac{(\psi'(s))^2}{\psi(s)}\,\mathrm{d}s < \infty. \end{aligned}$$

So the proof is completed. □

Remark. Grace and Lalli [3] studied the differential equation

$$(r(t)[x(t) + p(t)x(t-\tau)]')' + q(t)f(x(t-\sigma)) = 0 \tag{20}$$

subject to

$$\frac{f(x)}{x} \geqslant K \text{ for some } K > 0, \qquad \int^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty,$$

and showed that if there exists a function $\rho \in C^1([t_0, \infty), (0, \infty))$ such that

$$\int^{\infty} \left[K\rho(s)q(s)(1 - p(s - \sigma)) - \frac{(\rho'(s))^2 r(s - \sigma)}{4\rho(s)} \right] \mathrm{d}s = \infty, \tag{21}$$

then (20) is oscillatory. In our paper, if we choose $b_k = 0$, then (4) equals (21).

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