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Controlling rough paths

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Abstract

We formulate indefinite integration with respect to an irregular function as an algebraic problem which has a unique solution under some analytic constraints. This allows us to define a good notion of integral with respect to irregular paths with Hölder exponent greater than 1/3 (e.g. samples of Brownian motion) and study the problem of the existence, uniqueness and continuity of solution of differential equations driven by such paths. We recover Young's theory of integration and the main results of Lyons' theory of rough paths in Hölder topology.

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1. Introduction

This work has grown out from the attempt of the author to understand the integration theory of Lyons [7,8] which gives a meaning and nice continuity properties to integrals of the form

$$\int_s^t \langle \varphi(X_u), dX_u \rangle, \tag{1}$$

where φ a differential 1-form on some vector space V and $t \mapsto X_t$ is a path in V not necessarily of bounded variation. From the point of view of Stochastic Analysis Lyons' theory provide a path-wise formulation of stochastic integration and stochastic differential equations. The main feature of this theory is that a path in a

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vector space V should not be considered determined by a function from an interval $I \subset \mathbb{R}$ to V but, if this path is not regular enough, some additional information is needed which would play the rôle of the iterated integrals for regular paths: e.g. quantities like the rank two tensor:

$$\mathbb{X}^{2,\mu\nu}_{st} = \int_s^t \int_s^u dX^\mu_v \, dX^\nu_u \tag{2}$$

and its generalizations (see the works of Chen [10] for applications of iterated integrals to Algebraic Geometry and Lie Group Theory). For irregular paths the r.h.s. of Eq. (2) cannot in general be understood as a classical Lebesgue–Stieltjes integral. However if we have *any* reasonable definition for this integral then (under some mild regularity conditions) all the integrals of the form given in Eq. (1) can be defined to depend continuously on $X, \mathbb{X}^2$ and φ (for suitable topologies). A *rough* path is the original path together with its iterated integrals of low degree. The theory can then be extended to cover the case of more irregular paths (with Hölder exponents less than $1/3$) by generalization of the arguments (the more the path is irregular the more iterated integrals are needed to characterize a rough path).

With this work we would like to provide an alternative formulation of integration over rough paths which leads to the same results of that of Lyons' but to some extent is simpler and more straightforward. We will encounter an algebraic structure which is interesting by itself and corresponds to a kind of finite-difference calculus. In the original work of Lyons [7] roughness is measured in p-variation norm, instead here we prefer to work with Hölder-like (semi)norms, in Section 6 we prove that Brownian motion satisfy our requirements of regularity. In a recent work Friz [3] has established Hölder regularity of Brownian rough paths (according to Lyons' theory) and used this result to give an alternative proof of the support theorem for diffusions. This work has been extended later by Friz and Victoir [4] by interpreting Brownian rough paths as suitable processes on the free nilpotent group of step 2: regularity of Brownian rough paths can then be seen as a consequence of standard Hölder regularity results for stochastic processes on groups.

We will start by reformulating in Section 2 the classical integral as the unique solution of an algebraic problem (adjoined with some analytic condition to enforce uniqueness) and then generalizing this problem and building an abstract tool for its solution. As a first application we rediscover in Section 3 the integration theory of Young [11] which was the prelude to the more deep theory of Lyons. Essentially, Young's theory define the integral

$$\int_s^t f_u \, dg_u$$

when f is γ-Hölder continuous, g is ρ-Hölder continuous and $\gamma + \rho > 1$ (actually, the original argument was given in terms of p-variation norms). This will be mainly an exercise to familiarize with the approach before discussing the integration theory for more irregular paths in Section 4. We will define integration for a large class of paths whose increments are controlled by a fixed reference rough path. This is the main difference with the approach of Lyons. Next, to illustrate an application of the theory, we discuss the existence and uniqueness of solution of ordinary differential equation driven by irregular paths (Section 5). In particular, sufficient conditions will be given for the existence in the case of γ-Hölder paths with $\gamma > 1/3$ which are weaker than those required to get uniqueness. This point answer a question raised in Lyons [7]. In Section 6 we prove that Brownian motion and the second iterated integral provided by Itô or Stratonovich integration are Hölder regular rough paths for which the theory outlined above can be applied. Finally we show how to prove the main results of Lyons' theory (extension of multiplicative paths and the existence of a map from almost-multiplicative to multiplicative paths) within this approach. This last section is intended only for readers already acquainted with Lyons' theory (extensive accounts are present in literature, see e.g. [7,8]).

In Appendix A we collect some lengthy proofs.

2. Algebraic prelude

Consider the following observation. Let f be a bounded continuous function on $\mathbb{R}$ and x a function on $\mathbb{R}$ with continuous first derivative. Then there exists a unique couple (a, r) with $a \in C^1(\mathbb{R})$, $a_0 = 0$ and $r \in C(\mathbb{R}^2)$ such that

$$f_s(x_t - x_s) = a_t - a_s - r_{st} \tag{3}$$

and

$$\lim_{t \to s} \frac{|r_{st}|}{|t-s|} = 0. \tag{4}$$

This unique couple (a, r) is given by

$$a_t = \int_0^t f_u \, dx_u, \qquad r_{st} = \int_s^t (f_u - f_s) \, dx_u.$$

The indefinite integral $\int f \, dx$ is the unique solution a of the algebraic problem (3) with the additional requirement (4) on the remainder r. Since Eq. (3) make sense for arbitrary functions f, x it is natural to investigate the possible existence and uniqueness of regular solutions. This will lead to the generalization of the integral $\int f \, dx$ for functions x not necessarily of finite variation.

2.1. Framework

Let $\mathscr{C}$ be the algebra of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $\Omega\mathscr{C}_n$ $(n > 0)$ the subset of bounded continuous functions from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ which are zero on the main diagonal where all the arguments are equal, i.e. $R \in \Omega\mathscr{C}_n$ implies $R_{t_1 \dots t_n} = 0$ if $t_1 = t_2 = \cdots = t_n$. In this paper we will call elements from $\Omega\mathscr{C}_n$ (for any

$n > 0$) *processes* to distinguish them from *paths* which are elements of $\mathscr{C}$. The vector spaces $\Omega\mathscr{C}_n$ are $\mathscr{C}$-bimodules with left multiplication $(AB)_{t_1 \cdots t_{n+1}} := A_{t_1} B_{t_1 \cdots t_{n+1}}$ and right multiplication $(BA)_{t_1 \cdots t_{n+1}} := A_{t_{n+1}} B_{t_1 \cdots t_{n+1}}$ for all $(t_1, \ldots, t_{n+1}) \in \mathbb{R}^{n+1}$, $A \in \mathscr{C}$ and $B \in \Omega\mathscr{C}_n$. Moreover if $A \in \Omega\mathscr{C}_n$ and $B \in \Omega\mathscr{C}_m$ their external product $AB \in \Omega\mathscr{C}_{m+n-1}$ is defined as $(AB)_{t_1 \cdots t_{m+n-1}} = A_{t_1 \cdots t_n} B_{t_n \cdots t_{n+m-1}}$. In the following we will write $\Omega\mathscr{C}$ for $\Omega\mathscr{C}_1$.

The application $\delta : \mathscr{C} \to \Omega\mathscr{C}$ defined as

$$(\delta A)_{st} := A_t - A_s \tag{5}$$

is a derivation on $\mathscr{C}$ since $\delta(AB) = A\delta B + \delta AB = B\delta A + \delta BA$.

Let $\Omega\mathscr{C}^\gamma$ be the subspace of elements $X \in \Omega\mathscr{C}$ such that

$$||X||_\gamma := \sup_{t,s \in \mathbb{R}^2} \frac{|X_{st}|}{|t-s|^\gamma} < \infty$$

and let $\mathscr{C}^\gamma$ be the subspace of the elements $A \in \mathscr{C}$ such that $||\delta A||_\gamma < \infty$.

Define $\Omega\mathscr{C}_2^{\rho,\gamma}$ as the subspace of elements X of $\Omega\mathscr{C}_2$ such that

$$||X||_{\rho,\gamma} := \sup_{(s,u,t) \in \mathbb{R}^3} \frac{|X_{sut}|}{|u-s|^\rho |t-u|^\gamma} < \infty$$

Let $\Omega\mathscr{C}_2^z := \oplus_{\rho>0} \Omega\mathscr{C}_2^{\rho, z-\rho}$: an element $A \in \Omega\mathscr{C}_2^z$ is a finite linear combination of elements $A_i \in \Omega\mathscr{C}_2^{\rho_i, z-\rho_i}$ for some $\rho_i \in (0, z)$.

Define the linear operator $N : \Omega\mathscr{C} \to \Omega\mathscr{C}_2$ as

$$(NR)_{sut} := R_{st} - R_{ut} - R_{su}.$$

and let $\mathscr{Z}_2 := N(\Omega\mathscr{C})$ and $\mathscr{Z}_2^z := \Omega\mathscr{C}_2^z \cap \mathscr{Z}_2$.

We have that $\operatorname{Ker} N = \operatorname{Im} \delta$. Indeed $N\delta A = 0$ for all $A \in \mathscr{C}$ and it is easy to see that for each $R \in \Omega\mathscr{C}$ such that $NR = 0$ we can let $A_t = R_{t0}$ to obtain that $\delta A = R$.

If $F \in \mathscr{C}$ and $R \in \Omega\mathscr{C}$ then a straightforward computation shows that

$$N(FR)_{sut} = F_s N(R)_{sut} - \delta F_{su} R_{ut} = (FN(R) - \delta F R)_{sut};$$

$$N(RF)_{sut} = F_t N(R)_{sut} + R_{su} \delta F_{ut} = (N(R)F + R\delta F)_{sut}. \tag{6}$$

These equations suggest that the operators δ and N enjoy remarkable algebraic properties. Indeed they are just the first two members of a family of linear operators which acts as derivations on the modules $\Omega\mathscr{C}_k$, $k = 0, 1, \ldots$ and which can be characterized as the coboundaries of a cochain complex which we proceed to define.

2.2. A cochain complex

Consider the following chain complex: a *simple* chain of degree n is a string $[t_1 t_2 \cdots t_n]$ of real numbers and a chain of degree n is a formal linear combination of simple chains of the same degree with coefficients in $\mathbb{Z}$. The boundary operator ∂ is defined as

$$\partial[t_1\cdots t_n] = \sum_{i=1}^{n} (-1)^i [t_1\cdots \hat{t}_i \cdots t_n], \tag{7}$$

where $\hat{t}_i$ means that this element is removed from the string. For example

$$\partial[st] = -[t] + [s], \qquad \partial[sut] = -[su] + [ts] - [ut].$$

It is easy to verify that $\partial\partial = 0$. To this chain complex is adjoined in a standard way a complex of cochains (which are linear functionals on chains). A cochain A of degree n is such that, on simple chains of degree n, act as

$$\langle [t_1\cdots t_n], A \rangle = A_{t_1\cdots t_n}.$$

The coboundary ∂^* acts on cochains of degree n as

$$\begin{aligned}(\partial^* A)_{t_1\cdots t_{n+1}} &= \langle [t_1\cdots t_{n+1}], \partial^* A \rangle = \langle \partial[t_1\cdots t_{n+1}], A \rangle \\ &= \sum_{i=1}^{n+1} (-1)^i \langle \partial[t_1 \cdots \hat{t}_i \cdots t_{n+1}], A \rangle = \sum_{i=1}^{n+1} (-1)^i A_{t_1\cdots \hat{t}_i \cdots t_{n+1}} \end{aligned} \tag{8}$$

e.g. for cochains A, B of degree 1 and 2 respectively, we have

$$(\partial^* A)_{st} = A_s - A_t, \qquad (\partial^* B)_{sut} = B_{st} - B_{ut} - B_{su}$$

so that we have natural identifications of ∂^* with $-\delta$ when acting on 1-cochains and with N when acting on 2-cochains. We recognize also that elements of $\Omega\mathscr{C}_{n-1}$ ($\Omega\mathscr{C}_0 = \mathscr{C}$) are n-cochains and that we have the following complex of modules:

$$0 \to \mathbb{R} \to \mathscr{C} \xrightarrow{\partial^*} \Omega\mathscr{C} \xrightarrow{\partial^*} \Omega\mathscr{C}_2 \xrightarrow{\partial^*} \Omega\mathscr{C}_3 \to \cdots$$

As usual $\partial^*\partial^* = 0$ which means that the image of $\partial^*|_{\Omega\mathscr{C}_n}$ is in the kernel of $\partial^*|_{\Omega\mathscr{C}_{n+1}}$. Since $\operatorname{Ker} N = \operatorname{Im} \delta$ the above sequence is exact at $\Omega\mathscr{C}$. Actually, the sequence is exact at every $\Omega\mathscr{C}_n$: let A be an $n+1$-cochain such that $\partial^* A = 0$. Let us show that there exists an n-cochain B such that $A = \partial^* B$. Take

$$B_{t_1\cdots t_n} = (-1)^{n+1} A_{t_1\cdots t_n s},$$

where s is an arbitrary reference point. Then compute

$$
\begin{aligned}
(\partial^* B)_{t_1\cdots t_{n+1}} &= -B_{t_2\cdots t_{n+1}} + B_{t_1\hat{t}_2\cdots t_{n+1}} + \cdots + (-1)^{n+1} B_{t_1\cdots t_n} \\
&= (-1)^{n+1}[-A_{t_2\cdots t_{n+1}s} + A_{t_1\hat{t}_2\cdots t_{n+1}s} + \cdots + (-1)^{n+1} A_{t_1\cdots t_n s}] \\
&= (-1)^{n+1}[(\partial^* A)_{t_1 t_2\cdots t_{n+1}s} - (-1)^{n+2} A_{t_1\cdots t_{n+1}}] = A_{t_1\cdots t_{n+1}}.
\end{aligned}
$$

As an immediate corollary we can introduce the operator $N_2 : \Omega\mathscr{C}_2 \to \Omega\mathscr{C}_3$ such that $N_2 := \partial^*|_{\Omega\mathscr{C}_2}$ to characterize the image of N as the kernel of N_2. Note that, for example, N_2 satisfy a Leibnitz rule: if $A, B \in \Omega\mathscr{C}_2$,

$$
\begin{aligned}
N_2(AB)_{suvt} &= \partial^*(AB)_{suvt} = -(AB)_{uvt} + (AB)_{svt} - (AB)_{sut} + (AB)_{suv} \\
&= -A_{uv}B_{vt} + A_{sv}B_{vt} - A_{su}B_{ut} + A_{su}B_{uv} \\
&= (NA)_{suv}B_{vt} - A_{su}(NB)_{uvt} \\
&= (NAB - ANB)_{suvt}.
\end{aligned}
\tag{9}
$$

To understand the relevance of this discussion to our problem let us reformulate the observation at the beginning of this section as follows:

Problem 1. *Given two paths $F, X \in \mathscr{C}$ is it possible to find a (possibly) unique decomposition*

$$
F\delta X = \delta A - R, \tag{10}
$$

where $A \in \mathscr{C}$ and $R \in \Omega\mathscr{C}$?

To have uniqueness of this decomposition we should require that δA should be (in some sense) orthogonal to R. So we are looking to a canonical decomposition of $\Omega\mathscr{C} \simeq \delta\mathscr{C} \oplus \mathscr{B}$ where $\mathscr{B}$ is a linear subspace of $\Omega\mathscr{C}$ which should contain the remained R. This decomposition is equivalent to the possibility of splitting the short exact sequence

$$
0 \to \mathscr{C}/\mathbb{R} \xrightarrow{\delta} \Omega\mathscr{C} \xrightarrow{N} \mathscr{Z}_2 \to 0.
$$

We cannot hope to achieve the splitting in full generality and we must resort to consider an appropriate linear subspace $\mathscr{E}$ of $\Omega\mathscr{C}$ which contains $\delta\mathscr{C}$ and for which we can show that there exists a linear function $\Lambda_{\mathscr{E}} : N\mathscr{E} \to \mathscr{E}$ such that

$$
N\Lambda_{\mathscr{E}} = 1_{N\mathscr{E}}.
$$

Then $\Lambda_{\mathscr{E}}$ splits the short exact sequence

$$0 \to \mathscr{C}/\mathbb{R} \xrightarrow{\delta} \mathscr{E} \xrightarrow{N} N\mathscr{E} \to 0$$

which implies $\mathscr{E} \simeq \delta\mathscr{C} \oplus N\mathscr{E}$.

In this case, if $F\delta X \in \mathscr{E}$ we can recover δA as

$$\delta A = F\delta X - \Lambda_{\mathscr{E}} N(F\delta X). \tag{11}$$

To identify a subspace $\mathscr{E}$ for which the splitting is possible we note that

$$\operatorname{Im} \delta \cap \Omega\mathscr{C}^z = \{0\}$$

for all $z > 1$, indeed, if $X = \delta A$ for some $A \in \mathscr{C}$ and $X \in \Omega\mathscr{C}^z$ then $A \in \mathscr{C}^z$ which implies $A = \text{const}$ if $z > 1$.

Then we can reformulate the algebraic characterization of integration at the beginning of this section as the following problem:

Problem 2. *Given two paths $F, X \in \mathscr{C}$ is it possible to find $A \in \mathscr{C}$ and $R \in \Omega\mathscr{C}^z$ for some $z > 1$ such that the decomposition*

$$F\delta X = \delta A - R \tag{12}$$

holds?

Note that if such a decomposition exists then it is automatically unique since if $F\delta X = \delta A' - R'$ is another we have that $R - R' = \delta(A - A')$ but since $R - R' \in \Omega\mathscr{C}^z \cap \ker N$ we get $R = R'$ and thus $A = A'$ modulo a constant.

That Problem 2 cannot always be solved is clear from the following consideration: let $F = X$ and apply N to both sides of Eq. (12) to obtain

$$\delta X_{su} \delta X_{ut} = -NR_{sut}$$

for all $(s, u, t) \in \mathbb{R}^3$. Then

$$\delta X_{st} \delta X_{st} = -NR_{tst} = R_{st} + R_{st}$$

for all $(t, s) \in \mathbb{R}^2$. Now, if $R \in \Omega\mathscr{C}^z$ with $z > 1$ then

$$|\delta X_{st}|\,|\delta X_{st}| \leqslant 2||R||_z |t - s|^z \tag{13}$$

which implies that $X \in \mathscr{C}^{z/2}$. So unless this last condition is fulfilled we cannot solve Problem (12) with the required regularity on R.

A sufficient condition for a solution to Problem 2 to exists is given by the following result which states sufficient conditions on $A \in \Omega\mathscr{C}_2$ for which the algebraic problem

$$NR = A$$

has a unique solution $R \in \Omega\mathscr{C}/\delta\mathscr{C}$.

2.3. The main result

For every $A \in \mathscr{L}_2^z$ with $z > 1$ there exists a unique $R \in \Omega\mathscr{C}^z$ such that $NR = A$:

Proposition 1. *If $z > 1$ there exists a unique linear map $\Lambda : \mathscr{L}_2^z \to \Omega\mathscr{C}^z$ such that $N\Lambda = 1_{\mathscr{L}_2}$ and such that for all $A \in \mathscr{L}_2^z$ we have*

$$||\Lambda A||_z \leqslant \frac{1}{2^z - 2} \sum_{i=1}^{n} ||A_i||_{\rho_i, z-\rho_i}$$

if $A = \sum_{i=1}^n A_i$ with $n \geqslant 1$, $0 < \rho_i < z$ and $A_i \in \Omega\mathscr{C}_2^{\rho_i, z-\rho_i}$ for $i = 1, \dots, n$.

2.4. Localization

If $I \subset J$ denote with $A|_I$ the restriction on I of the function A defined on J. The operator Λ is local in the following sense:

Proposition 2. *If $I \subset \mathbb{R}$ is an interval and $A, B \in \mathscr{L}_2^z$ with $z > 1$ then*

$$A|_{I^3} = B|_{I^3} \Rightarrow \Lambda A|_{I^2} = \Lambda B|_{I^2}.$$

Proof. This follows essentially from the same argument which gives the uniqueness of Λ. Indeed if $Q = \Lambda A - \Lambda B$ we have that $NQ = A - B$ which vanish when restricted to I^2. So for $(t,s) \in I^2$, $t \leqslant u \leqslant s$ we have

$$Q_{ut} = Q_{st} - Q_{su}$$

but since $Q \in \Omega\mathscr{C}^z$ with $z > 1$ we get $Q|_{I^2} = 0$. □

Given an interval $I = [a,b] \subset \mathbb{R}$ and defining in an obvious way the corresponding spaces $\mathscr{C}^\gamma(I)$, $\Omega\mathscr{C}_n^\gamma(I)$, etc. we can introduce the operator $\Lambda_I : \mathscr{L}_2^z(I) \to \Omega\mathscr{C}^z(I)$ as $\Lambda_I A := \Lambda\tilde{A}|_{I^2}$ where $\tilde{A} \in \mathscr{L}_2^z$ is any arbitrary extension of the element $A \in \mathscr{L}_2^z(I)$. By the locality of Λ any choice of the extension $\tilde{A}$ will give the same result, moreover the specific choice $\tilde{A}_{sut} := A_{\tau(t),\tau(u),\tau(s)}$ where $\tau(t) := (t \wedge b) \vee a$ has the virtue to satisfy the following bound:

$$||\tilde{A}_i||_{\rho_i, z-\rho_i} \leqslant ||A_i||_{\rho_i, z-\rho_i, I},$$

where $||\cdot||_{\rho_i,z-\rho_i,I}$ is the norm on $\Omega\mathscr{C}_2^z(I)$ and $A=\sum_i A_i$ is a decomposition of A in $\Omega\mathscr{C}_2^z(I)$ so that we have

$$||\Lambda_I A||_{z,I} \leqslant \frac{1}{2^z-2}\sum_i ||A_i||_{\rho_i,z-\rho_i,I}.$$

We will write Λ instead of Λ_I whenever the interval I can be deduced from the context.

2.5. Notations

In the following we will have to deal with tensor products of vector spaces and we will use the "physicist" notation for tensors. We will use $V, V_1, V_2, \ldots$ to denote vector spaces which will be always finite dimensional.[1] Then, if V is a vector space, $A\in V$ will be denoted by A^μ, where μ is the corresponding vector index (in an arbitrary but fixed basis), ranging from 1 to the dimension of V, elements in V^* (the linear dual of V) are denoted by A_μ with lower indexes, elements in $V\otimes V$ will be denoted by $A^{\mu\nu}$, elements of $V^{\otimes 2}\otimes V^*$ as $A^{\mu\nu}_\kappa$, etc. Summation over repeated indexes is understood whenever not explicitly stated otherwise: $A_\mu B^\mu$ is the scalar obtained by contracting $A\in V^*$ with $B\in V$.

Symbols like $\bar\mu, \bar\nu, \ldots$ (a bar over a greek letter) will be vector multi-indexes, i.e. if $\bar\mu=(\mu_1,\ldots,\mu_n)$ then $A^{\bar\mu}$ is the element $A^{\mu_1,\ldots,\mu_n}$ of $V^{\otimes n}$. Given two multi-indexes $\bar\mu$ and $\bar\nu$ we can build another multi-index $\bar\mu\bar\nu$ which is composed of all the indices of $\bar\mu$ and $\bar\nu$ in sequence. With $|\bar\mu|$ we denote the degree of the multi-index $\bar\mu$, i.e. if $\bar\mu=(\mu_1,\ldots,\mu_n)$ then $|\bar\mu|=n$. Then for example $|\bar\mu\bar\nu|=|\bar\mu|+|\bar\nu|$. By convention we introduce also the empty multi-index denoted by $\emptyset$ such that $\bar\mu\emptyset=\emptyset\bar\mu=\bar\mu$ and $|\emptyset|=0$.

Symbols like $\mathscr{C}(V)$, $\Omega\mathscr{C}(V)$, $\mathscr{C}(I,V)$, etc. (where I is an interval) will denote paths and processes with values in the vector space V.

Moreover the symbol K will denote arbitrary strictly positive constants, maybe different from equation to equation and not depending on anything.

3. Young's theory of integration

Proposition 1 allows to solve Problem 2 when $F\in\mathscr{C}^\rho$, $X\in\mathscr{C}^\gamma$ with $\gamma+\rho>1$: in this case

$$N(F\delta X)_{sut} = -\delta F_{su}\delta X_{ut}$$

[1] In many of the arguments this will be not necessary, but to handle infinite-dimensional Banach spaces some care should be excercized in the definition of norms on tensor products. We prefer to skip this issue for the sake of clarity.

so that $N(F\delta X)\in\mathscr{Z}_2^{\gamma+\rho}$. Then since $N(F\delta X-\Lambda N(F\delta X))=0$ there exists a unique $A\in\mathscr{C}$ (modulo a constant) such that

$$\delta A = F\delta X - \Lambda N(F\delta X).$$

Proposition 3 (Young). *Fix an interval $I\subseteq\mathbb{R}$. If $F\in\mathscr{C}^\rho(I)$ and $X\in\mathscr{C}^\gamma(I)$ with $\gamma+\rho>1$ define*

$$\int_s^t F_u\, dX_u := [F\delta X - \Lambda N(F\delta X)]_{st}, \qquad s,t\in I. \tag{14}$$

Then we have

$$\left|\int_s^t (F_u - F_s)\, dX_u\right| \leqslant \frac{1}{2^{\gamma+\rho}-2}|t-s|^{\gamma+\rho}||F||_{\rho,I}||X||_{\gamma,I}, \qquad s,t\in I. \tag{15}$$

Proof. Is immediate observing that by definition

$$\int_s^t (F_u - F_s)\, dX_s = -[\Lambda N(F\delta X)]_{st} = [\Lambda(\delta F\delta X)]_{st}$$

and using the previous results. □

Another justification of this definition of the integral comes from the following convergence of discrete sums which also establish the equivalence of this theory of integration with that of Young.

Corollary 1. *In the hypothesis of the previous proposition we have*

$$\int_s^t F_u\, dX_u = \lim_{|\Pi|\to 0} \sum_{\{t_i\}\in\Pi} F_{t_i}(X_{t_{i+1}} - X_{t_i}), \qquad s,t\in I,$$

where the limit is taken over partitions $\Pi=\{t_0,t_1,\dots,t_n\}$ of the interval $[s,t]\subseteq I$ such that $t_0=s,\ t_n=t,\ t_{i+1}>t_i,\ |\Pi|=\sup_i|t_{i+1}-t_i|$.

Proof. For any partition Π write

$$S_\Pi = \sum_{i=0}^{n-1} F_{t_i}(X_{t_{i+1}} - X_{t_i}) = \sum_{i=0}^{n-1} (F\delta X)_{t_it_{i+1}} = \sum_{i=0}^{n-1} (\delta A + R)_{t_it_{i+1}}$$

with $R\in\Omega\mathscr{C}^{\gamma+\rho}(I)$ given by $R=\Lambda(\delta F\delta X)$ and such that (cf. Proposition 3):

$$||R||_{\gamma+\rho,I} \leqslant \frac{1}{2^{\gamma+\rho}-2}||F||_{\rho,I}||X||_{\gamma,I}.$$

Then

$$S_{\Pi} = A_t - A_s - \sum_{i=0}^{n-1} R_{t_i t_{i+1}} = \int_s^t F_u \, dX_u - \sum_{i=0}^{n-1} R_{t_i t_{i+1}}. \tag{16}$$

But now, since $\gamma + \rho > 1$,

$$\sum_{i=0}^{n-1} |R_{t_i t_{i+1}}| \leqslant ||R||_{\gamma+\rho,I} \sum_{i=0}^{n-1} |t_{i+1} - t_i|^{\gamma+\rho} \leqslant ||R||_{\gamma+\rho,I} |\Pi|^{\gamma+\rho-1} |t-s| \to 0$$

as $|\Pi| \to 0$. □

4. More irregular paths

In order to solve Problem 1 for a wider class of F and X we are led to dispense with the condition $R \in \Omega\mathscr{C}^z$ with $z > 1$ and thus loose the uniqueness of the decomposition: if the couple (A, R) solve the problem, then also $(A + B, R + \delta B)$ solve the problem with a nontrivial $B \in \mathscr{C}^z$. So our aim is actually to find a distinguished couple (A, R) which will be characterized by some additional conditions.

Up to now we have considered only paths with values in $\mathbb{R}$, since the general case of vector-valued paths can be easily derived; however, in the case of more irregular paths the vector features of the paths will play a prominent role so from now on we will consider paths with values in (finite-dimensional) Banach spaces $V, V_1, \ldots$.

Let $X \in \mathscr{C}^{\gamma}(V)$ a path with values in the Banach space V for some $\gamma > 0$ and *assume* that we are given a tensor process $\mathbb{X}^2$ in $\Omega\mathscr{C}^{2\gamma}(V^{\otimes 2})$ such that

$$N(\mathbb{X}^{2,\mu\nu}) = \delta X^{\mu} \delta X^{\nu}. \tag{17}$$

If $\gamma \leqslant 1/2$ we cannot obtain this process using Proposition 1 but (as we will see in Section 6) there are other natural ways to build such a process for special paths X. We can think at the arbitrary choice of $\mathbb{X}^2$ among all the possible solutions (with given regularity 2γ) of Eq. (17) as a way to resolve the ambiguity of the decomposition in Problem 1, since in this case

$$X^{\mu} \delta X^{\nu} = \delta I^{\mu\nu} - \mathbb{X}^{2,\mu\nu}$$

and so we are able to integrate any component of X with respect to each other and we can write

$$\int_s^t X_u^{\mu} \, dX_u^{\nu} = \delta I_{st}^{\mu\nu}$$

meaning that the integral on the l.h.s. is defined by the r.h.s., definition which depends on our choice of $\mathbb{X}^2$. Of course in this case Corollary 1 does not hold anymore and discrete sums of $X\delta X$ are not guaranteed to converge to $\int X\,dX$.

Note that in the scalar case the equation

$$X\delta X = \delta I - R$$

with $X \in \mathscr{C}^\gamma$ has always a solution given by $I_t = X_t^2/2 + \text{const}$ for which

$$\delta I_{st} = \frac{1}{2}X_t^2 - \frac{1}{2}X_s^2 = \frac{1}{2}X_t(X_t - X_s) + \frac{1}{2}X_s(X_t - X_s) = X_s\delta X_{st} + \frac{1}{2}(\delta X_{st})^2$$

giving the decomposition $\delta I = X\delta X + R$ with $R \in \Omega\mathscr{C}^{2\gamma}$. The same argument works for the symmetric part of the two-tensor $\mathbb{X}^2$: If $X \in \mathscr{C}^\gamma(V)$ there exists a two-tensor $S \in \Omega\mathscr{C}^{2\gamma}(V \otimes V)$ given by

$$S_{st}^{\mu\nu} = \frac{1}{2}\delta X_{st}^\mu \delta X_{st}^\nu$$

for which

$$NS^{\mu\nu} = \frac{1}{2}(\delta X^\mu \delta X^\nu + \delta X^\nu \delta X^\mu).$$

of course S is not unique as soon as $\gamma \leqslant 1/2$.

Since one of the feature of the integral we wish to retain is linearity we must agree that if A is a linear application from V to V and $Y_t^\mu = A_\nu^\mu X_t^\nu$ then the integral $\delta I = \int Y\,dX$ must be such that

$$Y^\mu \delta X^\nu = A_\kappa^\mu X^\kappa \delta X^\nu = \delta I^{\mu\nu} - A_\kappa^\mu \mathbb{X}^{2,\kappa\nu}$$

so

$$\delta I^{\mu\nu} = Y^\mu \delta X^\nu + A_\kappa^\mu \mathbb{X}^{2,\kappa\nu}$$

and we have fixed at once the values of all the integrals of linear functions of the path X w.r.t. X. Then consider a path Y which is only *locally* a linear function of X, i.e. such that

$$\delta Y^\mu = G_\nu^\mu \delta X^\nu + Q^\mu, \tag{18}$$

where Q is a "remainder" in $\Omega\mathscr{C}(V)$ and G is a path in $\mathscr{C}(V \otimes V^*)$. In order to be able to show that Y is integrable w.r.t. X we must find a solution R of the equation

$$NR^{\mu\nu} = \delta Y^\mu \delta X^\nu.$$

but then, using the local expansion given in Eq. (18),

$$\begin{aligned} NR^{\mu\nu} &= G^{\mu}_{\kappa}\delta X^{\kappa}\delta X^{\nu} + Q^{\mu}\delta X^{\nu} \\ &= G^{\mu}_{\kappa}N(\mathbb{X}^{2,\kappa\nu}) + Q^{\mu}\delta X^{\nu} \\ &= N(G^{\mu}_{\kappa}\mathbb{X}^{2,\kappa\nu}) + \delta G^{\mu}_{\kappa}\mathbb{X}^{2,\kappa\nu} + Q^{\mu}\delta X^{\nu}, \end{aligned}$$

where we have used Eq. (6) (the Leibnitz rule for N). To find a solution R is then equivalent to let

$$\widetilde{R}^{\mu\nu} = R^{\mu\nu} - G^{\mu}_{\kappa}\mathbb{X}^{2,\kappa\nu}$$

and solve

$$N\widetilde{R} = \delta G^{\mu}_{\kappa}\mathbb{X}^{2,\kappa\nu} + Q^{\mu}\delta X^{\nu}. \tag{19}$$

Sufficient conditions to apply Proposition 1 to solve Eq. (19) are that $G \in \mathscr{C}^{\eta-\gamma}(V \otimes V^*)$, $Q \in \Omega\mathscr{C}^{\eta}(V)$ with $\eta + \gamma = z > 1$. In this case there exists a unique $\widetilde{R} \in \Omega\mathscr{C}^{z}$ solving (19) and we have obtained the distinguished decomposition

$$Y^{\mu}\delta X^{\nu} = \delta I^{\mu\nu} - G^{\mu}_{\kappa}\mathbb{X}^{2,\kappa\nu} - \widetilde{R}^{\mu\nu}. \tag{20}$$

Note that the path Y lives a priori only in $\mathscr{C}^{\gamma}$ and this implies that uniqueness of the solution of Problem 2 can be achieved only if $\gamma > 1/2$. On the other hand the request that Y can be decomposed as in Eq. (18) with prescribed regularity on G and Q has allowed us to show that the ambiguity in the solution of Problem 1 can be reduced to the choice of a process $\mathbb{X}^2$ satisfying Eq. (17). Of course if $\gamma > 1/2$ there is only one solution to (17) with the prescribed regularity and decomposition (20) (into a gradient and a remainder) coincides with the unique solution of Problem 2.

Another way to look at this result is to consider the "non-exact" differential

$$F\delta X + G\mathbb{X}^2,$$

where F, G are arbitrary paths and ask in which case it admits a unique decomposition

$$F\delta X + G\mathbb{X}^2 = \delta A + R$$

as a sum of an exact differential plus a remainder term. Of course to have uniqueness is enough that $R \in \Omega\mathscr{C}^{z}$, $z > 1$. Compute

$$N(F\delta X + G\mathbb{X}^2) = -\delta F\delta X - \delta G\mathbb{X}^2 + G\delta X\delta X = (-\delta F + G\delta X)\delta X - \delta G\mathbb{X}^2,$$

so in order to have $R \in \Omega\mathscr{C}^{z}$, $z > 1$ condition (18) and suitable regularity of G and Q, are sufficient to apply Proposition 1.

4.1. Weakly controlled paths

The analysis laid out above leads to the following definition.

Definition 1. Fix an interval $I \subseteq \mathbb{R}$ and let $X \in \mathscr{C}^\gamma(I, V)$. A path $Z \in \mathscr{C}^\gamma(I, V)$ is said to be *weakly controlled by X in I with a remainder of order* η if there exists a path $Z' \in \mathscr{C}^{\eta-\gamma}(I, V \otimes V^*)$ and a process $R_Z \in \Omega\mathscr{C}^\eta(I, V)$ with $\eta > \gamma$ such that

$$\delta Z^\mu = Z'^{\mu\nu} \delta X^\nu + R_Z^\mu.$$

If this is the case we will write $(Z, Z') \in \mathscr{D}_X^{\gamma,\eta}(I, V)$ and we will consider on the linear space $\mathscr{D}_X^{\gamma,\eta}(I, V)$ the semi-norm

$$||Z||_{D(X,\gamma,\eta),I} := ||Z'||_{\infty,I} + ||Z'||_{\eta-\gamma,I} + ||R_Z||_{\eta,I} + ||Z||_{\gamma,I}.$$

(The last contribution is necessary to enforce $Z \in \mathscr{C}^\gamma(I, V)$ when I is unbounded).

The decomposition $\delta Z^\mu = Z'^{\mu\nu} \delta X^\nu + R^\mu$ is a priori not unique, so a path in $D_{\gamma,\eta}(I, X)$ must be understood as a pair (Z, Z') since then R_Z is uniquely determined. However we will often omit to specify Z' when it will be clear from the context.

The term *weakly controlled* is inspired by the fact that paths which are solution of differential equations controlled by X (see Section 5) belongs to the class of weakly controlled paths (w.r.t. X). In general, however, a weakly controlled path Z is uniquely determined knowing X and the "derivative" Z' only when $\eta > 1$.

Weakly controlled paths enjoy a transitivity property:

Lemma 1. *If* $Z \in \mathscr{D}_Y^{\gamma,\eta}(I, V)$ *and* $Y \in \mathscr{D}_X^{\gamma,\sigma}(I, V)$ *then* $Z \in \mathscr{D}_X^{\gamma,\min(\sigma,\eta)}(I, V)$ *and*

$$||(Z, Z')||_{D(X,\gamma,\delta),I} \leqslant K||Z||_{D(Y,\gamma,\eta),I}(1 + ||Y||_{D(X,\gamma,\sigma),I})(1 + ||X||_{\gamma,I})$$

where K is some fixed constant.

Proof. The proof is in Appendix A, Section A.2.1. □

Another important property of the class of weakly controlled paths is that it is stable under smooth maps. Let $C^{n,\delta}(V, V_1)$ the space of n-times differentiable maps from V to the vector space V_1 with δ-Hölder nth derivative and consider the norm

$$||\varphi||_{0,\delta} = ||\varphi||_\infty + ||\varphi||_\delta, \qquad ||\varphi||_{n,\delta} = ||\varphi||_\infty + \sum_{k=1}^n ||\partial^k \varphi||_\infty + ||\partial^n \varphi||_\delta,$$

where $\varphi \in C^{n,\delta}(V, V_1)$, $\partial^k \varphi$ is the kth derivative of φ seen as a function with values in $V_1 \otimes V^{*\otimes k}$ and

$$||\varphi||_\infty = \sup_{x \in V} |\varphi(x)|,$$

$$||\partial^n \varphi||_\delta = \sup_{x,y \in V} \frac{|\partial^n \varphi(x) - \partial^n \varphi(y)|}{|x-y|^\delta}.$$

Proposition 4. *Let $Y \in \mathscr{D}_X^{\gamma,\eta}(I, V)$ and $\varphi \in C^{1,\delta}(V, V_1)$, then the path Z such that $Z_t^\mu = \varphi(Y_t)^\mu$ is in $\mathscr{D}_X^{\gamma,\sigma}(I, V_1)$ with $\sigma = \min(\gamma(\delta+1), \eta)$. Its decomposition is*

$$\delta Z^\mu = \partial_\nu \varphi(Y)^\mu Y'^\nu_\kappa \delta X^\kappa + R_Z^\mu$$

with $R_Z \in \Omega\mathscr{C}^\sigma(I, V_1)$ and

$$||Z||_{D(X,\gamma,\sigma),I} \leqslant K||\varphi||_{1,\delta}(||Y||_{D(X,\gamma,\eta),I} + ||Y||^{1+\delta}_{D(X,\gamma,\eta),I} + ||Y||^{\sigma/\gamma}_{D(X,\gamma,\eta),I}). \tag{21}$$

If $\varphi \in C^{2,\delta}(V, V_1)$ we have also

$$||\varphi(Y) - \varphi(\widetilde{Y})||_{D(X,\gamma,(1+\delta)\gamma),I} \leqslant C||Y - \widetilde{Y}||_{D(X,\gamma,(1+\delta)\gamma),I} \tag{22}$$

for $Y, \widetilde{Y} \in \mathscr{D}_X^{\gamma,(1+\delta)\gamma}(I, V)$ with

$$C = K||\varphi||_{2,\delta}(1 + ||X||_{\gamma,I})(1 + ||Y||_{D(X,\gamma,(1+\delta)\gamma),I} + ||\widetilde{Y}||_{D(X,\gamma,(1+\delta)\gamma),I})^{1+\delta}.$$

Moreover if $\widetilde{Y} \in \mathscr{D}_{\widetilde{X}}^{\gamma,(1+\delta)\gamma}(I, V)$, $\widetilde{Z} = \varphi(\widetilde{Y})$ and

$$\delta Y^\mu = Y'^\mu_\nu \delta X^\nu + R_Y^\mu, \qquad \delta \widetilde{Y}^\mu = \widetilde{Y}'^{,\mu}_\nu \delta \widetilde{X}^\nu + R^\mu_{\widetilde{Y}},$$

$$\delta Z^\mu = Z'^\mu_\nu \delta X^\nu + R_Z^\mu, \qquad \delta \widetilde{Z}^\mu = \widetilde{Z}'^\mu_\nu \delta \widetilde{X}^\nu + R^\mu_{\widetilde{Z}},$$

with $Z'^\mu_{\nu,t} = \partial_\kappa \varphi(Y_t)^\mu Y'^\kappa_{\nu,t}$, $\widetilde{Z}'^\mu_{\nu,t} = \partial_\kappa \varphi(\widetilde{Y}_t)^\mu \widetilde{Y}'^\kappa_{\nu,t}$ then

$$\begin{aligned} &||Z' - \widetilde{Z}'||_\infty + ||Z' - \widetilde{Z}'||_{\delta\gamma,I} + ||R_Z - R_{\widetilde{Z}}||_{(1+\delta)\gamma,I} + ||Z - \widetilde{Z}||_{\gamma,I} \\ &\quad \leqslant C(||X - \widetilde{X}||_{\gamma,I} + \varepsilon_I) \end{aligned} \tag{23}$$

with

$$\varepsilon_I = ||Y' - \widetilde{Y}'||_{\infty,I} + ||Y' - \widetilde{Y}'||_{\delta\gamma,I} + ||R_Y - R_{\widetilde{Y}}||_{(1+\delta)\gamma,I} + ||Y - \widetilde{Y}||_{\gamma,I}.$$

Proof. The proof is given in Appendix A, Section A.2.2. □

4.2. Integration of weakly controlled paths

Let us give a reference path $X \in \mathscr{C}^{\gamma}(I, V)$ and an associated process $\mathbb{X}^2 \in \Omega\mathscr{C}^{2\gamma}(I, V \otimes V)$ satisfying the algebraic relationship

$$N\mathbb{X}^{2,\mu\nu}_{sut} = \delta X^{\mu}_{su} \delta X^{\nu}_{ut} \qquad s, u, t \in I. \tag{24}$$

Following Lyons we will call the couple $(X, \mathbb{X}^2)$ a *rough path* (of roughness $1/\gamma$).

We are going to show that weakly controlled paths can be integrated one against the other.

Take two paths Z, W in V weakly controlled by X with remainder of order η. By an argument similar to that at the beginning of this section we can obtain a unique decomposition of $Z\delta W$ as

$$Z^{\mu}\delta W^{\nu} = \delta A^{\mu\nu} - F^{\mu\mu'} G^{\nu\nu'} \mathbb{X}^{2,\mu'\nu'} + \Lambda N(Z^{\mu}\delta W^{\nu} + F^{\mu\mu'} G^{\nu\nu'} \mathbb{X}^{2,\mu'\nu'})$$

and we can state the following theorem:

Theorem 1. *For every $(Z, Z') \in D^{\gamma,\eta}_X(I, V)$ and $(W, W') \in D^{\gamma,\eta}_X(I, V)$ with $\eta + \gamma = \delta > 1$ define*

$$\int_s^t Z^{\mu}_u dW^{\nu}_u := Z^{\mu}_s \delta W^{\nu}_{st} + Z'^{\mu}_{\mu',s} W'^{\nu}_{\nu',s} \mathbb{X}^{2,\mu'\nu'}_{st} - [\Lambda N(Z^{\mu}\delta W^{\nu} + Z'^{\mu}_{\mu'} W'^{\nu}_{\nu'} \mathbb{X}^{2,\mu'\nu'})]_{st}, \quad s, t \in I \tag{25}$$

then this integral extends that defined in Proposition 3 *and the following bound holds*:

$$\left| \int_s^t (Z^{\mu}_u - Z^{\mu}_s) dW^{\nu}_u - Z'^{\mu}_{\mu',s} W'^{\nu}_{\nu',s} \mathbb{X}^{2,\mu'\nu'}_{st} \right| \leqslant \frac{1}{2^{\delta} - 2} |t - s|^{\delta} ||(Z, Z')||_{D(X,\gamma,\eta)} ||(W, W')||_{D(X,\gamma,\eta)}, \tag{26}$$

which implies the continuity of the bilinear application

$$((Z, Z'), (W, W')) \mapsto \left(\int_0^{\cdot} Z dW, Z W' \right)$$

from $\mathscr{D}^{\gamma,\eta}_X(V) \times \mathscr{D}^{\gamma,\eta}_X(V)$ to $\mathscr{D}^{\gamma,\min(2\gamma,\eta)}_X(V \otimes V)$.

Proof. Compute

$$\begin{aligned} Q^{\mu\nu}_{sut} &= N(Z^\mu \delta W^\nu + Z'^\mu_{\mu'} W'^\nu_{\nu'} \mathbb{X}^{2,\mu'\nu'})_{sut} \\ &= -\delta Z^\mu_{su} \delta W^\nu_{ut} + (Z'^\mu_{\mu'} W'^\nu_{\nu'})_s N\mathbb{X}^{2,\mu'\nu'}_{sut} - \delta(Z'^\mu_{\mu'} W'^\nu_{\nu'})_{su} \mathbb{X}^{2,\mu'\nu'}_{ut} \\ &= -Z'^\mu_{\mu',s} \delta X^{\mu'}_{su} W'^\nu_{\nu',u} \delta X^{\nu'}_{ut} - R^\mu_{Z,su} \delta W^\nu_{ut} - Z'^\mu_{\mu',s} \delta X^{\mu'}_{su} R^\nu_{W,ut} \\ &\quad - \delta(Z'^\mu_{\mu'} W'^\nu_{\nu'})_{su} \mathbb{X}^{2,\mu'}_{\nu',ut} + (Z'^\mu_{\mu'} W'^\nu_{\nu'})_s N\mathbb{X}^{2,\mu'\nu'}_{sut} \\ &= -R^\mu_{Z,su} \delta W^\nu_{ut} - Z'^\mu_{\mu',s} \delta X^{\mu'}_{su} R^\nu_{W,ut} \\ &\quad - \delta(Z'^\mu_{\mu'} W'^\nu_{\nu'})_{su} \mathbb{X}^{2,\mu'\nu'}_{ut} - Z'^\mu_{\mu',s} \delta X^{\mu'}_{su} \delta W'^\nu_{\nu',su} \delta X^{\nu'}_{ut} \end{aligned}$$

and observe that all the terms are in $\Omega\mathscr{C}^\delta_2(I, V^{\otimes 2})$ so that $Q \in \mathscr{Z}^\delta_2(I, V^{\otimes 2})$ is in the domain of Λ, then

$$\begin{aligned} \|\Lambda Q\|_{\delta,I} &\leqslant \frac{1}{2^\delta - 2} [\|R_Z\|_{\eta,I} \|W\|_{\gamma,I} + \|Z'\|_{\infty,I} \|X\|_{\gamma,I} \|R_W\|_{\eta,I} \\ &\quad + \|\mathbb{X}^2\|_{2\gamma,I} (\|Z'\|_{\infty,I} \|W'\|_{\eta-\gamma,I} + \|W'\|_{\infty,I} \|Z'\|_{\eta-\gamma,I}) \\ &\quad + \|Z'\|_{\infty,I} \|W'\|_{\eta-\gamma,I} \|X\|^2_{\gamma,I}] \\ &\leqslant \frac{1}{2^\delta - 2} (1 + \|X\|^2_{\gamma,I} + \|\mathbb{X}\|^2_{2\gamma,I}) \|(Z, Z')\|_{D(X,\gamma,\eta),I} \|(W, W')\|_{D(X,\gamma,\eta),I} \end{aligned}$$

and bound (26) together with the stated continuity easily follows.

To prove that this new integral extends the previous definition note that when $2\gamma > 1$ Eq. (24) has a unique solution and since $Z, W \in \mathscr{C}^\gamma(I, V)$ let $\tilde{A}_{st} = \int_s^t Z\, dW$ where the integral is understood in the sense of Proposition 3. Then we have

$$Z^\mu \delta W^\nu = \delta \tilde{A}^{\mu\nu} - \tilde{R}^{\mu\nu}$$

with $\tilde{R} \in \Omega\mathscr{C}^{2\gamma}(I, V \otimes V)$, at the same time

$$Z^\mu \delta W^\nu = \delta A^{\mu\nu} - Z'^\mu_{\mu'} W'^\nu_{\nu'} \mathbb{X}^{2,\mu'\nu'} - R^{\mu\nu}$$

with $R \in \Omega\mathscr{C}^\delta(I, V^{\otimes 2})$. Comparing these two expressions and taking into account that $2\gamma > 1$ we get $\delta A = \delta\tilde{A}$ and $\tilde{R}^{\mu\nu} = Z'^\mu_{\mu'} W'^\nu_{\nu'} \mathbb{X}^{2,\mu'\nu'} - R^{\mu\nu}$ proving the equivalence of the two integrals. $\square$

Note that, in the hypothesis of Theorem 1, we have

$$\mathbb{X}^{2,\mu\nu}_{st} = \int_s^t (X^\mu_u - X^\mu_s)\, dX^\nu_u.$$

Even if the notation does not make it explicit it is important to remark that the integral depends on the rough path $(X, \mathbb{X}^2)$, however if there is another rough path $(Y, \mathbb{Y}^2)$ and $X \in \mathscr{D}_Y^{\gamma,\eta}(I, V)$ we have shown that $\mathscr{D}_X^{\gamma,\eta}(I, V) \subseteq \mathscr{D}_Y^{\gamma,\eta}(I, V)$ (see Lemma 1) and the integral defined according to $(X, \mathbb{X}^2)$ is equal to that defined according to $(Y, \mathbb{Y}^2)$ if and only if we have

$$\mathbb{X}^{2,\mu\nu} = \int_s^t \delta X_{su}^\mu \, dX_u^\nu,$$

where this last integral is understood based on $(Y, \mathbb{Y}^2)$. Necessity is obvious, let us prove sufficiency. Let the decomposition of X according to Y be

$$\delta X^\mu = A_\nu^\mu \delta Y^\nu + R_X^\mu$$

and write

$$\delta Z^\mu = Z_\nu'^\mu \delta X^\nu + R_Z^\mu, \qquad \delta W^\mu = W_\nu'^\mu \delta W^\nu + R_W^\mu$$

then if

$$\delta I_{st}^{\mu\nu} = \int_s^t Z^\mu d_{(X,\mathbb{X}^2)} W^\nu$$

is the integral based on $(X, \mathbb{X}^2)$,

$$\delta \widetilde{I}_{st}^{\mu\nu} = \int_s^t Z^\mu d_{(Y,\mathbb{Y}^2)} W^\nu$$

the one based on $(Y, \mathbb{Y}^2)$; we have by definition of integral

$$\delta I^{\mu\nu} = Z^\mu \delta W^\nu + Z_\kappa'^\mu W_\rho'^{,\nu} \mathbb{X}^{2,\kappa\rho} + R_I^{\mu\nu},$$

$$\delta \widetilde{I}^{\mu\nu} = Z^\mu \delta W^\nu + Z_\kappa'^{,\mu} A_{\kappa'}^\kappa W_\rho'^\nu A_{\rho'}^\rho \mathbb{Y}^{2,\kappa'\rho'} + R_{\widetilde{I}}^{\mu\nu}$$

and

$$\mathbb{X}^{2,\kappa\rho} = A_{\kappa'}^\kappa A_{\rho'}^\rho \mathbb{Y}^{2,\kappa'\rho'} + R_{\mathbb{X}^2}^{\kappa\rho},$$

where $R_I, R_{\widetilde{I}}, R_{\mathbb{X}^2} \in \Omega\mathscr{C}^{\gamma+\eta}(V^{\otimes 2})$. Then

$$\begin{aligned}\delta(I^{\mu\nu} - \widetilde{I}^{\mu\nu}) &= Z_\kappa'^\mu W_\rho'^{,\nu}(\mathbb{X}^{2,\kappa\rho} - A_{\kappa'}^\kappa A_{\rho'}^\rho \mathbb{Y}^{2,\kappa'\rho'}) + R_I^{\mu\nu} - R_{\widetilde{I}}^{\mu\nu} \\ &= Z_\kappa'^\mu W_\rho'^\nu R_{\mathbb{X}^2}^{\kappa\rho} + R_I^{\mu\nu} - R_{\widetilde{I}}^{\mu\nu}\end{aligned}$$

but then $\delta(I - \widetilde{I}) \in \Omega\mathscr{C}^{\gamma+\eta}(I, V^{\otimes 2})$ with $\gamma + \eta > 1$ so it must be $\delta I = \delta\widetilde{I}$.

Given another rough path $(\widetilde{X}, \widetilde{\mathbb{X}}^2)$ and paths $\widetilde{W}, \widetilde{Z} \in \mathscr{D}^{\gamma,\eta}_{\widetilde{X}}(I, V)$ then it takes not so much effort to show that the difference

$$\Delta_{st} := \int_s^t Z\,dW - \int_s^t \widetilde{Z}\,d\widetilde{W}$$

(where the first integral is understood with respect to $(X, \mathbb{X}^2)$ and the second w.r.t. $(\widetilde{X}, \widetilde{\mathbb{X}}^2)$) can be bounded as

$$\|\Delta - Z\delta W + \widetilde{Z}\delta\widetilde{W} + \widetilde{W}'\widetilde{Z}'\widetilde{\mathbb{X}}^2 - W'Z'\mathbb{X}^2\|_{\delta,I} \leqslant \frac{1}{2^z - 2}(D_1 + D_2 + D_3), \quad (27)$$

where

$$D_1 = (1 + \|X\|_{\gamma,I}^2 + \|\mathbb{X}^2\|_{2\gamma,I})(\|(Z, Z')\|_{D(X,\gamma,\eta),I} + \|(\widetilde{Z}, \widetilde{Z}')\|_{D(\widetilde{X},\gamma,\eta),I})\varepsilon_W,$$

$$D_2 = (1 + \|X\|_{\gamma,I}^2 + \|\mathbb{X}^2\|_{2\gamma,I})(\|(W, W')\|_{D(X,\gamma,\eta),I} + \|(\widetilde{W}, \widetilde{W'})\|_{D(\widetilde{X},\gamma,\eta),I})\varepsilon_Z,$$

$$\begin{aligned} D_3 = {} & (\|(W, W')\|_{D(X,\gamma,\eta),I} + \|(\widetilde{W}, \widetilde{W'})\|_{D(\widetilde{X},\gamma,\eta),I}) \\ & \cdot (\|(Z, Z')\|_{D(X,\gamma,\eta),I} + \|(\widetilde{Z}, \widetilde{Z}')\|_{D(\widetilde{X},\gamma,\eta),I})(\|X - \widetilde{X}\|_{\gamma,I} + \|\mathbb{X}^2 - \widetilde{\mathbb{X}}^2\|_{2\gamma,I}) \end{aligned}$$

and

$$\varepsilon_Z = \|Z' - \widetilde{Z}'\|_{\infty,I} + \|Z' - \widetilde{Z}'\|_{\eta-\gamma,I} + \|R_Z - \widetilde{R}_Z\|_{\eta,I} + \|Z - \widetilde{Z}\|_{\gamma,I},$$

$$\varepsilon_W = \|W' - \widetilde{W}'\|_{\infty,I} + \|W' - \widetilde{W}'\|_{\eta-\gamma,I} + \|R_W - \widetilde{R}_W\|_{\eta,I} + \|W - \widetilde{W}\|_{\gamma,I}$$

so that the integral possess reasonable continuity properties also with respect to the reference rough path $(X, \mathbb{X}^2)$.

Remark 1. It is trivial but cumbersome to generalize the statement of Theorem 1 in the case of inhomogeneous degrees of smoothness, i.e. when we have $Z \in \mathscr{D}^{\gamma,\eta}_X(V)$, $W \in \mathscr{D}^{\rho,\eta'}_Y(V)$ with $X \in \mathscr{C}^\gamma(V)$, $Y \in \mathscr{C}^\rho(V)$ and there is a process $H \in \Omega\mathscr{C}^{\gamma+\rho}(V^{\otimes 2})$ which satisfy

$$NH^{\mu\nu} = \delta X^\mu \delta Y^\nu.$$

In this case the condition to be satisfied in order to be able to define the integral is $\min(\gamma + \eta', \rho + \eta) = \delta > 1$.

As in Section 3 we can give an approximation result of the integral defined in Theorem 1 as a limit of sums of increments:

Corollary 2. *In the hypothesis of the previous proposition we have*

$$\int_s^t Z_u^\mu dW_u^\nu = \lim_{|\Pi|\to 0} \sum_{i=0}^{n-1} (Z_{t_i}^\mu \delta W_{t_i,t_{i+1}}^\nu + Z'^{\mu}_{\mu',t_i} W'^{\nu}_{\nu',t_i} \mathbb{X}^{2,\mu'\nu'}_{t_i,t_{i+1}})$$

where the limit is taken over partitions $\Pi = \{t_0, t_1, \dots, t_n\}$ of the interval $[s,t]$ such that $t_0 = s,\ t_n = t,\ t_{i+1} > t_i,\ |\Pi| = \sup_i |t_{i+1} - t_i|$.

Proof. The proof is analogous to that of Corollary 1. □

Simpler bounds can be stated in the case where we are integrating a path controlled by X against X itself

Corollary 3. *When $W \in \mathscr{D}_X^{\gamma,\eta}(I, V_1 \otimes V^*)$ the integral*

$$\delta A_{st}^\mu = \int_s^t W_{\nu,u}^\mu \, dX_u^\nu$$

belongs to $\mathscr{D}_X^{\gamma,2\gamma}(I, V_1)$ and satisfy

$$||\delta A - W_\nu \delta X^\nu - W'_{\nu\kappa} \mathbb{X}^{2,\nu\kappa}||_{D(X,\gamma,\eta+\gamma),I} \leqslant \frac{1}{2^{\eta+\gamma} - 2} (||X||_{\gamma,I} + ||\mathbb{X}^2||_{2\gamma,I}) ||W||_{D(X,\gamma,\eta),I} \tag{28}$$

Moreover if $(\widetilde{X}, \widetilde{\mathbb{X}}^2)$ is another rough path and $\widetilde{W} \in \mathscr{D}_{\widetilde{X}}^{\gamma,\eta}(I, V_1 \otimes V^*)$ then

$$\delta B_{st}^\mu = \int_s^t W_{\nu,u}^\mu \, dX_u^\nu - \int_s^t \widetilde{W}_{\nu,u}^\mu \, d\widetilde{X}_u^\nu$$

and

$$\delta B^\mu = W_\nu^\mu \delta X^\nu - \widetilde{W}_\nu^\mu \delta \widetilde{X}^\nu - W'^{\mu}_{\nu\kappa} \mathbb{X}^{2,\nu\kappa} - \widetilde{W}'^{\mu}_{\nu\kappa} \widetilde{\mathbb{X}}^{2,\nu\kappa} + R_B^\mu$$

with R_B satisfying the bound

$$||R_B||_{\eta+\gamma,I} \leqslant \frac{1}{2^{\eta+\gamma} - 2} [C_{X,I} \varepsilon_{W,I} + (||W||_{D(X,\gamma,\eta),I} + ||\widetilde{W}||_{D(\widetilde{X},\gamma,\eta),I}) \rho_I] \tag{29}$$

with

$$\varepsilon_{W,I} = ||R_W - R_{\widetilde{W}}||_{\eta,I} + ||W' - \widetilde{W}'||_{\eta-\gamma,I}$$

and

$$\rho_I = ||X - \widetilde{X}||_\gamma + ||\mathbb{X}^2 - \widetilde{\mathbb{X}}^2||_{2\gamma,I}$$

$$C_{X,I} = ||X||_{\gamma,I} + ||\mathbb{X}^2||_{2\gamma,I} + ||\widetilde{X}||_{\gamma,I} + ||\widetilde{\mathbb{X}}^2||_{2\gamma,I}.$$

Proof. The integral path δA has the following decomposition:

$$\delta A^\mu = W^\mu_\nu \delta X^\nu + W'^\mu_{\nu\kappa}\mathbb{X}^{2,\nu\kappa} + R^\mu_A$$

with R_A satisfying

$$N R^\mu_A = \delta W'^\mu_{\nu\kappa}\mathbb{X}^{2,\nu\kappa} + R^\mu_{W,\nu}\delta X^\nu$$

then Eq. (28) follows immediately from the properties of Λ. Next, let $\delta\widetilde{A} = \int \widetilde{W}\, d\widetilde{X}$ and

$$\delta\widetilde{A}^\mu = \widetilde{W}^\mu_\nu \delta\widetilde{X}^\nu + \widetilde{W}'^\mu_{\nu\kappa}\widetilde{\mathbb{X}}^{2,\nu\kappa} + R^\mu_{\widetilde{A}}$$

then

$$N R^\mu_B = \delta W'^\mu_{\nu\kappa}\mathbb{X}^{2,\nu\kappa} + R^\mu_{W,\nu}\delta X^\nu - \delta\widetilde{W}'^\mu_{\nu\kappa}\widetilde{\mathbb{X}}^{2,\nu\kappa} + R^\mu_{\widetilde{W},\nu}\delta\widetilde{X}^\nu$$

and

$$\begin{aligned}||R_B||_{\eta+\gamma,I} \leqslant \frac{1}{2^{\eta+\gamma}-2}&[||W' - \widetilde{W}'||_{\eta-\gamma,I}||\mathbb{X}^2||_{2\gamma,I} + ||\widetilde{W}'||_{\eta-\gamma,I}||\mathbb{X}^2 - \widetilde{\mathbb{X}}^2||_{2\gamma,I} \\ &+ ||X - \widetilde{X}||_{\gamma,I}||R_W||_{\eta,I} + ||\widetilde{X}||_{\gamma,I}||R_W - R_{\widetilde{W}}||_{\eta,I}] \\ \leqslant \frac{1}{2^{\eta+\gamma}-2}&[C_{X,I}\varepsilon_{W,I} + (||W||_{D(X,\gamma,\eta),I} + ||\widetilde{W}||_{D(X,\gamma,\eta),I})\rho_I]. \qquad \square\end{aligned}$$

5. Differential equations driven by paths in $\mathscr{C}^\gamma(V)$

The continuity of the integral defined in Eq. (14) allows to prove existence and uniqueness of solutions of differential equations driven by paths in $\mathscr{C}^\gamma(V)$ for γ not too small.

Fix an interval $J \subseteq \mathbb{R}$ and let us given $X \in \mathscr{C}^\gamma(J, V)$ and a function $\varphi \in C(V, V \otimes V^*)$. A solution Y of the differential equation

$$dY^\mu_t = \varphi(Y_t)^\mu_\nu\, dX^\nu_t, \qquad Y_{t_0} = y, \quad t_0 \in J \tag{30}$$

in J will be a continuous path $Y \in \mathscr{C}^\gamma(V, J)$ such that

$$Y^\mu_t = y + \int_{t_0}^t \varphi(Y_u)^\mu_\nu\, dX^\nu_u. \tag{31}$$

for every $t \in J$. If $\gamma > 1/2$ sufficient conditions must be imposed on φ such that the integral in (31) can be understood in the sense of Proposition 3. If $1/3 < \gamma \leqslant 1/2$ the integral must be understood in the sense of Theorem 1. Then in this case we want to show that, given a driving rough path $(X, \mathbb{X}^2)$ it is possible to find a path $Y \in \mathscr{D}_X^{\gamma, 2\gamma}(V, J)$ that satisfy Eq. (31).

The strategy of the proof will consist in introducing a map $Y \mapsto G(Y)$ on suitable paths $Y \in \mathscr{C}(J, V)$ depending implicitly on X (and eventually on $\mathbb{X}^2$) such that

$$G(Y)_t = Y_{t_0} + \int_{t_0}^{t} \varphi(Y_u)^\mu_\nu \, dX^\nu_u. \tag{32}$$

Existence of solutions will follow from a fixed-point theorem applied to G acting on a suitable compact and convex subset of the Banach space of Hölder continuous functions on J (this require V to be finite dimensional). To show uniqueness we will prove that under stronger conditions on φ the map G is locally a strict contraction. Next we show also that the Itô map (in the terminology of Lyons [7]) $Y = F(y, \varphi, X)$ (or $Y = F(y, \varphi, X, \mathbb{X}^2)$) which sends the data of the differential equation to the corresponding solution $Y = G(Y)$, is a Lipschitz continuous map (in compact intervals J) in each of its argument, where on X and $\mathbb{X}^2$ we are considering the norms of $\mathscr{C}^\gamma(J, V)$ and $\Omega\mathscr{C}^{2\gamma}(J, V^{\otimes 2})$, respectively.

Note that, in analogy with the classical setting, the solution of the differential equation is "smooth" in the sense that it will be of the form

$$\delta Y = \varphi(Y)\delta X + R_Y \tag{33}$$

with $R_Y \in \Omega\mathscr{C}^z(V, J)$ with $z > 1$ in the case of $\gamma > 1/2$ and of the form

$$\delta Y = \varphi(Y)\delta X + \partial\varphi(Y)\varphi(Y)\mathbb{X}^2 + Q_Y \tag{34}$$

with $R_Y \in \Omega\mathscr{C}^z(V, J)$ with $z > 1$ in the case of $1/3 < \gamma \leqslant 1/2$.

Natural conditions for existence of solutions will be $\varphi \in C^\delta(V, V \otimes V^*)$ if $\gamma > 1/2$ and $(1+\delta)\gamma > 1$, while $\varphi \in C^{1,\delta}(V, V \otimes V^*)$ if $1/3 < \gamma \leqslant 1/2$ where $\delta \in (0,1)$ such that $(2+\delta)\gamma > 1$ while uniqueness will hold if $\varphi \in C^{1,\delta}(V, V \otimes V^*)$ or $\varphi \in C^{2,\delta}(V, V \otimes V^*)$ respectively with analogous conditions on δ.

Remark 2. Another equivalent approach to the definition of a differential equation in the non-smooth setting is to say that Y solves a differential equation driven by X if Eq. (33) or (34) is satisfied with remainders R_Y or Q_Y in $\Omega\mathscr{C}^z(V)$ for some z. This would have the natural meaning of describing the local dynamical behaviour of Y_t as the parameter t is changed in terms of the control X. This point of view has been explored previously in an unpublished work by Davie [1] which also gives some examples showing that the conditions on the vector field φ cannot be substantially relaxed.

Remark 3. In a recent work [6] Li and Lyons show that, under natural hypotesis on φ, the Itô map F can be differentiated with respect to the control path X (when extended to a rough path).

5.1. Some preliminary results

In the proofs of the propositions below it will be useful the following comparison of norms which holds for locally Hölder continuous paths:

Lemma 2. *Let $\eta > \gamma$, $b > a$ then $\Omega\mathscr{C}^\eta([a,b]) \subseteq \Omega\mathscr{C}^\gamma([a,b])$ and*

$$||X||_{\gamma,[a,b]} \leqslant |b-a|^{\eta-\gamma} ||X||_{\eta,[a,b]}$$

for any $X \in \Omega\mathscr{C}^\eta([a,b])$.

Proof. Easy:

$$||X||_{\gamma,[a,b]} = \sup_{t,s\in[a,b]} \frac{|X_{st}|}{|t-s|^\gamma} = \sup_{t,s\in[a,b]} \frac{|X_{st}|}{|t-s|^\eta} |t-s|^{\eta-\gamma} \leqslant |b-a|^{\eta-\gamma} \sup_{t,s\in[a,b]} \frac{|X_{st}|}{|t-s|^\eta}. \qquad \square$$

Moreover we will need to patch together local Hölder bounds for different intervals:

Lemma 3. *Let I, J be two adjacent intervals on $\mathbb{R}$ (i.e. $I \cap J \neq 0$) then if $X \in \Omega\mathscr{C}^\gamma(I, V)$, $X \in \Omega\mathscr{C}^\gamma(J, V)$ and $NX \in \Omega\mathscr{C}^{\gamma_1,\gamma_2}(I \cup J, V)$ with $\gamma = \gamma_1 + \gamma_2$, then we have $X \in \Omega\mathscr{C}^\gamma(I \cup J, V)$ with*

$$||X||_{\gamma,I\cup J} \leqslant 2(||X||_{\gamma,I} + ||X||_{\gamma,J}) + ||NX||_{\gamma_1,\gamma_2,I\cup J}. \tag{35}$$

Proof. See Appendix A, Section A.3.1. $\square$

5.2. Existence and uniqueness when $\gamma > 1/2$

First we will formulate the results for the case $\gamma > 1/2$ since they are simpler and require weaker conditions.

Proposition 5 (Existence $\gamma > 1/2$). *If $\gamma > 1/2$ and $\varphi \in C^\delta(V, V \otimes V^*)$ with $\delta \in (0,1)$ and $(1+\delta)\gamma > 1$ there exists a path $Y \in \mathscr{C}^\gamma(V)$ which solves Eq.* (30) (*where the integral is the one defined in Section* 3).

Proof. Consider an interval $I = [t_0, t_0 + T] \subseteq J$, $T > 0$ and note that $W = \varphi(Y)$ is in $\mathscr{C}^{\delta\gamma}(I, V \otimes V^*)$ with

$$||W||_{\delta\gamma,I} = ||\varphi(Y)||_{\delta\gamma,I} \leqslant ||\varphi||_\delta ||Y||_{\gamma,I}^\delta$$

so that if $(1+\delta)\gamma > 1$ it is meaningful, according to Proposition 3 to consider the application $\mathscr{C}^\gamma(I,V) \to \mathscr{C}^\gamma(I,V)$ defined as in Eq. (32). Moreover the path $Z = G(Y) \in \mathscr{C}^\gamma(I,V)$ satisfy

$$\delta Z^\mu = \varphi(Y)^\mu_\nu \delta X^\nu + Q^\mu_Z$$

with

$$||Q_Z||_{(1+\delta)\gamma,I} \leqslant \frac{1}{2^{(1+\delta)\gamma}-2} ||X||_{\gamma,I} ||\varphi(Y)||_{\delta\gamma,I} \leqslant \frac{1}{2^{(1+\delta)\gamma}-2} ||\varphi||_\delta ||X||_{\gamma,I} ||Y||^\delta_{\gamma,I},$$

then, using Lemma 2,

$$\begin{aligned} ||Z||_{\gamma,I} &\leqslant ||\varphi(Y)\delta X||_{\gamma,I} + ||Q_Z||_{\gamma,I} \\ &\leqslant ||\varphi||_{0,\delta} ||X||_{\gamma,I} + T^{\gamma\delta} ||Q_Z||_{(1+\delta)\gamma,I} \\ &\leqslant K C_{X,I} ||\varphi||_{0,\delta} (1 + T^{\delta\gamma} ||Y||^\delta_{\gamma,I}) \\ &\leqslant K C_{X,J} ||\varphi||_{0,\delta} (1 + T^{\delta\gamma} ||Y||^\delta_{\gamma,I}) \end{aligned}$$

with

$$C_{X,I} = ||X||_{\gamma,I}.$$

For any T let $A_T > 0$ be the solution to

$$A_T = K C_{X,J} ||\varphi||_{0,\delta} (1 + T^{\delta\gamma} A^\delta_T). \tag{36}$$

Then $||G(Y)||_{\gamma,I} \leqslant A_T$ whenever $||Y||_{\gamma,I} \leqslant A_T$ and moreover $G(Y)_{t_0} = Y_{t_0}$. Then for any $y \in V$, the application G maps the compact and convex set

$$Q_{y,[t_0,t_0+T]} = \{ Y \in \mathscr{C}^\gamma([t_0, t_0+T], V):\ Y_{t_0} = y, ||Y||_{\gamma,[t_0,t_0+T]} \leqslant A_T \} \tag{37}$$

into itself. Let us show that G on $Q_{y,[t_0,t_0+T]}$ is at least Hölder continuous with respect to the norm $||\cdot||_\gamma$. This will allow us to conclude (by the Leray–Schauder–Tychonoff theorem) the existence of a fixed-point in $Q_{y,[t_0,t_0+T]}$. To prove continuity take $Y, \widetilde{Y} \in Q_{y,I}$ and denote $\widetilde{Z} = G(\widetilde{Y})$ so that

$$\delta \widetilde{Z}^\mu = \varphi(\widetilde{Y})^\mu_\nu \delta X^\nu + \widetilde{Q}^\mu_Z$$

as for $Z = G(Y)$. Then

$$||Z - \widetilde{Z}||_{\gamma,I} \leqslant ||\varphi(Y) - \varphi(\widetilde{Y})||_{\infty,I} ||X||_{\gamma,I} + ||Q_Z - Q_{\widetilde{Z}}||_{\gamma,I} \tag{38}$$

but now taking $0<\alpha<1$ such that $(1+\alpha\delta)\gamma>1$

$$||Q_Z - Q_{\widetilde{Z}}||_{(1+\alpha\delta)\gamma,I} \leqslant \frac{1}{2^{(1+\alpha\delta)\gamma}-2}||X||_{\gamma,I}||\varphi(Y)-\varphi(\widetilde{Y})||_{\alpha\delta\gamma,I}.$$

To bound $||\varphi(Y)-\varphi(\widetilde{Y})||_{\alpha\delta\gamma,I}$ we interpolate between the following two bounds:

$$||\varphi(Y)-\varphi(\widetilde{Y})||_{0,I} \leqslant 2||\varphi(Y)-\varphi(\widetilde{Y})||_{\infty,I} \leqslant 2||\varphi||_\delta||\widetilde{Y}-Y||^\delta_{\infty,I}$$

and

$$||\varphi(Y)-\varphi(\widetilde{Y})||_{\delta\gamma,I} \leqslant ||\varphi(Y)||_{\delta\gamma,I} + ||\varphi(\widetilde{Y})||_{\delta\gamma,I} \leqslant ||\varphi||_\delta(||Y||^\delta_{\gamma,I} + ||\widetilde{Y}||^\delta_{\gamma,I}) \leqslant ||\varphi||_\delta 2A^\delta_T$$

obtaining

$$||\varphi(Y)-\varphi(\widetilde{Y})||_{\alpha\delta\gamma,I} \leqslant 2||\varphi||_\delta||\widetilde{Y}-Y||^{(1-\alpha)\delta}_{\infty,I}A^{\alpha\delta}_T$$

Eq. (38) becomes

$$\begin{aligned}||Z-\widetilde{Z}||_{\gamma,I} &\leqslant ||\varphi(Y)-\varphi(\widetilde{Y})||_{\infty,I}||X||_{\gamma,I} + T^{\alpha\delta\gamma}||Q_Z - Q_{\widetilde{Z}}||_{(1+\alpha\delta)\gamma,I} \\ &\leqslant K||\varphi||_\delta||X||_{\gamma,I}[||Y-\widetilde{Y}||^\delta_{\infty,I} + ||\widetilde{Y}-Y||^{(1-\alpha)\delta}_{\infty,I}A^{\alpha\delta}_T].\end{aligned}$$

Since $||Y-\widetilde{Y}||_{\infty,I} \leqslant ||Y-\widetilde{Y}||_{\gamma,I}$ (recall that $T<1$) we have that G is continuous on $Q_{y,I}$ for the topology induced by the norm $||\cdot||_{\gamma,I}$ (the paths all have a common starting point).

Since all these arguments does not depend on the location of the interval I we can patch together local solutions to get the existence of a global solution on all J. $\square$

Proposition 6 (Uniqueness $\gamma>1/2$). *Assume $\varphi\in C^{1,\delta}(V, V\otimes V^*)$ with $(1+\delta)\gamma>1$, then there exists a unique solution of Eq.* (30). *The Itô map $F(y,\varphi,X)$ is Lipschitz in the sense that satisfy the following bound*:

$$||F(y,\varphi,X)-F(\widetilde{y},\widetilde{\varphi},\widetilde{X})||_{\gamma,J} \leqslant M(||X-\widetilde{X}||_{\gamma,J} + ||\varphi-\widetilde{\varphi}||_{1,\delta} + |y-\widetilde{y}|)$$

for some constant M depending only on $||X||_{\gamma,J}$, $||\widetilde{X}||_{\gamma,J}$, $||\varphi||_{1,\delta}$, $||\widetilde{\varphi}||_{1,\delta}$ and J.

Proof. Let us continue to use the notations of the previous proposition. Let $Y, \widetilde{Y}$ be two paths in $\mathscr{C}^\gamma(J,V)$, and $X, \widetilde{X}\in\mathscr{C}^\gamma(J,V)$. Let $W=\varphi(Y)$, $\widetilde{W}=\varphi(\widetilde{Y})$, $Z=G(Y)$, $\widetilde{Z}=\widetilde{G}(\widetilde{Y})$ where $\widetilde{G}$ is the map corresponding to the driving path $\widetilde{X}$:

$$\widetilde{Y}\mapsto\widetilde{G}(\widetilde{Y})^\mu := \widetilde{Y}^\mu_{t_0} + \int_{t_0}^{\cdot}\varphi(\widetilde{Y}_u)^\mu_\nu\, d\widetilde{X}^\nu_u.$$

Then

$$\delta\widetilde{Z}^\mu = \varphi(\widetilde{Y}_s)^\mu \delta\widetilde{X}^\nu + Q^\mu_{\widetilde{Z}}.$$

Introduce the following shorthands:

$$\varepsilon_{Z,I} = \|Z - \widetilde{Z}\|_{\gamma,I}, \quad \varepsilon^*_{W,I} = \|W - \widetilde{W}\|_{\delta\gamma,I}, \quad \varepsilon_{Y,I} = \|Y - \widetilde{Y}\|_{\gamma,I}, \quad \varepsilon^*_{Y,I} = \|Y - \widetilde{Y}\|_{\delta\gamma,I};$$

$$\rho_I = \|X - \widetilde{X}\|_{\gamma,I} + |Y_0 - \widetilde{Y}_0| + \|\varphi - \widetilde{\varphi}\|_{1,\delta}$$

$$C_{X,I} = \|X\|_{\gamma,I} + \|\widetilde{X}\|_{\gamma,I} \qquad C_{Y,I} = \|Y\|_{\gamma,I} + \|\widetilde{Y}\|_{\gamma,I}.$$

With these notations, Lemma 5 states that, when $T<1$:

$$\varepsilon_{Z,I} \leqslant K C_{X,I} C^\delta_{Y,I}[(1 + \|\varphi\|_{1,\delta})\rho_I + \|\varphi\|_{1,\delta} T^{\gamma\delta} \varepsilon_{Y,I}]. \tag{39}$$

As we showed before in Proposition 5 there exists a constant A_T such that the set $Q_{y,I} := \{Y \in C^\gamma(I,V)\colon\ Y_{t_0} = y, \|Y\|_{\gamma,I} \leqslant A_T\}$ is invariant under G. Take $Y, \widetilde{Y} \in Q_{y,I}$ and $X = \widetilde{X}$. Then we have $\rho_I = 0$, $C_{Y,I} \leqslant 2A_T$ and

$$\varepsilon_{Z,I} \leqslant K \|\varphi\|_{1,\delta} C_{X,J} A^\delta_T T^{\gamma\delta} \varepsilon_{Y,I}.$$

Choosing T small enough such that $K\|\varphi\|_{1,\delta} C_{X,J} A^\delta_T T^{\gamma\delta} = \alpha < 1$ implies

$$\|G(Y) - G(\widetilde{Y})\|_{\gamma,I} = \varepsilon_{Z,I} \leqslant \alpha \|Y - \widetilde{Y}\|_{\gamma,I}.$$

The map G is then a strict contraction on $Q_{y,I}$ and has a unique fixed-point. Again, since the estimate does not depend on the location of $I \subset J$ we can extend the unique solution to all J. □

5.3. *Existence and uniqueness for $\gamma > 1/3$*

Proposition 7 (Existence $\gamma > 1/3$). *If $\gamma > 1/3$ and $\varphi \in C^{1,\delta}(V,V)$ with $(2+\delta)\gamma > 1$ there exists a path $Y \in \mathscr{D}^{\gamma,2\gamma}_X(V)$ which solves Eq.* (30) *where the integral is understood in the sense of Theorem* 1 *based on the couple $(X, \mathbb{X}^2)$.*

Proof. By Proposition 4 for any $Y \in \mathscr{D}^{\gamma,2\gamma}_X(J,V)$, the path $W = \varphi(Y)$ is in $\mathscr{D}^{\gamma,(1+\delta)\gamma}_X(J,V)$ with

$$\begin{aligned}\|W\|_{D(X,\gamma,(1+\delta)\gamma),I} &= \|\varphi(Y)\|_{D(X,\gamma,(1+\delta)\gamma),I} \leqslant K\|\varphi\|_{1,\delta}(\|Y\|_{*,I} + \|Y\|^{1+\delta}_{*,I} + \|Y\|^2_{*,I}) \\ &\leqslant 3K\|\varphi\|_{1,\delta}(1 + \|Y\|_{*,I})^2,\end{aligned} \tag{40}$$

where we introduced the notation $\|\cdot\|_{*,I} = \|\cdot\|_{D(X,\gamma,2\gamma),I}$.

Then we can integrate W against X as soon as $(2+\delta)\gamma > 1$ and define the map G as $G : \mathscr{D}_X^{\gamma,2\gamma}(I,V) \to \mathscr{D}_X^{\gamma,2\gamma}(I,V)$ with formula (32). Let Y be a path such that $Y'_{t_0} = \varphi(Y_{t_0})$.

The decomposition of Z (as above $Z = G(Y)$) reads

$$\delta Z^\mu = Z'^\mu_\nu \delta X^\nu + R_Z^\mu = \varphi(Y)^\mu_\nu \delta X^\nu + \partial^\kappa \varphi(Y)^\mu_\nu Y'^\kappa_\rho \mathbb{X}^{2,\nu\rho} + Q_Z^\mu$$

with (use Eq. (28))

$$||Q_Z||_{(2+\delta)\gamma,I} \leqslant K C_{X,I} ||\varphi(Y)||_{D(X,\gamma,(1+\delta)\gamma),I}, \tag{41}$$

where

$$C_{X,I} = 1 + ||X||_{\gamma,I} + ||\mathbb{X}^2||_{2\gamma,I}.$$

Our aim is to bound Z in $\mathscr{D}_X^{\gamma,2\gamma}(I,V)$. To achieve this we already have the good bound (41) for Q_Z so we need bounds for $||\partial_\kappa \varphi(Y)^\cdot_\nu Y'^\kappa_\rho \mathbb{X}^{2,\nu\rho}||_{2\gamma,I}$, $||\varphi(Y)||_{\gamma,I}$ and $||Z||_{\gamma,I}$. To simplify the arguments assume that $T<1$ since at the end we will need to take T small anyway.

Let us start with $||\partial_\kappa \varphi(Y)^\cdot_\nu Y'^\kappa_\rho \mathbb{X}^{2,\nu\rho}||_{2\gamma,I}$:

$$\begin{aligned} ||\partial_\kappa \varphi(Y)^\cdot_\nu Y'^\kappa_\rho \mathbb{X}^{2,\nu\rho}||_{2\gamma,I} &\leqslant ||\partial_\kappa \varphi(Y)^\cdot_\nu||_{\infty,I} ||Y'^\kappa_\rho||_{\infty,I} ||\mathbb{X}^{2,\nu\rho}||_{2\gamma,I} \\ &\leqslant ||\partial\varphi||_\infty (|Y'_{t_0}| + T^\gamma ||Y'||_{\gamma,I}) ||\mathbb{X}^{2,\nu\rho}||_{2\gamma,I} \\ &\leqslant ||\varphi||_{1,\delta} (||\varphi||_{1,\delta} + T^\gamma ||Y'||_{\gamma,I}) ||\mathbb{X}^{2,\nu\rho}||_{2\gamma,I}. \end{aligned} \tag{42}$$

Next, using the fact that

$$\begin{aligned} ||\partial\varphi(Y)||_{\infty,I} &\leqslant |\partial\varphi(Y_{t_0})| + ||\partial\varphi(Y)||_{0,I} \\ &\leqslant ||\varphi||_{1,\delta} + T^{\delta\gamma} ||\partial\varphi(Y)||_{\delta\gamma,I} \\ &\leqslant ||\varphi||_{1,\delta} + T^{\delta\gamma} ||\varphi(Y)||_{D(X,\gamma,(1+\delta)\gamma),I} \end{aligned}$$

obtain

$$\begin{aligned} ||\varphi(Y)||_{\gamma,I} &\leqslant ||X||_{\gamma,I} ||\partial\varphi(Y)||_{\infty,I} + ||R_{\varphi(Y)}||_{\gamma,I} \\ &\leqslant ||\varphi||_{1,\delta} ||X||_{\gamma,I} + T^{\delta\gamma} (||X||_{\gamma,I} ||\partial\varphi(Y)||_{D(X,\gamma,(1+\delta)\gamma),I} + ||R_{\varphi(Y)}||_{(1+\delta)\gamma,I}) \\ &\leqslant C_{X,I} (||\varphi||_{1,\delta} + T^{\delta\gamma} ||\varphi(Y)||_{D(X,\gamma,(1+\delta)\gamma),I}). \end{aligned} \tag{43}$$

To finish consider

$$\|Z\|_{\gamma,I} \leqslant \|Z'\delta X\|_{\gamma,I} + \|R_Z\|_{\gamma,I}$$
$$\leqslant \|\varphi(Y)\|_{\infty,I}\|X\|_{\gamma,I} + \|\partial\varphi(Y)Y'\mathbb{X}^2\|_{2\gamma,I} + \|Q_Z\|_{2\gamma,I}. \tag{44}$$

Putting together the bounds given in Eqs. (41), (42), (43) and Eq. (44) we get

$$\|Z\|_{*,I} = \|\varphi(Y)\|_\infty + \|\varphi(Y)\|_{\gamma,I} + \|\partial_\kappa\varphi(Y)^{\cdot}_{\nu} Y'^{\kappa}_{\rho}\mathbb{X}^{2,\nu\rho}\|_{2\gamma,I} + \|Q_Z\|_{2\gamma,I} + \|Z\|_{\gamma,I}$$
$$\leqslant 2(1+\|X\|_{\gamma,I})\|\varphi(Y)\|_\infty + \|\varphi(Y)\|_{\gamma,I} + 2\|\partial_\kappa\varphi(Y)^{\cdot}_{\nu} Y'^{\kappa}_{\rho}\mathbb{X}^{2,\nu}_{\rho}\|_{2\gamma,I}$$
$$+\ 2T^{\delta\gamma}\|Q_Z\|_{(2+\delta)\gamma,I}$$
$$\leqslant KC_{X,I}(\|\varphi\|_{1,\delta} + \|\varphi\|^2_{1,\delta} + T^{\delta\gamma}\|\varphi\|_{1,\delta}\|Y\|_{*,I} + T^{\delta\gamma}\|\varphi(Y)\|_{D(X,\gamma,(1+\delta)\gamma),I}) \tag{45}$$

Eq. (40) is used to conclude that

$$\|G(Y)\|_{*,I} \leqslant K\|\varphi\|_{1,\delta}C_{X,I}(1+\|\varphi\|_{1,\delta} + T^{\delta\gamma}(1+\|Y\|_{*,I}))^2$$
$$\leqslant K\|\varphi\|_{1,\delta}C_{X,J}(1+\|\varphi\|_{1,\delta} + T^{\delta\gamma}(1+\|Y\|_{*,I}))^2. \tag{46}$$

There exists T_* such that for any $T < T_*$ the equation

$$A_T = K\|\varphi\|_{1,\delta}C_{X,J}(1+\|\varphi\|_{1,\delta} + T^{\delta\gamma}(1+A_T))^2$$

has at least a solution $A_T > 0$. Then we get that $\|G(Y)\|_{*,I} \leqslant A_T$ whenever $\|Y\|_{*,I} \leqslant A_T$. Let us now prove that in the set

$$Q'_{y,I} = \{Y \in \mathscr{D}^{\gamma,2\gamma}_X(I,V):\ Y_{t_0} = y,\ Y'_{t_0} = \varphi(y),\ \|Y\|_{*,I} \leqslant A_T\}$$

the map G is continuous (in the topology induced by the $\|\cdot\|_{*,I}$ norm). Take $Y, \widetilde{Y} \in Q'_{y,I}$ with $Z = G(Y)$, $\widetilde{Z} = G(\widetilde{Y})$ and

$$\delta\widetilde{Z}^\mu = \widetilde{Z}'^{\mu}_{\nu}\delta X^\nu + R^\mu_{\widetilde{Z}} = \varphi(\widetilde{Y})^\mu_\nu \delta X^\nu + \partial^\kappa\varphi(\widetilde{Y})^\mu_\nu \widetilde{Y}'^{\kappa}_{\rho}\mathbb{X}^{2,\nu\rho} + Q^\mu_{\widetilde{Z}}.$$

Take $0 < \alpha < 1$ and $(2+\alpha\delta)\gamma > 1$: a bound similar to Eq. (45) exists for $\|Z-\widetilde{Z}\|_{*,I}$:

$$\|Z-\widetilde{Z}\|_{*,I} \leqslant 2(1+\|X\|_{\gamma,I})\|\varphi(Y)-\varphi(\widetilde{Y})\|_\infty + \|\varphi(Y)-\varphi(\widetilde{Y})\|_{\gamma,I}$$
$$+\ 2\|(\partial_\kappa\varphi(Y)^{\cdot}_{\nu} Y'^{\kappa}_{\rho} - \partial_\kappa\varphi(\widetilde{Y})^{\cdot}_{\nu}\widetilde{Y}'^{\kappa}_{\rho})\mathbb{X}^{2,\nu}_{\rho}\|_{2\gamma,I} + 2\|Q_Z - Q_{\widetilde{Z}}\|_{(2+\alpha\delta)\gamma,I}$$

$$\leqslant KC_{X,I}[||\varphi(Y)-\varphi(\widetilde{Y})||_{\gamma,I}+||\partial\varphi(Y)+\partial\varphi(\widetilde{Y})||_{\infty,I}A_T+||Y'-\widetilde{Y}'||_{\infty,I}||\varphi||_\infty] \\ +\ 2||Q_Z-Q_{\widetilde{Z}}||_{(2+\alpha\delta)\gamma,I}$$

when $||Y-\widetilde{Y}||_{*,I}\leqslant\epsilon<1$ we have

$$||\varphi(Y)-\varphi(\widetilde{Y})||_{\gamma,I}+||\partial\varphi(Y)+\partial\varphi(\widetilde{Y})||_{\infty,I}A_T+||Y'-\widetilde{Y}'||_{\infty,I}||\varphi||_\infty\leqslant K||\varphi||_{1,\delta}(1+A_T)\epsilon^\delta,$$

moreover, we can bound $||Q_Z-Q_{\widetilde{Z}}||_{(2+\alpha\delta)\gamma,I}$ as

$$||Q_Z-Q_{\widetilde{Z}}||_{(2+\alpha\delta)\gamma,I}\leqslant\frac{1}{2^{(2+\alpha\delta)\gamma}-2}C_{X,I}[||R_W-R_{\widetilde{W}}||_{(1+\alpha\delta)\gamma,I}+||\partial\varphi(Y)-\partial\varphi(\widetilde{Y})||_{\alpha\delta\gamma,I}]$$

with $W=\varphi(Y)$, $\widetilde{W}=\varphi(\widetilde{Y})$. Both of the terms in the r.h.s. will be bounded by interpolation: the first between

$$||R_W-R_{\widetilde{W}}||_{(1+\delta)\gamma,I}\leqslant||\varphi(Y)||_{D(X,\gamma,(1+\delta)\gamma)}+||\varphi(\widetilde{Y})||_{D(X,\gamma,(1+\delta)\gamma)}$$

and

$$\begin{aligned}||R_W-R_{\widetilde{W}}||_{\gamma,I}&=||(\delta\varphi(Y)-\delta\varphi(\widetilde{Y}))-(\partial\varphi(Y)-\partial\varphi(\widetilde{Y}))\delta X||_{\gamma,I}\\&\leqslant||\varphi(Y)-\varphi(\widetilde{Y})||_{\gamma,I}+C_{X,I}||\partial\varphi(Y)-\partial\varphi(\widetilde{Y})||_{\infty,I}\\&\leqslant||\varphi||_{1,\delta}\epsilon+C_{X,I}||\varphi||_{1,\delta}\epsilon^\delta\end{aligned}$$

while the second between

$$||\partial\varphi(Y)-\partial\varphi(\widetilde{Y})||_{\delta\gamma,I}\leqslant||\partial\varphi(Y)||_{\delta\gamma,I}\leqslant+||\partial\varphi(\widetilde{Y})||_{\delta\gamma,I}$$

and

$$||\partial\varphi(Y)-\partial\varphi(\widetilde{Y})||_{0,I}\leqslant2||\partial\varphi(Y)-\partial\varphi(\widetilde{Y})||_{\infty,I}\leqslant||\varphi||_{1,\delta}||Y-\widetilde{Y}||^\delta_{\infty,I}\leqslant||\varphi||_{1,\delta}\epsilon^\delta.$$

These estimates are enough to conclude that $||Z-\widetilde{Z}||_{*,I}$ goes to zero whenever $||\widetilde{Y}-Y||_{*,I}$ does.

Reasoning as in Proposition 5 we can prove that a solution exists in $\mathscr{D}^{\gamma,2\gamma}_X(I,V)$ for any $I\subseteq J$ such that $|I|$ is sufficiently small. Cover J by a sequence $I_1,\dots,I_n$ of intervals of size $T<T_*$. Patching together local solutions we have a continuous solution $\bar{Y}$ defined on all J with

$$\delta\bar{Y}=\bar{Y}'\delta X+R_{\bar{Y}},$$

where $R_{\bar{Y}}\in\bigcup_i\Omega\mathscr{C}^{2\gamma}(I_i,V)$ and $\bar{Y}'\in\bigcup_i\Omega\mathscr{C}^{\gamma}(I_i,V)$. It remains to prove that $\bar{Y}\in\mathscr{D}^{\gamma,2\gamma}_X(J,V)$. Since the restriction of $\bar{Y}$ on I_i is in Q_{y,I_i} for some $y\in V$ we have that (with abuse of notation) $||\bar{Y}||_{*,I_i}\leqslant A_T$ for any i.

Using Lemma 3 iteratively we can obtain that

$$||\bar{Y}||_{\gamma,J} \leqslant 2^{n+1} \sup_i ||\bar{Y}||_{\gamma,I_i} \leqslant 2^{n+1} A_T$$

and by the same token

$$||\bar{Y}'||_{\gamma,J} \leqslant 2^{n+1} A_T.$$

Next consider $R_{\bar{Y}}$: write $J_k = \bigcup_{i=1}^k I_i$ and by the very same lemma get ($J_{i+1} = J_i \cup I_{i+1}$)

$$\begin{aligned} ||R_{\bar{Y}}||_{2\gamma,J_{i+1}} &\leqslant 2||R_{\bar{Y}}||_{2\gamma,J_i} + 2||R_{\bar{Y}}||_{2\gamma,I_{i+1}} + ||\delta\bar{Y}'\delta X||_{\gamma,\gamma,J_{i+1}} \\ &\leqslant 2||R_{\bar{Y}}||_{2\gamma,J_i} + 2||R_{\bar{Y}}||_{2\gamma,I_{i+1}} + ||\bar{Y}'||_{\gamma,J}||X||_{\gamma,J} \end{aligned}$$

since $NR_{\bar{Y}} = -\delta\bar{Y}'\delta X$. By induction over i we end up with

$$||R_{\bar{Y}}||_{2\gamma,J} \leqslant 2^{n+1} \sup_i ||R_{\bar{Y}}||_{2\gamma,I_i} + n||\bar{Y}'||_{\gamma,J}||X||_{\gamma,J} \leqslant (2^{n+1} + 2^{2n+2}n)A_T$$

and this is enough to conclude that $\bar{Y} \in \mathscr{D}_X^{\gamma,2\gamma}(J,V)$. □

Proposition 8 (Uniqueness $\gamma > 1/3$). *If $\gamma > 1/3$ and $\varphi \in C^{2,\delta}(V,V)$ with $(2+\delta)\gamma > 1$ there exists a unique path $Y \in \mathscr{D}_X^{\gamma,2\gamma}(J,V)$ which solves Eq.* (30) *based on the couple $(X,\mathbb{X}^2)$. Moreover the Itô map $F(y,\varphi,X,\mathbb{X}^2)$ is Lipschitz continuous in the following sense. Let $Y = F(y,\varphi,X,\mathbb{X}^2)$ and $\widetilde{Y} = F(\widetilde{y},\widetilde{\varphi},\widetilde{X},\widetilde{\mathbb{X}}^2)$ where $(X,\mathbb{X}^2)$ and $(\widetilde{X},\widetilde{\mathbb{X}}^2)$ are two rough paths, then defining*

$$\varepsilon_{Y,I} = ||Y' - \widetilde{Y}'||_{\infty,I} + ||Y' - \widetilde{Y}'||_{\gamma,I} + ||R_Y - R_{\widetilde{Y}}||_{2\gamma,I} + ||\varphi - \widetilde{\varphi}||_{2,\delta}$$

$$\rho_I = |Y_{t_0} - \widetilde{Y}_{t_0}| + ||X - \widetilde{X}||_{\gamma,I} + ||\mathbb{X}^2 - \widetilde{\mathbb{X}}^2||_{2\gamma,I}$$

and

$$C_{X,I} = (1 + ||X||_{\gamma,I} + ||\widetilde{X}||_{\gamma,I} + ||\mathbb{X}^2||_{2\gamma,I} + ||\widetilde{\mathbb{X}}^2||_{2\gamma,I})$$

$$C_{Y,I} = (1 + ||Y||_{*,I} + ||\widetilde{Y}||_{*,I}).$$

We have that there exists a constant M depending only on $C_{X,J}$, $C_{Y,J}$, $||\varphi||_{2,\delta}$ and $||\widetilde{\varphi}||_{2,\delta}$ such that

$$\varepsilon_{Y,J} \leqslant M\rho_J.$$

Proof. The strategy will be the same as in the proof of Proposition 6. Take two paths $Y, \widetilde{Y} \in \mathscr{D}_X^{\gamma,2\gamma}(J,V)$ and let as above $Z = G(Y)$, $\widetilde{Z} = \widetilde{G}(\widetilde{Y})$. Write the decomposition for each of the paths $Y, \widetilde{Y}, Z, \widetilde{Z}$ as

$$\delta Y^\mu = Y'^\mu_\nu \delta X^\nu + R^\mu_Y, \qquad \delta \widetilde{Y}^\mu = \widetilde{Y}'^\mu_\nu \delta \widetilde{X}^\nu + R^\mu_{\widetilde{Y}},$$

and

$$\delta Z = Z' \delta X + R_Z = \varphi(Y)\delta X + \partial\varphi(Y)\mathbb{X}^2 + Q_Z,$$

$$\delta \widetilde{Z} = \widetilde{Z}' \delta \widetilde{X} + R_{\widetilde{Z}} = \widetilde{\varphi}(\widetilde{Y})\delta \widetilde{X} + \partial\widetilde{\varphi}(\widetilde{Y})\widetilde{\mathbb{X}}^2 + Q_{\widetilde{Z}}.$$

The key point is to bound $\varepsilon_{Z,I}$ defined as

$$\varepsilon_{Z,I} = ||\varphi(Y) - \widetilde{\varphi}(\widetilde{Y})||_{\infty,I} + ||\varphi(Y) - \widetilde{\varphi}(\widetilde{Y})||_{\gamma,I} + ||R_Z - R_{\widetilde{Z}}||_{2\gamma,I}$$

and the result of Lemma 6 (in Appendix A) tells us that, when $T<1$, $\varepsilon_{Z,I}$ can be bounded by

$$\varepsilon_{Z,I} \leqslant K[(1 + ||\varphi||_{2,\delta})C^2_{X,I}C^3_{Y,I}\rho_I + ||\varphi||_{2,\delta}T^{\delta\gamma}C^3_{X,I}C^2_{Y,I}\varepsilon_{Y,I}]. \tag{47}$$

Taking $Y_0 = \widetilde{Y}_0$, $\widetilde{X} = X$, $\widetilde{\mathbb{X}}^2 = \mathbb{X}^2$ and $\varphi = \widetilde{\varphi}$ we have $\rho_I = \rho_J = 0$. As shown in the proof of Proposition 7 if $T < T_*$ for any $y \in V$ there exists a set $Q_{y,I} \subset \mathscr{D}^{\gamma,2\gamma}_X(I, V)$ invariant under G. Moreover if $Y, \widetilde{Y} \in Q_{y,I}$ for some y then $||Y||_{*,I} \leqslant A_T$, $||\widetilde{Y}||_{*,I} \leqslant A_T$ and letting

$$\bar{C}_{Y,T} = 1 + 2A_T$$

we can rewrite Eq. (47) as

$$\varepsilon_{Z,I} \leqslant K||\varphi||_{2,\delta}T^{\delta\gamma}C^3_{X,J}\bar{C}^2_{Y,T}\varepsilon_{Y,I}.$$

So choosing T small enough such that

$$T^{\delta\gamma}C^3_{X,J}\bar{C}^2_{Y,T} = \alpha < 1 \tag{48}$$

we have

$$||G(Y) - G(\widetilde{Y})||_{*,I} = \varepsilon_{Z,I} \leqslant \alpha\varepsilon_{Y,I} = \alpha||Y - \widetilde{Y}||_{*,I}.$$

Then G is a strict contraction in $\mathscr{D}^{\gamma,2\gamma}_X(I, V)$ and thus has a unique fixed-point. Again, patching together local solutions we get a global one defined on all J and belonging to $\mathscr{D}^{\gamma,2\gamma}_X(J, V)$.

Now let us discuss the continuity of the Itô map $F(y, \varphi, X, \mathbb{X}^2)$. Let $Y, \widetilde{Y}$ be the solutions based on $(X, \mathbb{X}^2)$ and $(\widetilde{X}, \widetilde{\mathbb{X}}^2)$ respectively. We have $Y = G(Y) = Z$, $\widetilde{Y} = \widetilde{G}(\widetilde{Y}) = \widetilde{Z}$ so that $\varepsilon_{Z,I} = \varepsilon_{Y,I}$ for any interval $I \subset J$ and we can use Eq. (47)

to write

$$\varepsilon_{Y,I} = \varepsilon_{Z,I} \leqslant K[(1 + ||\varphi||_{2,\delta}) C_{X,I}^2 C_{Y,I}^3 \rho_I + ||\varphi||_{2,\delta} T^{\delta\gamma} C_{X,I}^3 C_{Y,I}^2 \varepsilon_{Y,I}].$$

Fix T small enough for (48) to hold so that

$$\varepsilon_{Y,I} \leqslant (1-\alpha)^{-1} K (1 + ||\varphi||_{2,\delta}) C_{X,J}^2 C_{Y,J}^3 \rho_I = M_1 \rho_I.$$

Cover J with intervals $I_1, \dots, I_n$ of width T and let $J_k = \bigcup_{i=1}^k I_k$ with $J_n = J$.

To patch together the bounds for different I_i into a global bound for $\varepsilon_{Y,J}$ we use again Lemma 3 to estimate

$$\begin{aligned}
||R_Y - R_{\widetilde{Y}}||_{2\gamma,J_{i+1}} &\leqslant ||R_Y - R_{\widetilde{Y}}||_{2\gamma,J_i} + ||R_Y - R_{\widetilde{Y}}||_{2\gamma,I_{i+1}} + ||\delta Y' \delta X - \delta \widetilde{Y}' \delta \widetilde{X}||_{\gamma,\gamma,J_{i+1}} \\
&\leqslant ||R_Y - R_{\widetilde{Y}}||_{2\gamma,J_i} + ||R_Y - R_{\widetilde{Y}}||_{2\gamma,I_{i+1}} \\
&\quad + ||Y' - \widetilde{Y}'||_{2\gamma,J_{i+1}} ||X||_{\gamma,J} + ||\widetilde{Y}'||_{\gamma,J} ||X - \widetilde{X}||_{\gamma,J_{i+1}}
\end{aligned}$$

then we obtain easily that

$$\varepsilon_{Y,J_{i+1}} \leqslant C_{X,J}(\varepsilon_{Y,J_i} + \varepsilon_{Y,I_{i+1}}) + C_{Y,J} \rho_J.$$

Proceeding by induction we get

$$\begin{aligned}
\varepsilon_{Y,J_n} &\leqslant \left(C_{X,J} n + \sum_{k=1}^n C_{X,J}^k\right) \sup_i \varepsilon_{Y,I_i} + n C_{Y,J} \rho_J \\
&\leqslant \left[2 \sum_{k=1}^n C_{X,J}^k M_1 + n C_{Y,J}\right] \rho_J
\end{aligned}$$

which implies that there exists a constant M depending only on $C_{X,J}$, $C_{Y,J}$, $||\varphi||_{2,\delta}$ such that

$$\varepsilon_{Y,J} \leqslant M \rho_J. \qquad \square$$

6. Some probability

So far we have developed our arguments using only analytic and algebraic properties of paths. In this section we show how probability theory provides concrete examples of non-smooth paths for which the theory outlined above applies.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space where is defined a standard Brownian motion X with values in $V = \mathbb{R}^n$ (endowed with the Euclidean scalar product). It is well known that X is almost surely locally Hölder continuous for any exponent $\gamma < 1/2$, so that we can fix $\gamma < 1/2$ and choose a version of X living in $\mathscr{C}^\gamma(I, V)$ on any bounded interval I. In this case solutions $\mathbb{X}^2$ of Eq. (17) can be obtained by stochastic integration: let

$$W^{\mu\nu}_{\text{Itô},st} := \int_s^t (X^\mu_u - X^\mu_s)\, \hat{d}X^\nu_u,$$

where the hat indicates that the integral is understood in Itô's sense with respect to the forward filtration $\mathscr{F}_t = \sigma(X_s; s \leqslant t)$. Then it is easy to show that, for any $s, u, t \in \mathbb{R}$

$$W^{\mu\nu}_{\text{Itô},st} - W^{\mu\nu}_{\text{Itô},su} - W^{\mu\nu}_{\text{Itô},ut} = (X^\mu_u - X^\mu_s)(X^\nu_t - X^\nu_u) \tag{49}$$

which means that

$$NW^{\mu\nu}_{\text{Itô}} = \delta X^\mu \delta X^\nu.$$

Then we can choose a continuous version $\mathbb{X}^2_{\text{Itô}}$ of $(t,s) \mapsto W_{\text{Itô,st}}$ for which Eq. (49) holds a.s. for all $t, u, s \in \mathbb{R}$. It remains to show that $\mathbb{X}^2_{\text{Itô}} \in \Omega\mathscr{C}^{2\gamma}(I, V^{\otimes 2})$ (for any $\gamma < 1/2$ and bounded interval I).

To prove this result we will develop a small variation on a well-known argument first introduced by Garsia, Rodemich and Rumsey (cf. [5,9]) to control Hölder-like seminorms of continuous stochastic processes with a corresponding integral norm.

Fix an interval $T \subset \mathbb{R}$. A Young function ψ on $\mathbb{R}^+$ is an increasing, convex function such that $\psi(0) = 0$.

Lemma 4. *For any process $R \in \Omega\mathscr{C}(T)$ let*

$$U = \int_{T \times T} \psi\left(\frac{|R_{st}|}{p(|t-s|/4)}\right) dt\, ds,$$

where $p : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function with $p(0) = 0$ and ψ is a Young function. Assume there exists a constant C such that

$$\sup_{(u,v,r) \in [s,t]^3} |NR_{uvr}| \leqslant \psi^{-1}\left(\frac{C}{|t-s|^2}\right) p(|t-s|/4), \tag{50}$$

for any couple $s < t$ such that $[s,t] \subset T$. Then

$$|R_{st}| \leqslant 16 \int_0^{|t-s|} \left[\psi^{-1}\left(\frac{U}{r^2}\right) + \psi^{-1}\left(\frac{C}{r^2}\right)\right] dp(r) \tag{51}$$

for any $s, t \in T$.

Proof. See Appendix A, Section A.4. □

Remark 4. Lemma 4 reduces to well-known results in the case $NR = 0$ since we can take $C = 0$. Condition (50) is not very satisfying and we conjecture that an integral control over NR would suffice to obtain (51). However in its current formulation it is enough to prove the following useful corollary.

Corollary 4. *For any $\gamma > 0$ and $p \geqslant 1$ there exists a constant C such that for any $R \in \Omega\mathscr{C}$*

$$||R||_{\gamma,T} \leqslant C(U_{\gamma+2/p,p}(R,T) + ||NR||_{\gamma,T}), \tag{52}$$

where

$$U_{\gamma,p}(R,T) = \left[\int_{T\times T}\left(\frac{|R_{st}|}{|t-s|^{\gamma}}\right)^p dt\, ds\right]^{1/p}.$$

Proof. In the previous proposition take $\psi(x) = x^p$, $p(x) = x^{\gamma+2/p}$; the conclusion easily follows. □

In the case of $\mathbb{X}^2$ we have, fixed $T = [t_0, t_1] \in \mathbb{R}$, $t_0 < t_1$, and using the scaling properties of Brownian motion,

$$\begin{aligned}\mathbb{E}[U_{\gamma+2/p,p}(\mathbb{X}^2_{\text{Itô}}, T)^p] &= \mathbb{E}\int_{[t_0,t_1]^2} \frac{|\mathbb{X}^2_{\text{Itô},uv}|^p}{|u-v|^{p\gamma+2}}\, du\, dv \\ &= \mathbb{E}|\mathbb{X}^2_{\text{Itô},0\,1}|^p \int_{[t_0,t_1]^2} |u-v|^{p(1-\gamma-2/p)}\, du\, dv < \infty\end{aligned}$$

for any $\gamma < 1$ and $p > 1/(1-\gamma)$ so that, a.s. $U_{\gamma+2/p,p}(\mathbb{X}^2_{\text{Itô}}, T)$ is finite for any $\gamma < 1$ and p sufficiently large. Since

$$\sup_{(u,v,w):s\leqslant u\leqslant v\leqslant w\leqslant t} |(N\mathbb{X}^2_{\text{Itô}})_{uvw}| \leqslant \sup_{(u,v,w):s\leqslant u\leqslant v\leqslant w\leqslant t} |\delta X_{uv}||\delta X_{vw}| \leqslant ||X||^2_{\gamma,T}|t-s|^{2\gamma}$$

for any $t_0 \leqslant s \leqslant t \leqslant t_1$, we have from (52) that for any $\gamma < 1/2$, a.s.

$$||\mathbb{X}^2_{\text{Itô},st}(\omega)|| \leqslant C_{\gamma,T}(\omega)|t-s|^{2\gamma}$$

for any $t, s \in I$, where $C_{\gamma,T}$ is a suitable random constant. Then for any $\gamma < 1/2$ and bounded interval $I \subset \mathbb{R}$ we can choose a version such that $\mathbb{X}^2_{\text{Itô}} \in \Omega\mathscr{C}^{2\gamma}(I, V^{\otimes 2})$.

We can introduce

$$\mathbb{X}^{2,\mu\nu}_{\text{Strat.},st} := \int_s^t (X^\mu_u - X^\mu_s) \circ \hat{d} X^\nu_u,$$

where the integral is understood in Stratonovich sense, then by well-known results in stochastic integration, we have

$$\mathbb{X}^{2,\mu\nu}_{\text{Strat.},st} = \mathbb{X}^{2,\mu\nu}_{\text{Itô},st} + \frac{g^{\mu\nu}}{2}(t-s),$$

where $g^{\mu\nu} = 1$ if $\mu = \nu$ and $g^{\mu\nu} = 0$ otherwise. It is clear that, also in this case, we can select a continuous version of $\mathbb{X}^2_{\text{Strat.},st}$ which lives in $\Omega\mathscr{C}^{2\gamma}$ and such that $N\mathbb{X}^2_{\text{Strat.}} = \delta X \delta X$.

The connection between stochastic integrals and the integral we defined in Section 4 starting from a couple $(X, \mathbb{X}^2)$ is clarified in the next corollary:

Corollary 5. *Let* $\varphi \in C^{1,\delta}(V, V \otimes V^*)$ *with* $(1+\delta)\gamma > 1$, *then the Itô stochastic integral*

$$\delta I^\mu_{\text{Itô},st} = \int_s^t \varphi(X_u)^\mu_\nu \hat{d} X^\nu_u$$

has a continuous version which is a.s. equal to

$$\delta I^\mu_{\text{rough},st} = \int_s^t \varphi(X_u)^\mu_\nu \, dX^\nu_u$$

where the integral is understood in the sense of Theorem 1 *based on the rough path* $(X, \mathbb{X}^2_{\text{Itô}})$ *moreover the Stratonovich integral*

$$\delta I^\mu_{\text{Strat.},st} = \int_s^t \varphi(X_u)^\mu_\nu \circ \hat{d} X^\nu_u$$

is a.s. equal to the integral

$$\delta J^\mu_{st} = \int_s^t \varphi(X_u)^\mu_\nu \, dX^\nu_u$$

defined based on the couple $(X, \mathbb{X}^2_{\text{Strat.}})$ *and the following relation holds*

$$\delta J^\mu_{st} = \delta I^\mu_{\text{rough},st} + \frac{g^{\nu\kappa}}{2} \int_s^t \partial_\kappa \varphi(X_u)^\mu_\nu \, du$$

Proof. Recall that the Itô integral $\delta I_{\text{Itô}}$ is the limit in probability of the discrete sums

$$S^\mu_\Pi = \sum_i \varphi(X_{t_i})^\mu_\nu (X^\nu_{t_{i+1}} - X^\nu_{t_i})$$

while the integral δI_{rough} is the classical limit as $|\Pi| \to 0$ of

$$S'^\mu_\Pi = \sum_i [\varphi(X_{t_i})^\mu_\nu (X^\nu_{t_{i+1}} - X^\nu_{t_i}) + \partial_\kappa \varphi(X_{t_i})^\mu_\nu \mathbb{X}^{2,\kappa\nu}_{\text{Itô},t_i t_{i+1}}]$$

(cfr. Corollary 2). Then it will suffice to show that the limit in probability of

$$R^\mu_\Pi = \sum_i \partial_\kappa \varphi(X_{t_i})^\mu_\nu \mathbb{X}^{2,\kappa\nu}_{\text{Itô},t_i t_{i+1}}$$

is zero. Since we assume $\partial\varphi$ bounded it will be enough to show that $R_\Pi \to 0$ in $L^2(\Omega)$.

By a standard argument, using the fact that R_Π is a discrete martingale, we have

$$\mathbb{E}|R_\Pi|^2 = \sum_i \mathbb{E}|\partial_\kappa \varphi(X_{t_i})_\nu \mathbb{X}^{2,\kappa\nu}_{\text{Itô},t_i t_{i+1}}|^2 \leqslant ||\varphi||_{1,\delta} \sum_i \mathbb{E}|\mathbb{X}^2_{\text{Itô},t_i t_{i+1}}|^2$$
$$= ||\varphi||_{1,\delta} \mathbb{E}|\mathbb{X}^2_{\text{Itô},01}|^2 \sum_i |t_{i+1} - t_i|^2 \leqslant ||\varphi||_{1,\delta} \mathbb{E}|\mathbb{X}^2_{\text{Itô},01}|^2 |\Pi||t-s|$$

which implies that $\mathbb{E}|R_\Pi|^2 \to 0$ as $|\Pi| \to 0$.

As far as the integral δJ is concerned, we have that it is the classical limit of

$$S''^{\mu}_{\Pi} = \sum_i [\varphi(X_{t_i})^\mu_\nu (X^\nu_{t_{i+1}} - X^\nu_{t_i}) + \partial_\kappa \varphi(X_{t_i})^\mu_\nu \mathbb{X}^{2,\kappa\nu}_{\text{Strat.},t_i t_{i+1}}]$$
$$= \sum_i \left[\varphi(X_{t_i})^\mu_\nu (X^\nu_{t_{i+1}} - X^\nu_{t_i}) + \partial_\kappa \varphi(X_{t_i})^\mu_\nu \mathbb{X}^{2,\kappa\nu}_{\text{Itô},t_i t_{i+1}} + \frac{g^{\kappa\nu}}{2} \partial_\kappa \varphi(X_{t_i})^\mu_\nu (t_{i+1} - t_i)\right]$$
$$= S'^{\mu}_{\Pi} + \frac{g^{\kappa\nu}}{2} \sum_i \partial_\kappa \varphi(X_{t_i})^\mu_\nu (t_{i+1} - t_i)$$

so that

$$\delta I^\mu_{\text{rough},st} = \delta J^\mu_{st} - \frac{g^{\kappa\nu}}{2} \int_s^t \partial_\kappa \varphi(X_u)^\mu_\nu \, du$$

as claimed and then, by the relationship between Itô and Stratonovich integration:

$$\delta I^\mu_{\text{Itô},st} = \delta I^\mu_{\text{Strat.},st} - \frac{g^{\kappa\nu}}{2} \int_s^t \partial_\kappa \varphi(X_u)^\mu_\nu \, du$$

we get $\delta J = \delta I_{\text{Strat.}}$. □

7. Relationship with Lyons' theory of rough paths

The general abstract result given in Proposition 1 can also be used to provide alternative proofs of the main results in Lyons' theory of rough paths [7], i.e. the extension of multiplicative paths to any degree and the construction of a multiplicative path from an almost-multiplicative one. The main restriction is that we only consider control functions $\omega(t,s)$ (cfr. Lyons [7] for details and definitions) which are given by

$$\omega(t,s) = K|t-s|$$

for some constant K.

Given an integer n, $T^{(n)}(V)$ denote the truncated tensor algebra up to degree $n : T^{(n)}(V) \coloneqq \oplus^n_{k=0} V^{\otimes k}$, $V^{\otimes 0} = \mathbb{R}$. A tensor-valued path $Z : I^2 \to T^{(n)}(V)$ is of *finite p-variation* if

$$||Z^{\bar\mu}||_{|\bar\mu|/p} \leqslant K^{|\bar\mu|}, \qquad \forall \bar\mu : |\bar\mu| \leqslant n, \tag{53}$$

where $\bar{\mu}$ is a tensor multi-index. A path Z of degree n and finite p-variation is *almost multiplicative* (of roughness p) if $Z^{\emptyset} \equiv 1,\ n \geqslant \lfloor p \rfloor$ and

$$NZ^{\bar{\mu}} = \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} Z^{\bar{\nu}} Z^{\bar{\kappa}} + R^{\bar{\mu}} \tag{54}$$

with $R^{\bar{\mu}} \in \Omega\mathscr{C}_2^z(I, T^{(n)}(V))$ for some $z>1$ uniformly for all $\bar{\mu}$. By convention the summation $\sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}}$ does not include the terms where either $\bar{\nu} = \emptyset$ or $\bar{\kappa} = \emptyset$.

A path Z is *multiplicative* if $Z^{\emptyset} \equiv 1$ and

$$NZ^{\bar{\mu}} = \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} Z^{\bar{\nu}} Z^{\bar{\kappa}}. \tag{55}$$

Then the key result is contained in the following proposition:

Proposition 9. *If Z is an almost-multiplicative path of degree n and finite p-variation, $n \geqslant \lfloor p \rfloor$, then there exists a unique multiplicative path $\widetilde{Z}$ in $T^{(\lfloor p \rfloor)}(V)$ with finite p-variation such that*

$$\|Z^{\bar{\mu}} - \widetilde{Z}^{\bar{\mu}}\|_z \leqslant K \tag{56}$$

for some $z>1$ and all multi-index $\bar{\mu}$ such that $|\bar{\mu}| \leqslant \lfloor p \rfloor$.

Proof. Let us prove that there exists a multiplicative path $\widetilde{Z}$ such that

$$Z = \widetilde{Z} + Q \tag{57}$$

with $Q \in \Omega\mathscr{C}^z,\ z>1$. We proceed by induction: if $|\bar{\mu}| = 1$:

$$NZ^{\bar{\mu}}_{sut} = R^{\bar{\mu}}_{sut}$$

which, given that $R^{\bar{\mu}} \in \Omega\mathscr{C}_2^z,\ z>1$, implies that exists a unique $\widetilde{Z}^{\bar{\mu}}$ such that $N\widetilde{Z}^{\bar{\mu}} = 0$ and

$$Z^{\bar{\mu}} = \widetilde{Z}^{\bar{\mu}} + \Lambda R^{\bar{\mu}} = \widetilde{Z}^{\bar{\mu}} + Q^{\bar{\mu}}$$

with $Q^{\bar{\mu}} \in \Omega\mathscr{C}^z$. Then assume that Eq. (57) is true up to degree $j-1$ and let us show that it is true also for a multi-index $\bar{\mu}$ of degree j:

$$\begin{aligned} NZ^{\bar{\mu}} &= \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} Z^{\bar{\nu}} Z^{\bar{\kappa}} + R^{\bar{\mu}} \\ &= \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} (\widetilde{Z}^{\bar{\nu}} + Q^{\bar{\nu}})(\widetilde{Z}^{\bar{\kappa}} + Q^{\bar{\kappa}}) + R^{\bar{\mu}} \end{aligned}$$

$$= \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} \widetilde{Z}^{\bar{\nu}} \widetilde{Z}^{\bar{\kappa}} + \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} [Q^{\bar{\nu}} \widetilde{Z}^{\bar{\kappa}} + \widetilde{Z}^{\bar{\nu}} Q^{\bar{\kappa}} + Q^{\bar{\nu}} Q^{\bar{\kappa}}] + R^{\bar{\mu}}$$

$$= \sum_{\bar{\nu}\,\bar{\kappa}=\bar{\mu}} \widetilde{Z}^{\bar{\nu}} \widetilde{Z}^{\bar{\kappa}} + \widetilde{R}^{\bar{\mu}}.$$

If we can prove that $\widetilde{R}^{\bar{\mu}}$ is in the image of N, then writing

$$\widetilde{Z}^{\bar{\mu}} = Z^{\bar{\mu}} - \Lambda \widetilde{R}^{\bar{\mu}} = Z^{\bar{\mu}} + Q^{\bar{\mu}}$$

we obtain the multiplicative property for $\widetilde{Z}^{\bar{\mu}}$

$$N\widetilde{Z}^{\bar{\mu}} = \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} \widetilde{Z}^{\bar{\nu}}_{ut} \widetilde{Z}^{\bar{\kappa}}_{su}$$

with $|\bar{\mu}| = j$, and we are done since uniqueness is obvious. To prove $\widetilde{R}^{\bar{\mu}} \in \operatorname{Im} N$ we must show that $N_2 \widetilde{R}^{\bar{\mu}} = 0$:

$$N_2 \widetilde{R}^{\bar{\mu}} = N_2 \left[NZ^{\bar{\mu}} - \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} \widetilde{Z}^{\bar{\nu}} \widetilde{Z}^{\bar{\kappa}} \right] = N_2 \left[\sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} \widetilde{Z}^{\bar{\nu}} \widetilde{Z}^{\bar{\kappa}} \right]$$

$$= \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} N\widetilde{Z}^{\bar{\nu}} \widetilde{Z}^{\bar{\kappa}} - \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} \widetilde{Z}^{\bar{\nu}} N \widetilde{Z}^{\bar{\kappa}}$$

$$= \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} \sum_{\bar{\sigma}\bar{\tau}=\bar{\nu}} \widetilde{Z}^{\bar{\sigma}} \widetilde{Z}^{\bar{\tau}} \widetilde{Z}^{\bar{\kappa}} - \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} \sum_{\bar{\sigma}\bar{\tau}=\bar{\kappa}} \widetilde{Z}^{\bar{\nu}} \widetilde{Z}^{\bar{\sigma}} \widetilde{Z}^{\bar{\tau}} = 0,$$

where we used the Leibniz rule for N_2 (see Eq. (9)).

To finish we can take for the constant K in Eq. (56) the maximum of $||Q^{\bar{\mu}}||_z$ for all $|\bar{\mu}| \leqslant \lfloor p \rfloor$. □

Proposition 10. *Let Z be a multiplicative path of degree n and finite p-variation such that*

$$\sum_{\bar{\mu}:|\bar{\mu}|=k} ||Z^{\bar{\mu}}||_{k/p} \leqslant C \frac{\alpha^k}{k!} \tag{58}$$

for all $k \leqslant n$ and with $\alpha, C > 0$; then if $(n+1) > p$ and C is small enough (*see Eq.* (60)) *there exists a unique multiplicative extension of Z to any degree and Eq.* (58) *holds for every k.*

Proof. By induction we can assume that Z is a multiplicative path of degree k for which Eq. (58) holds up to degree k and prove that it can be extended to degree $k+1$ with the same bound. Note that $k \geqslant n$ and then $(k+1) > p$. For $|\bar{\mu}| = k+1$ we should have

$$NZ^{\bar{\mu}} = \sum_{\bar{\nu}\bar{\kappa}=\bar{\mu}} Z^{\bar{\nu}} Z^{\bar{\kappa}} \in \mathscr{Z}_2^{(k+1)/p}. \tag{59}$$

Since $(k+1)>p$, this equation has a unique solution $Z^{\bar\mu}\in\Omega\mathscr{C}^{(k+1)/p}(T^{k+1}(V))$. Then observe that, from Eq. (59)

$$Z^{\bar\mu}_{st}=Z^{\bar\mu}_{ut}+Z^{\bar\mu}_{su}+\sum_{\bar\nu\bar\kappa=\bar\mu}Z^{\bar\nu}_{su}Z^{\bar\kappa}_{ut}$$

and taking as u the mid-point between t and s we can bound $Z^{\bar\mu}$ as follows:

$$\sum_{|\bar\mu|=k+1}||Z^{\bar\mu}_{st}||_{(k+1)/p}\leqslant\frac{2}{2^{(k+1)/p}}\sum_{|\bar\mu|=k+1}||Z^{\bar\mu}_{st}||_{(k+1)/p}+C^2\alpha^{k+1}\sum_{i=1}^{k}\frac{2^{-i/p}}{i!}\frac{2^{-(k+1-i)/p}}{(k+1-i)!}.$$

Now,

$$\sum_{i=0}^{k+1}\frac{2^{-i/p}}{i!}\frac{2^{-(k+1-i)/p}}{(k+1-i)!}\leqslant\sum_{i=0}^{k+1}\frac{2^{-i}}{i!}\frac{2^{-(k+1-i)}}{(k+1-i)!}+2\sum_{i=0}^{\lfloor p\rfloor}\frac{(2^{-(k+1-i)/p}2^{-i/p}-2^{-(k+1-i)}2^{-i})}{i!(k+1-i)!}$$

$$=\frac{1}{(k+1)!}\left[1+2\sum_{i=0}^{\lfloor p\rfloor}\frac{(k+1)!}{i!(k+1-i)!}(2^{-(k+1)/p}-2^{-(k+1)})\right]$$

$$\leqslant\frac{1+D_pk^{\lfloor p\rfloor}2^{-(k+1)/p}}{(k+1)!}$$

which gives

$$\sum_{|\bar\mu|=k+1}||Z^{\bar\mu}_{st}||_{(k+1)/p}\leqslant C^2\frac{(2^{(k+1)/p}-2)}{2^{(k+1)/p}}\frac{(1+D_pk^{\lfloor p\rfloor}2^{-(k+1)/p})\alpha^{k+1}}{(k+1)!}\leqslant C\frac{\alpha^{k+1}}{(k+1)!}$$

whenever C is such that

$$0<C\leqslant\min_{k\geqslant n}\frac{2^{(k+1)/p}}{(2^{(k+1)/p}-2)(1+D_pk^{\lfloor p\rfloor}2^{-(k+1)/p})}. \tag{60}$$

This concludes the proof of the induction step. □

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Appendix A. Some proofs

A.1. Proof of Proposition 1

The basic technique to prove the existence of the map Λ is borrowed form [2]. Let $\eta(x)$ be a smooth function on $\mathbb{R}$ with compact support and $\eta_\alpha(x) := \alpha^{-1}\eta(x/\alpha)$.

Define

$$(\Lambda_\beta A)_{st} := -\int_s^t dx \int\int d\tau\, d\sigma \mathscr{F}_\beta(x,s;\tau,\sigma) A_{\tau x\sigma},$$

where

$$\mathscr{F}_\beta(x,s;\tau,\sigma) := [\eta_\beta(x-\tau) - \eta_\beta(s-\tau)]\partial_x \eta_\beta(x-\sigma)$$

and the integrals in τ and σ are extended over all $\mathbb{R}$.

Given that $A \in \mathscr{Z}_2$ there exists $R \in \Omega\mathscr{C}$ such that $NR = A$ and

$$\begin{aligned}(\Lambda_\beta A)_{st} &= -\int_s^t dx \int\int d\tau\, d\sigma \mathscr{F}_\beta(x,s;\tau,\sigma)(R_{\tau\sigma} - R_{\tau x} - R_{x\sigma}) \\ &= -\int_s^t dx \int\int d\tau\, d\sigma \mathscr{F}_\beta(x,s;\tau,\sigma) R_{\tau\sigma}\end{aligned}$$

since the other terms vanish after the integrations in τ or σ. Then the following decomposition holds:

$$\Lambda_\beta A = \tilde{R}_\beta + \delta\Phi_\beta(R), \tag{A.1}$$

where

$$(\tilde{R}_\beta)_{st} := \int\int d\tau\, d\sigma \eta_\beta(s-\tau)[\eta_\beta(t-\sigma) - \eta_\beta(s-\sigma)] R_{\tau\sigma}$$

and

$$\delta\Phi_\beta(R)_{st} := -\int_s^t dx \int\int d\sigma\, d\tau \eta_\beta(x-\tau)\partial_x \eta_\beta(x-\sigma) R_{\tau\sigma}.$$

In Eq. (A.1) the l.h.s. depends only on $A = NR$ while each of the terms in the r.h.s. depends explicitly on R. We have $N\Lambda_\beta A = N\tilde{R}_\beta$ and since $\lim_{\beta\to 0}\tilde{R}_\beta = R$ pointwise we have that $\lim_{\beta\to 0} N\Lambda_\beta A = NR = A$. So every accumulation point X of $\Lambda_\beta A$ will solve the equation $NX = A$. Moreover if it exists $X \in \Omega\mathscr{C}^z$ with $z > 1$ and $NX = A$ then it is unique and $\lim_{\beta\to 0}\Lambda_\beta R = X$ in $\Omega\mathscr{C}^1$ since in this case

$$\Lambda_\beta A = \tilde{R}_\beta + \delta\Phi_\beta(R) = \tilde{X}_\beta + \delta\Phi_\beta(X)$$

and it is easy to prove that $\Phi_\beta(X) \to 0$ in $\mathscr{C}^1$.

Now we will prove that $\lim_{\beta\to 0}\Lambda_\beta A$ exists when $A\in\mathscr{Z}_2^z$ with $z>1$.

Define $f_\tau:\mathbb{R}^2\times\mathbb{R}_+\to V$ as $f_\tau(x,y,\alpha):=\eta_\alpha(x-\tau)$ and $g_\sigma:\mathbb{R}^2\times\mathbb{R}_+\to V$ as $g_\sigma(x,y,\alpha):=\eta_\alpha(y-\sigma)$. Apply Stokes Theorem to the exact differential 2-form $\omega:=df_\tau\wedge dg_\sigma=d(f_\tau dg_\sigma)$ on $D:=\Delta_{t,s}\times[\beta,\beta']$ where $\Delta_{t,s}=\{(x,y)\in\mathbb{R}^2: s<x<y<t\}$. Then

$$\int_{\partial D}\omega=\int_D d\omega=0,$$

where the boundary $\partial D=-c_1+c_2+c_3$ is composed of $c_1=\Delta_{t,s}\times\{\beta\}$, $c_2=\Delta_{t,s}\times\{\beta'\}$, $c_3=\partial\Delta_{t,s}\times[\beta,\beta']$. So

$$\int_{\Delta_{t,s}}\omega|_{\alpha=\beta}=\int_{\Delta_{t,s}}\omega|_{\alpha=\beta'}+\int_{\partial\Delta_{t,s}\times[\beta,\beta']}\omega$$

giving

$$\int_s^t\mathscr{F}_\beta(x,s;\tau,\sigma)\,dx=\int_s^t\mathscr{F}_{\beta'}(x,s;\tau,\sigma)\,dx+\int_\beta^{\beta'}d\alpha\int_s^t\mathscr{K}(\alpha,x,t,s;\tau,\sigma)\,dx$$

with

$$\begin{aligned}\mathscr{K}(\alpha,x,t,s;\tau,\sigma)&=\partial_\alpha[\eta_\alpha(x-\sigma)-\eta_\alpha(s-\sigma)]\partial_x\eta_\alpha(x-\tau)\\&\quad+\partial_\alpha[\eta_\alpha(t-\tau)-\eta_\alpha(x-\tau)]\partial_x\eta_\alpha(x-\sigma).\end{aligned}$$

Then

$$\Lambda_\beta A_{st}=\Lambda_{\beta'}A_{st}-\int_\beta^{\beta'}d\alpha\int_s^t dx\int\int d\tau\,d\sigma\,\mathscr{K}(\alpha,x,t,s;\tau,\sigma)R_{\tau\sigma}.\tag{A.2}$$

Assume we can write $A=\sum_{i=1}^n A_i$ where $A_i\in\Omega\mathscr{C}_2^{\rho_i,z-\rho_i}$ for a choice of n and $\rho_i>0$, $i=1,\dots,n$. Write $\rho_i'=z-\rho_i$.

Then consider

$$\begin{aligned}I(\alpha)&=-\int_s^t dx\int\int d\tau\,d\sigma\mathscr{K}(\alpha,x,t,s;\tau,\sigma)R_{\tau\sigma}\\&=\int\int d\tau\,d\sigma\{\partial_\alpha\eta_\alpha(s-\sigma)[\eta_\alpha(t-\tau)-\eta_\alpha(s-\tau)]\\&\quad-\partial_\alpha\eta_\alpha(t-\tau)[\eta_\alpha(t-\sigma)-\eta_\alpha(s-\sigma)]\}R_{\tau\sigma}\\&\quad+\int_s^t dx\int\int d\tau\,d\sigma[\partial_\alpha\eta_\alpha(x-\tau)\partial_x\eta(x-\sigma)-\partial_\alpha\eta_\alpha(x-\sigma)\partial_x\eta(x-\tau)]R_{\tau\sigma}\end{aligned}$$

$$= \int\int d\tau\, d\sigma \partial_\alpha \eta_\alpha(\sigma)\eta_\alpha(\tau)[R_{t+\tau,s+\sigma} - R_{s+\tau,s+\sigma} - R_{t+\tau,t+\sigma} + R_{t+\tau,s+\sigma}]$$

$$+ \int_s^t dx \int\int d\tau\, d\sigma \partial_\alpha \eta_\alpha(\tau)\partial_\sigma \eta(\sigma)[R_{x+\sigma,x+\tau} - R_{x+\tau,x+\sigma}]$$

$$= \int\int d\tau\, d\sigma \partial_\alpha \eta_\alpha(\sigma)\eta_\alpha(\tau)[NR_{t+\tau,s+\tau,s+\sigma} + NR_{t+\tau,t+\sigma,s+\sigma}]$$

$$+ \int_s^t dx \int\int d\tau\, d\sigma \partial_\alpha \eta_\alpha(\tau)\partial_\sigma \eta(\sigma)[NR_{x+\sigma,x,x+\tau} - NR_{x+\tau,x,x+\sigma}]$$

so that we can bound

$$|I(\alpha)| \leqslant \int\int d\tau\, d\sigma |\partial_\alpha \eta_\alpha(\sigma)||\eta_\alpha(\tau)|[|NR_{t+\tau,s+\tau,s+\sigma}| + |NR_{t+\tau,t+\sigma,s+\sigma}|]$$

$$+ \int_s^t dx \int\int d\tau\, d\sigma |\partial_\alpha \eta_\alpha(\tau)||\partial_\sigma \eta(\sigma)|[|NR_{x+\sigma,x,x+\tau}| + |NR_{x+\tau,x,x+\sigma}|]$$

$$\leqslant \sum_{i=1}^n ||A_i||_{\rho_i,\rho_i'} \int\int d\tau\, d\sigma |\partial_\alpha \eta_\alpha(\sigma)||\eta_\alpha(\tau)|[|t-s|^{\rho_i}|\tau-\sigma|^{\rho_i'} + |\tau-\sigma|^{\rho_i}|t-s|^{\rho_i'}]$$

$$+ \sum_{i=1}^n ||A_i||_{\rho_i,\rho_i'} \int_s^t dx \int\int d\tau\, d\sigma |\partial_\alpha \eta_\alpha(\tau)||\partial_\sigma \eta(\sigma)|[|\sigma|^{\rho_i}|\tau|^{\rho_i'} + |\tau|^{\rho_i}|\sigma|^{\rho_i'}],$$

where each term can be bounded as follows:

$$\int\int d\tau\, d\sigma |\partial_\alpha \eta_\alpha(\sigma)||\eta_\alpha(\tau)||\tau-\sigma|^a = \alpha^{a-1} \int\int d\tau\, d\sigma |\eta(\sigma) - \sigma\eta'(\sigma)||\eta(\tau)|\,|\tau-\sigma|^a \leqslant K\alpha^{a-1},$$

$$\int\int d\tau |\partial_\alpha \eta_\alpha(\tau)||\tau|^a = \alpha^{a-1} \int\int d\tau |\eta(\tau) - \tau\eta'(\tau)||\tau|^a \leqslant K^{1/2}\alpha^{a-1}$$

for a suitable constant $K > 0$ and obtain

$$|I(\alpha)| \leqslant K \sum_{i=1}^n (\alpha^{\rho_i - 1}|t-s|^{\rho_i'} + \alpha^{\rho_i' - 1}|t-s|^{\rho_i})||A_i||_{\rho_i,\rho_i'} + K|t-s|\sum \alpha^{z-2}||A_i||_{\rho_i,\rho_i'}.$$

Upon integration in α we get:

$$\int_0^1 |I(\alpha)|\, d\alpha \leqslant K \sum_{i=1}^n ||A_i||_{\rho_i,\rho_i'}$$

if $|t-s| \leqslant 1$. By dominated convergence of the integral in Eq. (A.2),

$$\lim_{\beta\to 0} \Lambda_\beta A =: \Lambda A$$

exists (in $\Omega\mathscr{C}$ uniformly in bounded intervals). If we also observe that

$$|(\Lambda_{\beta'}A)_{st}| \leqslant K(\beta')^{-1}|t-s| \sum_{i=1}^{n} ||A_i||_{\rho_i,\rho'_i}$$

we get that

$$|(\Lambda A)_{t,s}| \leqslant K \sum_{i=1}^{n} ||A_i||_{\rho_i,\rho'_i}$$

for $|t-s| \leqslant 1$.

Finally, let $J_{t,s}(x) \coloneqq s+(t-s)(0 \vee (x \wedge 1))$ and $(J^*_{t,s}X)_{u,v,w} \coloneqq X_{J_{t,s}(u),J_{t,s}(v),J_{t,s}(w)}$ for all $X \in \Omega\mathscr{C}_2$. Then

$$||J^*_{t,s}X||_{\gamma,\gamma'} \leqslant |t-s|^{\gamma+\gamma'}||X||_{\gamma,\gamma'}.$$

Since $\Lambda_\beta A_{t,s} = (J^*_{t,s}\Lambda_{|t-s|\beta}A)_{0,1} = \Lambda_{|t-s|\beta}(J^*_{t,s}A)_{0,1}$ and

$$|(\Lambda(J^*_{t,s}R))_{1,0}| \leqslant K \sum_{i=1}^{n} ||J^*_{t,s}A_i||_{\rho_i,\rho'_i}$$

this is enough to obtain the desired regularity:

$$|(\Lambda A)_{t,s}| \leqslant K|t-s|^z \sum_{i=1}^{n} ||A_i||_{\rho_i,\rho'_i}.$$

The constant K can be chosen to be equal to $1/(2^z-2)$. Let $\Phi = \sum_{i=1}^n ||A_i||_{\rho_i,\rho'_i}$. and $R = \Lambda A$ and since $NR = A$ write

$$R_{st} = R_{ut} + R_{su} + \sum_i A_{i,sut}$$

with $t > u > s$ and $u = s + |t-s|/2$. Then estimate

$$\begin{aligned}|R_{st}| &\leqslant |R_{ut}| + |R_{su}| + \sum_i |A_{i,sut}| \\ &\leqslant ||R||_z(|t-u|^z + |u-s|^z) + \sum_i ||A_i||_{\rho_i,\rho'_i}|u-s|^{\rho_i}|t-u|^{\rho'_i} \\ &= \frac{2||R||_z + \Phi}{2^z}|t-s|^z\end{aligned}$$

so that

$$||R||_z \leqslant \frac{1}{2^z-2}\Phi.$$

A.2. Some proofs for Section 4

A.2.1. Proof of Lemma 1

Proof. Write down the decomposition for Z and Y:

$$\delta Z^\mu = F^\mu_\nu \delta Y^\nu + R^\mu_{ZY},$$

$$\delta Y^\mu = G^\mu_\nu \delta X^\nu + R^\mu_Y$$

where $F \in \mathscr{C}^{\eta-\gamma}(I, V \otimes V^*)$, $G \in \mathscr{C}^{\sigma-\gamma}(I, V)$, $R_{ZY} \in \Omega\mathscr{C}^\eta(I, V)$ and $R_Y \in \Omega\mathscr{C}^\sigma(I, V)$, then

$$\delta Z^\mu = F^\mu_\nu G^{\nu\kappa} \delta X^\kappa + R^\mu_{ZY} + F^\mu_\nu R^\nu_Y = Z'^\mu_\kappa \delta X^\kappa + R^\mu_{ZX}$$

with $Z'^\mu_\kappa = F^\mu_\nu G^\nu_\kappa$ and $R^\mu_{ZX} = R^\mu_{ZY} + F^\mu_\nu R^\nu_Y$. Let $\delta = \min(\sigma, \eta)$ and note that for R_{ZY} we have

$$||R_{ZY}||_{\eta,I} \leqslant ||Z||_{D(Y,\gamma,\eta),I},$$

$$||R_{ZY}||_{\gamma,I} \leqslant ||Z||_{\gamma,I} + ||F||_{\infty,I} ||Y||_{\gamma,I} \leqslant ||Z||_{D(Y,\gamma,\eta),I}(1 + ||Y||_{\gamma,I})$$

and by interpolation we obtain $(a = (\eta - \delta)/(\eta - \gamma) \leqslant 1)$

$$||R_{ZY}||_{\delta,I} \leqslant ||R_{ZY}||^{1-a}_{\eta,I} ||R_{ZY}||^a_{\gamma,I} \leqslant ||Z||_{D(Y,\gamma,\eta),I}(1 + ||Y||_{\gamma,I})^a \leqslant ||Z||_{D(Y,\gamma,\eta),I}(1 + ||Y||_{\gamma,I})$$

and similarly

$$||R_Y||_{\delta,I} \leqslant ||Y||_{D(X,\gamma,\sigma),I}(1 + ||X||_{\gamma,I});$$

moreover,

$$||F||_{0,I} = \sup_{t,s \in I} |F_t - F_s| \leqslant \sup_{t,s \in I} (|F_t| + |F_s|) = 2||F||_{\infty,I} \leqslant 2||Z||_{D(Y,\gamma,\eta),I}$$

so, again by interpolation, we find

$$||F||_{\delta-\gamma,I} \leqslant ||Z||_{D(Y,\gamma,\eta),I} 2^{1-(\delta-\gamma)/(\sigma-\gamma)} \leqslant 2||Z||_{D(Y,\gamma,\eta),I}$$

and

$$||G||_{\delta-\gamma,I} \leqslant 2||Y||_{D(X,\gamma,\sigma),I}.$$

To finish bound the norm of Z, Z' as

$$||(Z, Z')||_{D(X,\gamma,\delta),I} = ||Z'||_{\infty,I} + ||Z'||_{\delta-\gamma,I} + ||R_{ZX}||_{\delta,I} + ||Z||_{\gamma,I}$$

$$\leqslant ||F||_{\infty,I} ||G||_{\infty,I} + ||F||_{\delta-\gamma,I} ||G||_{\infty,I}$$

$$+ \|F\|_{\infty,I}\|G\|_{\delta-\gamma,I} + \|R_{ZY}\|_{\delta,I} + \|F\|_{\infty,I}\|R_Y\|_{\delta,I} + \|Z\|_{\gamma,I}$$
$$\leqslant K\|Z\|_{D(Y,\gamma,\eta),I}(1+\|Y\|_{D(X,\gamma,\sigma),I})(1+\|X\|_{\gamma,I}). \qquad \square$$

A.2.2. Proof of Proposition 4

Let $y(r) = (Y_t - Y_s)r + Y_s$ so that

$$\begin{aligned} Z_t^\mu - Z_s^\mu &= \varphi(y(1))^\mu - \varphi(y(0))^\mu = \int_0^1 \partial_\nu \varphi(y(r))^\mu y'(r)^\nu \, dr \\ &= (Y_t^\nu - Y_s^\nu) \int_0^1 \partial_\nu \varphi(y(r))^\mu \, dr \\ &= \partial_\nu \varphi(Y_s)^\mu (Y_t^\nu - Y_s^\nu) + (Y_t^\nu - Y_s^\nu) \int_0^1 [\partial_\nu \varphi(y(r))^\mu - \partial_\nu \varphi(Y_s)^\mu] \, dr \end{aligned}$$

Then if $\delta Y^\mu = Y'^\mu_\nu \delta X^\nu + R^\mu$ we have

$$\begin{aligned} Z_t^\mu - Z_s^\mu &= \partial_\nu \varphi(Y_s)^\mu Y'^\nu_{\kappa,s}(X_t^\kappa - X_s^\kappa) + \partial_\nu \varphi(Y_s)^\mu R^\nu_{st} + (Y_t^\nu - Y_s^\nu) \int_0^1 [\partial_\nu \varphi(y(r))^\mu - \partial_\nu \varphi(Y_s)^\mu] \, dr \\ &= Z'^\mu_{\kappa,s}(X_t^\kappa - X_s^\kappa) + R^\mu_{Z,st} \end{aligned} \tag{A.3}$$

with $Z'^\mu_{\kappa,s} = \partial_\nu \varphi(Y_s)^\mu Y'^\nu_{\kappa,s}$,

$$\begin{aligned} \|Z'\|_{\sigma-\gamma} &\leqslant \|\partial\varphi(Y_\cdot)\|_{\sigma-\gamma}\|Y'\|_\infty + \|\partial\varphi(Y_\cdot)\|_\infty\|Y'\|_{\sigma-\gamma} \\ &\leqslant (\|\partial\varphi(Y_\cdot)\|_{\delta\gamma} + \|\partial\varphi(Y_\cdot)\|_0)\|Y'\|_\infty + \|\partial\varphi(Y_\cdot)\|_\infty(\|Y'\|_{\eta-\gamma} + \|Y'\|_0) \\ &\leqslant \|\varphi\|_{1,\delta}(\|Y\|_\gamma^\delta + 2)\|Y'\|_\infty + 2\|\varphi\|_{1,\delta}(\|Y'\|_{\eta-\gamma} + 2\|Y'\|_\infty) \\ &\leqslant K\|\varphi\|_{1,\delta}(\|Y\|_{D(X,\gamma,\eta)} + \|Y\|^{1+\delta}_{D(X,\gamma,\eta)}). \end{aligned}$$

As far as R_Z is concerned we have

$$\begin{aligned} |R_{Z,st}| &= |Y_t - Y_s|\left[\int_0^1 |\partial\varphi(y(r)) - \partial\varphi(Y_s)| dr\right] \\ &\leqslant \|\varphi\|_{1,\delta}\left|\int_0^1 r^\delta dr\right| |Y_t - Y_s|^{1+\delta} \leqslant K\|\varphi\|_{1,\delta}\|Y\|_\gamma^{1+\delta}|t-s|^{\gamma(1+\delta)}; \end{aligned}$$

and

$$|R_{Z,st}| = |Y_t - Y_s|\left[\int_0^1 |\partial\varphi(y(r)) - \partial\varphi(Y_s)| dr\right] \leqslant K\|\varphi\|_{1,\delta}\|Y\|_\gamma |t-s|^\gamma.$$

Interpolating these two inequalities we get

$$||R_Z||_\sigma \leqslant K||\varphi||_{1,\delta}||Y||_\gamma^{\sigma/\gamma} \leqslant K||\varphi||_{1,\delta}||Y||_{D(X,\gamma,\sigma)}^{\sigma/\gamma}$$

which together with the obvious bound

$$||Z||_\gamma \leqslant ||\varphi||_{1,\delta}||Y||_\gamma$$

implies

$$||Z||_{D(X,\gamma,\sigma)} \leqslant K||\varphi||_{1,\delta}(||Y||_{D(X,\gamma,\eta)} + ||Y||_{D(X,\gamma,\eta)}^{1+\delta} + ||Y||_{D(X,\gamma,\eta)}^{\sigma/\gamma}).$$

If $\delta\widetilde{Y}^\mu = \widetilde{Y}'^\mu_\nu \delta X^\nu + \widetilde{R}^\mu$ is another path, $\widetilde{Z}_t = \varphi(\widetilde{Y}_t)$ and $H = Z - \widetilde{Z}$ we have (see Eq. (A.3)):

$$\delta H^\mu = H'^\mu_\kappa \delta X^\kappa + A^\mu + B^\mu \tag{A.4}$$

with

$$H'^\mu_\kappa = \partial_\nu\varphi(Y)^\mu Y'^\nu_\kappa - \partial_\nu\varphi(\widetilde{Y})^\mu \widetilde{Y}'^\nu_\kappa,$$

$$A^\mu_{st} = \partial_\nu\varphi(Y_s)^\mu R^\nu_{st} - \partial_\nu\varphi(\widetilde{Y}_s)^\mu \widetilde{R}^\nu_{st}$$

and

$$\begin{aligned} B^\mu_{st} &= \delta Y^\nu_{st} \int_0^1 [\partial_\nu\varphi(y(r))^\mu - \partial^\nu\varphi(y(0))^\mu]\,dr - \delta\widetilde{Y}^\nu_{st}\int_0^1 [\partial_\nu\varphi(\widetilde{y}(r))^\mu - \partial_\nu\varphi(\widetilde{y}(0))^\mu]\,dr \\ &= \delta(Y - \widetilde{Y})^\nu_{st} \int_0^1 [\partial_\nu\varphi(y(r))^\mu - \partial_\nu\varphi(y(0))^\mu]\,dr \\ &\quad + \delta\widetilde{Y}^\nu_{st}\int_0^1 [\partial_\nu\varphi(y(r))^\mu - \partial_\nu\varphi(\widetilde{y}(r))^\mu - \partial_\nu\varphi(y(0))^\mu + \partial_\nu\varphi(\widetilde{y}(0))^\mu]\,dr. \end{aligned}$$

Let $y(r,r') = (y(r) - \widetilde{y}(r))r' + \widetilde{y}(r)$ and bound the second integral as

$$\begin{aligned} &\left|\int_0^1 dr[\partial_\nu\varphi(y(r))^\mu - \partial_\nu\varphi(\widetilde{y}(r))^\mu - \partial_\nu\varphi(y(0))^\mu + \partial_\nu\varphi(\widetilde{y}(0))^\mu]\right| \\ &= \left|\int_0^1 dr\int_0^1 dr'[\partial_\kappa\partial_\nu\varphi(y(r,r'))^\mu - \partial_\kappa\partial_\nu\varphi(y(0,r'))^\mu](y(r) - \widetilde{y}(r))^\kappa\right| \\ &\leqslant ||\varphi||_{2,\delta}\int_0^1 dr\int_0^1 dr'|y(r,r') - y(0,r')|^\delta|y(r) - \widetilde{y}(r)| \\ &\leqslant K||\varphi||_{2,\delta}(||Y||_\gamma + ||\widetilde{Y}||_\gamma)^\delta||Y - \widetilde{Y}||_\infty|t-s|^{\gamma\delta}, \end{aligned}$$

then

$$||B||_{(1+\delta)\gamma} \leqslant ||Y - \widetilde{Y}||_\gamma||\varphi||_{2,\delta}||Y||_\gamma^\delta + K||\widetilde{Y}||_\gamma||\varphi||_{2,\delta}(||Y||_\gamma + ||\widetilde{Y}||_\gamma)^\delta||Y - \widetilde{Y}||_\infty$$

and in the same way it is possible to obtain

$$||B||_\gamma \leqslant ||\varphi||_{2,\delta}(||Y - \widetilde{Y}||_\gamma + ||\widetilde{Y}||_\gamma ||Y - \widetilde{Y}||_\infty).$$

Moreover

$$||H'||_\infty \leqslant ||\varphi||_{2,\delta} ||Y' - \widetilde{Y}'||_\infty + ||Y'||_\infty ||\varphi||_{2,\delta} ||Y - \widetilde{Y}||_\infty,$$

$$||H'||_{\gamma\delta} \leqslant ||\varphi||_{2,\delta} ||Y' - \widetilde{Y}'||_{\gamma\delta} + ||Y'||_{\gamma\delta} ||\varphi||_{2,\delta} ||Y - \widetilde{Y}||_\infty$$

and

$$\begin{aligned} ||A||_\gamma &\leqslant ||\varphi||_{2,\delta} ||R - \widetilde{R}||_\gamma + ||R||_\gamma ||\varphi||_{2,\delta} ||Y - \widetilde{Y}||_\infty \\ &\leqslant ||\varphi||_{2,\delta}(||Y' - \widetilde{Y}'||_\infty ||X||_\gamma + ||Y - \widetilde{Y}||_\gamma) + (||Y'||_\infty ||X||_\gamma + ||Y||_\gamma)||\varphi||_{2,\delta} ||Y - \widetilde{Y}||_\infty \end{aligned}$$

$$||A||_{(1+\delta)\gamma} \leqslant ||\varphi||_{2,\delta} ||R - \widetilde{R}||_{(1+\delta)\gamma} + ||R||_{(1+\delta)\gamma} ||\varphi||_{2,\delta} ||Y - \widetilde{Y}||_\infty.$$

And collecting all these results together we end up with

$$||Z - \widetilde{Z}||_{D(X,\gamma,(1+\delta)\gamma)} \leqslant C||Y - \widetilde{Y}||_{D(X,\gamma,(1+\delta)\gamma)}$$

with

$$C = K||\varphi||_{2,\delta}(1 + ||X||_\gamma)(1 + ||Y||_{D(X,\gamma,(1+\delta)\gamma)} + ||\widetilde{Y}||_{D(X,\gamma,(1+\delta)\gamma)})^{1+\delta}.$$

To finish consider the case in which $\delta \widetilde{Y}^\mu = \widetilde{Y}'^\mu_\nu \delta \widetilde{X}^\nu + \widetilde{R}^\mu_{\widetilde{Y}}$ is a path controlled by $\widetilde{X}$. If we let again $\widetilde{Z}_t = \varphi(\widetilde{Y}_t)$ and $H = Z - \widetilde{Z}$ we have

$$\delta H^\mu = \partial_\nu \varphi(\widetilde{Y}_\cdot)^\mu \widetilde{Y}'^\nu_\kappa \delta(X^\kappa - \widetilde{X}^\kappa) + H'^\mu_\kappa \delta X^\kappa + A^\mu + B^\mu$$

where the only difference with the expression in Eq. (A.4) is in the first term in the r.h.s. then

$$||Z - \widetilde{Z}||_\gamma + ||Z' - \widetilde{Z}'||_{\delta\gamma} + ||R_Z - R_{\widetilde{Z}}||_{(1+\delta)\gamma} + ||Z' - \widetilde{Z}'||_\infty \leqslant C(\varepsilon + ||X - \widetilde{X}||_\gamma)$$

with

$$\varepsilon = ||Y - \widetilde{Y}||_\gamma + ||Y' - \widetilde{Y}'||_{\delta\gamma} + ||R_Y - R_{\widetilde{Y}}||_{(1+\delta)\gamma} + ||Y' - \widetilde{Y}'||_\infty$$

and this concludes the proof of Proposition 4. □

A.3. Some proofs and lemmata used in Section 5

A.3.1. Proof of Lemma 3

Proof. Take $u \in I \cap J$:

$$\begin{aligned}\sup_{t\in I\setminus J, s\in J\setminus I}\frac{|X_{st}|}{|t-s|^\gamma} &\leqslant \sup_{t\in I\setminus J, s\in J\setminus I}\frac{|X_{ut}|+|X_{su}|+|(NX)_{sut}|}{|t-s|^\gamma}\\ &\leqslant \sup_{t\in I\setminus J, s\in J\setminus I}\frac{|X_{ut}|}{|t-s|^\gamma}+\sup_{t\in I\setminus J, s\in J\setminus I}\frac{|X_{su}|}{|t-s|^\gamma}+\sup_{t\in I\setminus J, s\in J\setminus I}\frac{|(NX)_{sut}|}{|t-s|^\gamma}\\ &\leqslant \sup_{t\in I\setminus J, s\in J\setminus I}\frac{|X_{ut}|}{|t-u|^\gamma}+\sup_{t\in I\setminus J, s\in J\setminus I}\frac{|X_{su}|}{|u-s|^\gamma}+\sup_{t\in I\setminus J, s\in J\setminus I}\frac{|(NX)_{sut}|}{|t-u|^{\gamma_2}|s-u|^{\gamma_2}}\\ &\leqslant ||X||_{\gamma,I}+||X||_{\gamma,J}+||X||_{\gamma_1,\gamma_2,I\cup J}\end{aligned}$$

then

$$\begin{aligned}||X||_{\gamma,I\cup J} = \sup_{t,s\in I\cup J}\frac{|X_{st}|}{|t-s|^\gamma} &\leqslant \sup_{t,s\in I}\frac{|X_{st}|}{|t-s|^\gamma}+\sup_{t,s\in J}\frac{|X_{st}|}{|t-s|^\gamma}+\sup_{t\in I\setminus J, s\in J\setminus I}\frac{|X_{st}|}{|t-s|^\gamma}\\ &\leqslant 2(||X||_{\gamma,I}+||X||_{\gamma,J})+||X||_{\gamma_1,\gamma_2,I\cup J}\end{aligned}$$

as claimed. □

A.3.2. Lemmata for some bounds on the map G

With the notation in the proof of Proposition 5 we have

Lemma A.1. *For any interval $I=[t_0,t_0+T]\subseteq J$ such that $T<1$ the following bound holds*

$$\varepsilon_{Z,I} \leqslant K C_{X,I} C_{Y,I}^{\delta}[(1+||\varphi||_{1,\delta})\rho_I + T^{\gamma\delta}\varepsilon_{Y,I}] \tag{A.5}$$

Proof. Consider first the case when $\varphi=\widetilde{\varphi}$. Eq. (A.6) is a statement of continuity of the integral defined in Proposition 3 is a bounded bilinear application $(A,B)\mapsto \int A\,dB$ then it is also continuous in both arguments and it is easy to check that

$$||Q_Z - Q_{\widetilde{Z}}||_{(1+\delta)\gamma,I} \leqslant K(C_{X,I}\varepsilon^*_{W,I} + C_{Y,I}\rho_I), \tag{A.6}$$

where we used the shorthands (defined in the proof of Proposition 6):

$$\varepsilon_{Z,I}=||Z-\widetilde{Z}||_{\gamma,I},\quad \varepsilon^*_{W,I}=||W-\widetilde{W}||_{\delta\gamma,I},\quad \varepsilon_{Y,I}=||Y-\widetilde{Y}||_{\gamma,I},\quad \varepsilon^*_{Y,I}=||Y-\widetilde{Y}||_{\delta\gamma,I};$$

$$\rho_I = ||X - \widetilde{X}||_{\gamma,I} + |Y_0 - \widetilde{Y}_0|$$

$$C_{X,I} = ||X||_{\gamma,I} + ||\widetilde{X}||_{\gamma,I}$$

$$C_{Y,I} = ||Y||_{\gamma,I} + ||\widetilde{Y}||_{\gamma,I}.$$

Observe that

$$\begin{aligned} ||\varphi(Y) - \varphi(\widetilde{Y})||_{\infty,I} &\leqslant |\varphi(Y_0) - \varphi(\widetilde{Y}_0)| + T^{\delta\gamma}||\varphi(Y) - \varphi(\widetilde{Y})||_{\delta\gamma,I} \\ &\leqslant ||\varphi||_{1,\delta}\rho_I + T^{\delta\gamma}\varepsilon^*_{W,I}, \end{aligned}$$

$$\begin{aligned} \varepsilon_{Z,I} &\leqslant ||\varphi(Y)\delta X - \varphi(\widetilde{Y})\delta\widetilde{X}||_{\gamma,I} + ||Q_Z - Q_{\widetilde{Z}}||_{\gamma,I} \\ &\leqslant ||\varphi(Y) - \varphi(\widetilde{Y})||_{\infty,I}||X||_{\gamma,I} + ||\varphi(\widetilde{Y})||_{\infty,I}||X - \widetilde{X}||_{\gamma,I} + T^{\delta\gamma}||Q_Z - Q_{\widetilde{Z}}||_{(1+\delta)\gamma,I} \\ &\leqslant ||\varphi||_{1,\delta}\rho_I C_{X,I} + T^{\delta\gamma}\varepsilon^*_{W,I} + KT^{\delta\gamma}(C_{X,I}\varepsilon^*_{W,I} + C_{Y,I}\rho_I) \\ &\leqslant ||\varphi||_{1,\delta}\rho_I(C_{X,I} + 1 + KC^{\delta}_{Y,I}) + T^{\gamma\delta}\varepsilon^*_{W,I}(C_{X,I} + KC_{Y,I}). \end{aligned}$$

It remains to bound $\varepsilon^*_{W,I}$: Write

$$\varphi(x) - \varphi(y) = \int_0^1 d\alpha\partial\varphi(\alpha x + (1-\alpha)y)(x-y) = R\varphi(x,y)(x-y)$$

then

$$||R\varphi||_{\infty} = \sup_{x,y\in V} |R\varphi(x,y)| \leqslant ||\varphi||_{1,\delta}$$

and

$$\begin{aligned} |R\varphi(x,y) - R\varphi(x',y')| &= \left|\int_0^1 (\partial\varphi(\alpha x + (1-\alpha)y) - \partial\varphi(\alpha x' + (1-\alpha)y')\, d\alpha\right| \\ &\leqslant ||\varphi||_{1,\delta}\int_0^1 |\alpha(x-x') + (1-\alpha)(y-y')|^{\delta}\, d\alpha \\ &\leqslant ||\varphi||_{1,\delta}(|x-x'|^{\delta} + |y-y'|^{\delta}) \end{aligned}$$

so that

$$\begin{aligned} \varepsilon^*_{W,I} &= ||\varphi(Y) - \varphi(\widetilde{Y})||_{\delta\gamma,I} = ||R\varphi(Y,\widetilde{Y})(Y - \widetilde{Y})||_{\delta\gamma,I} \\ &\leqslant ||R\varphi(Y,\widetilde{Y})||_{\infty,I}||Y - \widetilde{Y}||_{\delta\gamma,I} + ||R\varphi(Y,\widetilde{Y})||_{\delta\gamma,I}||Y - \widetilde{Y}||_{\infty,I} \end{aligned}$$

$$\leqslant ||\varphi||_{1,\delta}||Y-\widetilde{Y}||_{\delta\gamma,I}+||Y-\widetilde{Y}||_{\infty,I}||\varphi||_{1,\delta}(||Y||^{\delta}_{\gamma,I}+||\widetilde{Y}||^{\delta}_{\gamma,I})$$

$$\leqslant K||\varphi||_{1,\delta}C^{\delta}_{Y,I}\varepsilon^{*}_{Y,I}$$

$$\leqslant K||\varphi||_{1,\delta}C^{\delta}_{Y,I}\varepsilon_{Y,I}$$

concluding:

$$\varepsilon_{Z,I}\leqslant K||\varphi||_{1,\delta}C_{X,I}C^{\delta}_{Y,I}(\rho_I+T^{\gamma\delta}\varepsilon_{Y,I}). \tag{A.7}$$

The general case in which $\varphi\neq\widetilde{\varphi}$ can be easily derived from Eq. (A.7) and the continuity of the integral, giving:

$$\varepsilon_{Z,I}\leqslant KC_{X,I}C^{\delta}_{Y,I}[(1+||\varphi||_{1,\delta})\rho_I+T^{\gamma\delta}\varepsilon_{Y,I}]. \qquad \square$$

Using the notation in the proof of Proposition 7 we have

Lemma A.2. *For any interval $I=[t_0,t_0+T]\subseteq J$ such that $T<1$ the following bound holds*

$$\varepsilon_{Z,I}\leqslant K||\varphi||_{2,\delta}(C^2_{X,I}C^3_{Y,I}\rho_I+T^{\delta\gamma}C^3_{X,I}C^2_{Y,I}\varepsilon_{Y,I})+K||\varphi-\widetilde{\varphi}||_{2,\delta}C_{X,I}C^2_{Y,I}. \tag{A.8}$$

Proof. To begin assume that $\varphi=\widetilde{\varphi}$. Let $W=\varphi(Y)$, $\widetilde{W}=\varphi(\widetilde{Y})$ and write their decomposition as

$$\delta W^{\mu}=W'^{\mu}_{\nu}\delta X^{\nu}+R^{\mu}_{W}, \qquad \delta\widetilde{W}^{\mu}=\widetilde{W}'^{\mu}_{\nu}\delta\widetilde{X}^{\nu}+R^{\mu}_{\widetilde{W}},$$

with $W'^{\mu}_{\nu}=\partial_{\kappa}\varphi(Y)^{\mu}Y'^{\kappa}_{\nu}$, $\widetilde{W}'^{\mu}_{\nu}=\partial_{\kappa}\varphi(\widetilde{Y})^{\mu}\widetilde{Y}'^{\kappa}_{\nu}$. Moreover let

$$\varepsilon^{*}_{W,I}=||W'-\widetilde{W}'||_{\infty,I}+||W'-\widetilde{W}'||_{\delta\gamma,I}+||R_W+R_{\widetilde{W}}||_{(1+\delta)\gamma,I}+||W-\widetilde{W}||_{\gamma,I}$$

Using the bound (29) we have

$$||Q-Q_{\widetilde{Z}}||_{(2+\delta)\gamma}\leqslant K(D_1+D_2) \tag{A.9}$$

$$D_1=C_X\varepsilon^{*}_{W,I}$$

$$D_2=(||\varphi(Y)||_{D(X,\gamma,(1+\delta)\gamma),I}+||\varphi(\widetilde{Y})||_{D(\widetilde{X},\gamma,(1+\delta)\gamma),I})(||X-\widetilde{X}||_{\gamma,I}+||\mathbb{X}^2-\widetilde{\mathbb{X}}^2||_{2\gamma,I})$$

$$\leqslant K||\varphi||_{2,\delta}C^2_{Y,I}\rho_I$$

where we used Eq. (40) to bound $||\varphi(Y)||_{D(X,\gamma,(1+\delta)\gamma),I}$ and $||\varphi(\widetilde{Y})||_{D(\widetilde{X},\gamma,(1+\delta)\gamma),I}$ in terms of $C_{Y,I}$.

By Proposition 4 we have

$$\varepsilon^*_{W,I} \leqslant K||\varphi||_{2,\delta} C_{X,I} C^{1+\delta}_{Y,I}(||X-\widetilde{X}||_{\gamma,I} + \varepsilon^*_{Y,I}) \leqslant K||\varphi||_{2,\delta} C_{X,I} C^2_{Y,I}(\rho_I + \varepsilon^*_{Y,I}) \qquad \text{(A.10)}$$

with

$$\varepsilon^*_{Y,I} = ||Y' - \widetilde{Y}'||_\infty + ||Y' - \widetilde{Y}'||_{\delta\gamma} + ||R_Y - R_{\widetilde{Y}}||_{(1+\delta)\gamma} + ||Y - \widetilde{Y}||_\gamma$$

and

$$C_I = K||\varphi||_{2,\delta} C_{X,I} C^{1+\delta}_{Y,I}.$$

Taking $T<1$ we can bound $\varepsilon^*_{Y,I} \leqslant \varepsilon_{Y,I} + ||Y-\widetilde{Y}||_{\gamma,I}$ and

$$\begin{aligned}\varepsilon^*_{Y,I} &\leqslant ||Y' - \widetilde{Y}'||_\infty + ||Y' - \widetilde{Y}'||_\gamma + ||R_Y - R_{\widetilde{Y}}||_{2\gamma} + C_{X,I}\varepsilon_{Y,I} + C_{Y,I}||X-\widetilde{X}||_{\gamma,I} \\ &\leqslant 2C_{X,I}\varepsilon_{Y,I} + C_{Y,I}\rho_I, \qquad \text{(A.11)}\end{aligned}$$

where we used the following majorization for $||Y-\widetilde{Y}||_{\gamma,I}$:

$$\begin{aligned}||Y-\widetilde{Y}||_{\gamma,I} &\leqslant ||Y'\delta X - \widetilde{Y}'\delta\widetilde{X}||_{\gamma,I} + ||R_Y - R_{\widetilde{Y}}||_{\gamma,I} \\ &\leqslant ||Y' - \widetilde{Y}'||_{\infty,I}||X||_{\gamma,I} + (||Y'||_{\infty,I} + ||\widetilde{Y}'||_{\infty,I})||X-\widetilde{X}||_{\gamma,I} + ||R_Y - R_{\widetilde{Y}}||_{2\gamma,I} \\ &\leqslant C_{X,I}\varepsilon_{Y,I} + C_{Y,I}\rho_I. \qquad \text{(A.12)}\end{aligned}$$

Eq. (A.11) together with Eq. (A.10) imply

$$\varepsilon^*_{W,I} \leqslant K||\varphi||_{2,\delta}(C_{X,I}C^3_{Y,I}\rho_I + C^2_{X,I}C^2_{Y,I}\varepsilon_{Y,I})$$

and so

$$\begin{aligned}||Q_Z - Q_{\widetilde{Z}}||_{(2+\delta)\gamma} &\leqslant KC_X\varepsilon_{W,I} + K||\varphi||_{2,\delta}C^2_{Y,I}\rho_I \\ &\leqslant K(C_XC_I(1+2C_Y) + ||\varphi||_{2,\delta}C^2_{Y,I})\rho_I + 2KC_IC^2_X\varepsilon_{Y,I} \\ &\leqslant K||\varphi||_{2,\delta}(C^2_XC^3_Y\rho_I + C^3_XC^2_Y\varepsilon_{Y,I}) \qquad \text{(A.13)}\end{aligned}$$

$$\begin{aligned}\varepsilon_{Z,I} &= ||\varphi(Y) - \varphi(\widetilde{Y})||_{\infty,I} + ||\varphi(Y) - \varphi(\widetilde{Y})||_{\gamma,I} + ||R_Z - R_{\widetilde{Z}}||_{2\gamma,I} \\ &\leqslant |\varphi(Y_0) - \varphi(\widetilde{Y}_0)| + 2||\varphi(Y) - \varphi(\widetilde{Y})||_{\gamma,I} + ||R_Z - R_{\widetilde{Z}}||_{2\gamma,I}.\end{aligned}$$

Proceed step by step:

$$\begin{aligned}||\partial\varphi(Y)-\partial\varphi(\widetilde{Y})||_{\infty,I} &\leqslant |\partial\varphi(Y_{t_0})-\partial\varphi(\widetilde{Y}_{t_0})| + T^\gamma||\partial\varphi(Y)-\partial\varphi(\widetilde{Y})||_{\gamma,I}\\ &\leqslant ||\varphi||_{2,\delta}|Y_{t_0}-\widetilde{Y}_{t_0}| + T^\gamma||\varphi||_{2,\delta}||Y-\widetilde{Y}||_{\gamma,I}\\ &\leqslant T^\gamma||\varphi||_{2,\delta}C_{X,I}\varepsilon_{Y,I} + 2||\varphi||_{2,\delta}C_{Y,I}\rho_I.\end{aligned}$$

Next:

$$\begin{aligned}||R_Z-R_{\widetilde{Z}}||_{2\gamma,I} &\leqslant ||\partial\varphi(Y)\mathbb{X}^2-\partial\varphi(\widetilde{Y})\widetilde{\mathbb{X}}^2||_{2\gamma,I} + ||Q_Z-Q_{\widetilde{Z}}||_{2\gamma,I}\\ &\leqslant ||\partial\varphi(Y)-\partial\varphi(\widetilde{Y})||_{\infty,I}(||\mathbb{X}^2||_{2\gamma,I}+||\widetilde{\mathbb{X}}^2||_{2\gamma,I})\\ &\quad + (||\partial\varphi(Y)||_{\infty,I}+||\partial\varphi(\widetilde{Y})||_{\infty,I})||\mathbb{X}^2-\widetilde{\mathbb{X}}^2||_{2\gamma,I}\\ &\quad + T^{\delta\gamma}||Q_Z-Q_{\widetilde{Z}}||_{(2+\delta)\gamma,I}\\ &\leqslant K||\varphi||_{2,\delta}(\rho_I C_X^2C_Y^3 + \varepsilon_{Y,I}T^{\delta\gamma}C_X^3C_Y^2)\end{aligned}$$

and

$$\begin{aligned}||\varphi(Y)-\varphi(\widetilde{Y})||_{\gamma,I} &\leqslant ||\partial\varphi(Y)\delta X-\partial\varphi(\widetilde{Y})\delta\widetilde{X}||_{\gamma,I} + ||R_W-R_{\widetilde{W}}||_{\gamma,I}\\ &\leqslant ||\partial\varphi(Y)-\partial\varphi(\widetilde{Y})||_{\infty,I}(||X||_{\gamma,I}+||\widetilde{X}||_{\gamma,I})\\ &\quad + (||\partial\varphi(Y)||_{\infty,I}+||\partial\varphi(\widetilde{Y})||_{\infty,I})||X-\widetilde{X}||_{\gamma,I} + T^\gamma||R_W-R_{\widetilde{W}}||_{2\gamma,I}\\ &\leqslant (||\varphi||_{2,\delta}|Y_{t_0}-\widetilde{Y}_{t_0}| + T^\gamma||\varphi||_{2,\delta}C_{X,I}\varepsilon_{Y,I}\\ &\quad + ||\varphi||_{2,\delta}C_{Y,I}||X-\widetilde{X}||_{\gamma,I})(||X||_{\gamma,I}+||\widetilde{X}||_{\gamma,I})\\ &\quad + 2||\varphi||_{2,\delta}||X-\widetilde{X}||_{\gamma,I} + T^\gamma\varepsilon^*_{W,I}\\ &\leqslant K||\varphi||_{2,\delta}(C_{X,I}C_{Y,I}^3\rho_I + T^\gamma C_{X,I}^2C_{Y,I}^2\varepsilon_{Y,I}).\end{aligned}$$

Finally we have

$$\varepsilon_{Z,I}\leqslant K||\varphi||_{2,\delta}(C_{X,I}^2C_{Y,I}^3\rho_I + T^{\delta\gamma}C_{X,I}^3C_{Y,I}^2\varepsilon_{Y,I}). \tag{A.14}$$

When $\varphi\neq\widetilde{\varphi}$ rewrite the difference $Z-\widetilde{Z}$ as

$$Z_t-\widetilde{Z}_t = Y_{t_0}-\widetilde{Y}_{t_0} + \int_{t_0}^t[\varphi(Y)-\varphi(\widetilde{Y})]\,dX + \int_{t_0}^t[\varphi(\widetilde{Y})-\widetilde{\varphi}(\widetilde{Y})]\,dX$$

the contribution to $\varepsilon_{Z,I}$ from the first integral is bounded by Eq. (A.14) while the last integral can be bounded by $K||\varphi-\widetilde{\varphi}||_{2,\delta}C_{X,I}C_{Y,I}^2$ (cf. Eq. (46)) giving the final result (A.5). □

A.4. Proof of Lemma 4

Proof. Let $B(u,r) = \{w \in T : |w-u| \leqslant r\}$. Observe that by the monotonicity and convexity of ψ for any couple of measurable sets $A, B \subset T$ we have

$$\left| \int_{A\times B} R_{st} \frac{dt\,ds}{|A||B|} \right| \leqslant p(d(A,B)/4)\psi^{-1}\left(\int_{A\times B} \psi\left(\frac{|A_{st}|}{p(d(t,s)/4)} \right) \frac{dtds}{|A||B|} \right)$$

$$\leqslant p(d(A,B)/4)\psi^{-1}\left(\frac{U}{|A||B|} \right), \tag{A.15}$$

where $d(A,B) = \sup_{t\in A, s\in B}|t-s|$. Let

$$\bar{R}(t,r_1,r_2) = \int_{B(t,r_1)} \frac{du}{|B(t,r_1)|} \int_{B(t,r_2)} \frac{dv}{|B(t,r_2)|} R_{uv}.$$

Take $t,s \in T$, $a = |t-s|$, define the decreasing sequence of numbers $\lambda_n \downarrow 0$ as $\lambda_0 = a$, λ_{n+1} such that

$$p(\lambda_n) = 2p(\lambda_{n+1})$$

then

$$p((\lambda_n+\lambda_{n+1})/4) \leqslant p(\lambda_n) = 2p(\lambda_{n+1})$$
$$= 4p(\lambda_{n+1}) - 2p(\lambda_{n+1})$$
$$= 4[p(\lambda_{n+1}) - p(\lambda_{n+2})].$$

Using Eq. (A.15) and the fact that $|B(t,\lambda_i)| \geqslant \lambda_i$ for every $i \geqslant 0$ we have

$$|\bar{R}(t,\lambda_{n+1},\lambda_n)| \leqslant p((\lambda_n+\lambda_{n+1})/4)\psi^{-1}\left(\frac{U}{\lambda_n\lambda_{n+1}} \right)$$
$$\leqslant 4[p(\lambda_{n+1}) - p(\lambda_{n+2})]\psi^{-1}\left(\frac{U}{\lambda_n\lambda_{n+1}} \right)$$
$$\leqslant 4\int_{\lambda_{n+2}}^{\lambda_{n+1}} \psi^{-1}\left(\frac{U}{r^2} \right) dp(r).$$

Take a sequence $\{t_i\}_{i=0}^{\infty}$ of variables in T and note that, for every $n \geqslant 0$,

$$R_{tt_n} = R_{tt_{n+1}} + R_{t_{n+1}t_n} + (NR)_{tt_{n+1}t_n}$$

so that, by induction,

$$R_{tt_0} = R_{tt_{n+1}} + \sum_{i=0}^{n} [R_{t_{i+1}t_i} + (NR)_{tt_{i+1}t_i}].$$

Average each t_i over the ball $B(t, \lambda_i)$ and bound as follows:

$$\bar{R}(t, 0, \lambda_0) = \bar{R}(t, 0, \lambda_{n+1}) + \sum_{i=0}^{n} \bar{R}(t, \lambda_{i+1}, \lambda_i) + \sum_{i=0}^{n} \bar{B}(t, \lambda_{i+1}, \lambda_i), \tag{A.16}$$

where

$$\bar{B}(t, \lambda_{i+1}, \lambda_i) = \int_{B(t,\lambda_{i+1})} \frac{dv}{|B(t, \lambda_{i+1})|} \int_{B(t,\lambda_i)} \frac{du}{|B(t, \lambda_i)|} NR_{tvu}$$

which, using (50), can be majorized by

$$\begin{aligned} |\bar{B}(t, \lambda_{i+1}, \lambda_i)| &\leqslant \psi^{-1}\left(\frac{C}{\lambda_i^2}\right) p(\lambda_i/2) \leqslant 4\psi^{-1}\left(\frac{C}{\lambda_i^2}\right)[p(\lambda_{i+1}) - p(\lambda_{i+2})] \\ &\leqslant 4 \int_{\lambda_{i+2}}^{\lambda_{i+1}} \psi^{-1}\left(\frac{C}{r^2}\right) dp(r). \end{aligned}$$

Then, taking the limit as $n \to \infty$ in Eq. (A.16), using the continuity of R and that $R_{tt} = 0$, we get

$$\begin{aligned} |\bar{R}(t, 0, \lambda_0)| &\leqslant \sum_{i=0}^{\infty} 4 \int_{\lambda_{i+2}}^{\lambda_{i+1}} \psi^{-1}\left(\frac{U}{r^2}\right) dp(r) + \sum_{i=0}^{\infty} 4 \int_{\lambda_{i+2}}^{\lambda_{i+1}} \psi^{-1}\left(\frac{C}{r^2}\right) dp(r) \\ &\leqslant 4 \int_0^{\lambda_1} \left[\psi^{-1}\left(\frac{U}{r^2}\right) + \psi^{-1}\left(\frac{C}{r^2}\right)\right] dp(r) \\ &\leqslant 4 \int_0^{|t-s|} \left[\psi^{-1}\left(\frac{U}{r^2}\right) + \psi^{-1}\left(\frac{C}{r^2}\right)\right] dp(r) \end{aligned} \tag{A.17}$$

and of course the analogous estimate

$$|\bar{R}(s, 0, \lambda_0)| \leqslant 4 \int_0^{|t-s|} \left[\psi^{-1}\left(\frac{U}{r^2}\right) + \psi^{-1}\left(\frac{C}{r^2}\right)\right] dp(r). \tag{A.18}$$

Moreover,

$$R_{st} = R_{su} + R_{uv} + R_{vt} + NR_{sut} + NR_{uvt}$$

so

$$|R_{st}| \leqslant |R_{su}| + |R_{vt}| + |R_{uv}| + \sup_{r \in [s,t]} |NR_{srt}| + \sup_{r \in [u,t]} |NR_{urt}|.$$

By averaging u over the ball $B(s, a)$ and v over the ball $B(t, a)$ we get

$$\int_{B(s,a)} \frac{du}{|B(s, a)|} \int_{B(t,a)} \frac{dv}{|B(t, a)|} |R_{uv}| \leqslant p(3a/4) \psi^{-1}\left(\frac{U}{4a^2}\right) \leqslant \int_0^{|t-s|} \psi^{-1}\left(\frac{U}{r^2}\right) dp(r)$$

and

$$\int_{B(s,a)} \frac{du}{|B(s,a)|} \sup_{r\in[u,t]} |NR_{urt}| \leqslant p(a/2)\psi^{-1}\left(\frac{C}{a^2}\right) \leqslant \int_0^{|t-s|} \psi^{-1}\left(\frac{C}{r^2}\right) dp(r).$$

Putting all together we end up with

$$|R_{st}| \leqslant 10 \int_0^{|t-s|} \left[\psi^{-1}\left(\frac{U}{r^2}\right) + \psi^{-1}\left(\frac{C}{r^2}\right)\right] dp(r).$$

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