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SHOOTING METHODS FOR 1D STEADY-STATE FREE BOUNDARY PROBLEMS

A. MARQUINA*
Departamento de Matemática Aplicada, Universitat de València
Dr. Moliner, 50 46100-Burjassot (Valencia) Spain
V. MARTÍNEZ
Departamento de Matemáticas e Informática, Universitat Jaume I
Campus Penyeta Roja, 12071-Castelló Spain

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Abstract—In this note, we present two numerical methods based on shooting methods to solve steady-state diffusion-absorption models.

1. INTRODUCTION

The purpose of this note is to show how to use the shooting method for the approximation to the solution of some free boundary problems. In this note, we present two numerical methods based on shooting methods [1], for the approximation to the solution of models for the diffusion and absorption of oxygen in tissues. The models are a steady-state variant of one originally studied by Crank and Gupta [2] and previously studied by Epperson [3].

The differential equation related to this problem has as its simplest expression:

$$\begin{cases} -u''(x) = f(x), & 0 < x < s, \\ u(0) = 1, & \\ u(s) = u'(s) = 0, & \end{cases} \tag{P1}$$

where $u(x)$ is the concentration of oxygen at the point x, $f(x)(< 0)$ is the rate at which oxygen is absorbed and $x = s$ represents the furthest penetration of oxygen.

Our aim is to look for approximations to $u(x)$ and the free-boundary $x = s$. A closed form solution for $u(x)$ is (see [3]),

$$u(x) = -\int_x^s (t - x) f(t)\, dt.$$

This solution satisfies the problem :

$$\begin{cases} -u''(x) = f(x), & 0 < x < s, \\ u(s) = 0, & \\ u'(s) = 0. & \end{cases} \tag{P1s}$$

Now, it remains to find s such that $u(0) = 1$. The free-boundary $x = s$ we are looking for is a positive real root of the function

$$F(s) = 1 + \int_0^s t f(t)\, dt. \tag{1}$$

Epperson has given the following two theorems for this model:

THEOREM 1. *A pair $(s, u) \in (0, +\infty) \times C^2(0, s)$ is a solution of $(P1)$ if and only if s is a root of $F(s)$ and u is a solution of $(P1s)$.*

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THEOREM 2. *If $f(x) < 0$ for $x \in (0, +\infty)$ and there exists a $x_0 > 0$ such that $f(x) \leq \sigma < 0$ for $x \geq x_0$, then there is a unique $s_0 > 0$ such that $F(s_0) = 0$.*

Our main goal is to construct numerical methods based on the shooting method, which are applicable to a wider class of models (compared to the models discussed before). In this note, we propose the following two methods.

2. METHOD I (THE SHOOTING METHOD WITH MOVING INITIAL POSITION)

- **Data** $\varepsilon > 0, s_1, s_2$.
- $n = 1$.
- $s = s_1$.
- $P1s$.
- $u_1(x) = u(x)$.
- If $|u_1(0) - 1| < \varepsilon$ then **end.**
- **Begin repeat iteration.**
    - $n = n + 1$.
    - $s = s_n$.
    - $P1s$.
    - $u_n(x) = u(x)$.
    - If $|u_n(0) - 1| < \varepsilon$ then **end.**
    - $s_{n+1} = A(s_n, s_{n-1}, u_n(0), u_{n-1}(0))$.
- **End repeat iteration.**
- **End.**

- **Subroutine** $P1s$.
    - Solve ($P1s$) with explicit Runge-Kutta-3 from right to left.
    - Denote the solution by $u(x)$.
- **End.**

Here, we have defined $G(s_n) = u_n(0)$. A denotes the iteration of the secant method in order to compute the root of $F(s) = 1 - G(s)$, that is,

$$A(s_n, s_{n-1}, u_n(0), u_{n-1}(0)) = s_n - \left(u_n(0) - 1\right) \frac{s_n - s_{n-1}}{u_n(0) - u_{n-1}(0)}.$$

3. METHOD II (THE SHOOTING METHOD WITH MOVING TARGET)

This method is based on solving the problem:

$$\begin{cases} -u''(x) = f(x), & 0 < x < s, \\ u(0) = 1, \\ u'(0) = \gamma. \end{cases} \qquad (P1\gamma s)$$

The algorithm is the following:

- **Data** $\varepsilon > 0$, $s_1, s_2, \gamma_1, \gamma_2$.
- $n = 1$.
- $s = s_1$, $\gamma = \gamma_1$.
- $P1\gamma s$.
- $u_{11}(x) = u(x)$.
- If $|u_{11}(s_1)| + |u'_{11}(s_1)| < \varepsilon$ then **end.**
- **Begin repeat iteration.**
    - $n = n + 1$.
    - $s = s_n$, $\gamma = \gamma_n$.
    - $P1\gamma s$.
    - $u_{nn}(x) = u(x)$.

- If $|u_{nn}(s_n)| + |u'_{nn}(s_n)| < \varepsilon$ then end.
- $s = s_n$, $\gamma = \gamma_{n-1}$.
- $P1\gamma s$.
- $u_{n-1n}(x) = u(x)$.
- If $|u_{n-1n}(sn)| + |u'_{n-1n}(sn)| < \varepsilon$ then end.
- $s = s_{n-1}$, $\gamma = \gamma_n$.
- $P1\gamma s$.
- $u_{nn-1}(x) = u(x)$.
- If $|u_{nn-1}(s_{n-1})| + |u'_{nn-1}(s_{n-1})| < \varepsilon$ then end.
- $\gamma_{n+1} = B_{\gamma_{nn-1}}$.
- $s_{n+1} = B_{s_{nn-1}}$.

- **End repeat iteration.**
- **End.**

- **Subroutine** $P1\gamma s$.
  - Solve ($P1\gamma s$) with explicit Runge-Kutta-3 from left to right.
  - Denote the solution by $u(x)$.
- **End.**

Here, we have defined $G1(\gamma_n, s_m) = u_{nm}(s_m)$ and $G2(\gamma_n, s_m) = u'_{nm}(s_m)$. Thus, we must look for the roots of the system:

$$\begin{cases} G1(\gamma, s) = 0, \\ G2(\gamma, s) = 0. \end{cases} \tag{2}$$

Such roots are computed by a two-dimensional secant method, defined by:

$$B_{\gamma_{nn-1}} = \gamma_n - \frac{H_{\gamma_{nn-1}}}{J_{nn-1}}, \qquad B_{s_{nn-1}} = s_n - \frac{H_{s_{nn-1}}}{J_{nn-1}},$$

where

$$J_{nn-1} = \begin{vmatrix} \frac{G1(\gamma_n,s_n)-G1(\gamma_{n-1},s_n)}{\gamma_n-\gamma_{n-1}} & \frac{G1(\gamma_n,s_n)-G1(\gamma_n,s_{n-1})}{s_n-s_{n-1}} \\ \frac{G2(\gamma_n,s_n)-G2(\gamma_{n-1},s_n)}{\gamma_n-\gamma_{n-1}} & \frac{G2(\gamma_n,s_n)-G2(\gamma_n,s_{n-1})}{s_n-s_{n-1}} \end{vmatrix},$$

$$H_{\gamma_{nn-1}} = \begin{vmatrix} G1(\gamma_n, s_n) & \frac{G1(\gamma_n,s_n)-G1(\gamma_n,s_{n-1})}{s_n-s_{n-1}} \\ G2(\gamma_n, s_n) & \frac{G2(\gamma_n,s_n)-G2(\gamma_n,s_{n-1})}{s_n-s_{n-1}} \end{vmatrix},$$

$$H_{s_{nn-1}} = \begin{vmatrix} \frac{G1(\gamma_n,s_n)-G1(\gamma_{n-1},s_n)}{\gamma_n-\gamma_{n-1}} & G1(\gamma_n, s_n) \\ \frac{G2(\gamma_n,s_n)-G2(\gamma_{n-1},s_n)}{\gamma_n-\gamma_{n-1}} & G2(\gamma_n, s_n) \end{vmatrix}.$$

For the sake of comparison, we also include Epperson's method structured in the same way as our methods (I and II).

4. EPPERSON'S METHOD

We consider the problem:

$$\begin{cases} -u''(x) = f(x), & 0 < x < s, \\ u(0) = 1, \\ u(s) = 0. \end{cases} \tag{P2s}$$

The algorithm is the following:

- **Data** $\varepsilon > 0, s_1$.
- $n = 1$.

- **Begin repeat iteration.**
  - $s = s_n$.
  - **Quads.**
  - $F(s_n) = Fs$.
  - If $|F(s_n)| < \varepsilon$ then **exit iteration.**
  - $s_{n+1} = C(s_n, F(s_n))$.
  - $n = n + 1$.
- **End repeat iteration.**
- $s = s_n$.
- Solve $(P2s)$ by a difference scheme.
- Denote the solution by $u_n(x)$.
- **End.**

- **Subroutine Quads.**
  - Compute $I(s) = \int_0^s t f(t)\, dt$ with an approximate quadrature rule.
  - Denote $Fs = 1 + I(s) \approx F(s)$.
- **End.**

Here, C denotes the iteration of the Newton's method, where $F(s_n)$ is computed by means of a quadrature and $F'(s_n) = s_n\, f(s_n)$,

$$C(s_n, F(s_n)) = s_n - \frac{F(s_n)}{s_n\, f(s_n)}.$$

Our methods apply to a wider class of models characterized by the following formulation:

$$\begin{cases} -u''(x) = f(x, u, u'), \\ u(0) = 1, \\ u(s) = u'(s) = 0. \end{cases}$$

The absorption term depends either on the concentration or on the derivative of the concentration. Such models become more complicated and Epperson's method cannot apply. As an example, we will study the following model, where the absorption depends only on the concentration:

$$\begin{cases} -u''(x) + q(x)\, u(x) = f(x), & 0 < x < s, \\ u(0) = 1, \\ u(s) = u'(s) = 0. \end{cases} \tag{P3}$$

In order to be consistent with models discussed before, we will assume that $q(x) \geq 0$ and $f(x) \leq \sigma < 0$ for some σ, and that both functions are smooth enough. Here, $q(x) \geq 0$ means that the absorption is a nondecreasing function of the concentration.

Similar to problem $(P1)$, a solution to $(P3)$ satisfying the boundary conditions $u(s) = u'(s) = 0$ must verify the following integral identity:

$$u(s;x) = \int_0^s (t - x)\big[q(t)\, u(s;t) - f(t)\big]\, dt.$$

The free-boundary $x = s$ is a positive real root of the function

$$H(s) = 1 + \int_0^s t\big[f(t) - q(t)\, u(s;t)\big]\, dt.$$

Now, we consider the following problem:

$$\begin{cases} -u''(x) + q(x)\, u(x) = f(x), & 0 < x < s, \\ u(s) = 0, \\ u'(s) = 0. \end{cases} \tag{P3s}$$

For every $s > 0$, the problem $(P3s)$ has a unique solution (see [4, Chapter 3]). Then, we have the following theorem:

Theorem 3. *A pair $(s,u) \in (0,+\infty) \times C^2(0,s)$ is a solution of $(P3)$ if and only if s is a positive root of $H(s)$ and u is a solution of $(P3s)$. Moreover, this solution will be unique if the root s is unique.*

In order to prove that there is a positive root of $H(s)$, we require a lemma:

Lemma 1. *If $u(s;x)$ is a solution of $(P3s)$ with $q(x) \geq 0$ and $f(x) \leq \sigma < 0$ for $x \in (0,+\infty)$, then*

(i) *$u(s;x) \geq 0$ for $x \in (0,+\infty)$;*

(ii) *$u(s;x)$ is nonincreasing for $s < x$;*

(iii) *$u(s;x)$ is increasing for $s > x$.*

Proof. $u(s;x)$ may be written as

$$u(s;x) = -\int_x^s g(\xi,x)\, f(\xi)\, d\xi$$

where $g(\xi,x)$ is the unique solution of the problem (see [4, Chapter 3]):

$$\begin{cases} -u''(x) + q(x)\, u(x) = 0, \\ u(\xi) = 0, \\ u'(\xi) = -1. \end{cases} \tag{$P3H\xi$}$$

(i) For each $s > 0$, the reader can easily check
  (a) $g(\xi,x) > 0$ for $0 \leq x \leq s$ and $x \leq \xi \leq s$.
  (b) $g(\xi,x) < 0$ for $x \geq s$ and $s \leq \xi \leq x$.
  These relations together with $f(x) < 0$ imply that $u(s;x) \geq 0$ for $x > 0$.

(ii) Since the partial derivative of $u(s;x)$ with respect to s, is

$$u_s(s;x) = -g(s,x)\, f(s),$$

from (a), we obtain $g(s;x) < 0$ for $s < x$, and, therefore,

$$u_s(s;x) < 0, \qquad \text{for } s < x$$

is satisfied. Thus, $u(s;x)$ will be nonincreasing for $s < x$.

(iii) It is analogous.

Now, we are ready to prove the following theorem:

Theorem 4. *If $f(x) < 0$ for $x \in (0,+\infty)$ and there exists a $x_0 > 0$ such that $f(x) \leq \sigma < 0$ for $x \geq x_0$ and $q(x) \geq 0$ for $x \in (0,+\infty)$, then there is a unique $s_0 > 0$ such that $H(s_0) = 0$.*

Proof. We have $H(0) = 1$. From Lemma 1:

$$H(s) = 1 + \int_0^s t\big[f(t) - q(t)\, u(s;t)\big]\, dt \leq 1 + \int_0^s t f(t)\, dt = F(s),$$

($F(s)$ is the function defined in (1)). Then, $H(s) \leq F(s)$ is satisfied for $s \in (0,+\infty)$. Using Taylor's formula (as in [3, pp. 390]) there exists a $s_1 > 0$ such that $H(s_1) < 0$. Therefore, there is a $s_0 > 0$ with $H(s_0) = 0$. On the other hand, we have

$$H'(s) = s f(s) + \int_0^s t q(t)\, u_s'(s;t)\, dt.$$

From Lemma 1 together with $q(x) \geq 0$, we get $H'(s) < 0$. Hence, $H(s)$ is nonincreasing and s_0 is unique.

5. EXAMPLES

We include some examples for which exact solutions are known and, therefore, we present for each one two tables of absolute pointwise errors, corresponding to methods I and II.

EXAMPLE 1. The problem:

$$\begin{cases} -u''(x) = -\sqrt{x}, & 0 < x < s, \\ u(0) = 1, \\ u(s) = u'(s) = 0 \end{cases}$$

has the following exact solution: $u(x) = \frac{4}{15}(x^{5/2} - s^{5/2}) - \frac{2}{3}s^{3/2}(x - s)$ and $s = (5/2)^{(2/5)} \approx 1.44269990$.

Table 1.
Method I
$n = 20,\ s_1 = 1,\ s_2 = 3$

x_i	$\tilde{u}_i$	$Error$
0.072	$0.91703647E+00$	$0.29866030E-05$
0.144	$0.83543874E+00$	$0.30065815E-05$
0.216	$0.75580691E+00$	$0.28876223E-05$
0.289	$0.67859005E+00$	$0.27376561E-05$
0.361	$0.60416460E+00$	$0.25766622E-05$
0.433	$0.53286152E+00$	$0.24106796E-05$
0.505	$0.46497972E+00$	$0.22420790E-05$
0.577	$0.40079388E+00$	$0.20719580E-05$
0.649	$0.34055958E+00$	$0.19008866E-05$
0.721	$0.28451682E+00$	$0.17291871E-05$
0.793	$0.23289253E+00$	$0.15570538E-05$
0.866	$0.18590256E+00$	$0.13846099E-05$
0.938	$0.14375310E+00$	$0.12119371E-05$
1.010	$0.10664191E+00$	$0.10390915E-05$
1.082	$0.74759269E-01$	$0.86611265E-06$
1.154	$0.48288769E-01$	$0.69302913E-06$
1.226	$0.27407964E-01$	$0.51986213E-06$
1.298	$0.12288939E-01$	$0.34662763E-06$
1.371	$0.30987827E-02$	$0.17333788E-06$
1.443	$-.37263782E-25$	$0.24154714E-11$

Table 2.
Method II
$n = 20,\ s_1 = 1,\ s_2 = 3,\ \gamma_1 = -1.5,\ \gamma_2 = -3.7$

x_i	$\tilde{u}_i$	$Error$
1.371	$0.30987827E-02$	$0.17333788E-06$
1.298	$0.12288939E-01$	$0.34662763E-06$
1.226	$0.27407964E-01$	$0.51986213E-06$
1.154	$0.48288769E-01$	$0.69302913E-06$
1.082	$0.74759269E-01$	$0.86611265E-06$
1.010	$0.10664191E+00$	$0.10390915E-05$
0.938	$0.14375310E+00$	$0.12119371E-05$
0.866	$0.18590256E+00$	$0.13846099E-05$
0.793	$0.23289253E+00$	$0.15570538E-05$
0.721	$0.28451682E+00$	$0.17291871E-05$
0.649	$0.34055958E+00$	$0.19008866E-05$
0.577	$0.40079388E+00$	$0.20719580E-05$
0.505	$0.46497972E+00$	$0.22420790E-05$
0.433	$0.53286152E+00$	$0.24106796E-05$
0.361	$0.60416460E+00$	$0.25766622E-05$
0.289	$0.67859005E+00$	$0.27376561E-05$
0.216	$0.75580691E+00$	$0.28876223E-05$
0.144	$0.83543874E+00$	$0.30065815E-05$
0.072	$0.91703647E+00$	$0.29866030E-05$
0.000	$0.10000000E+01$	$0.18240628E-21$

EXAMPLE 2. The problem:

$$\begin{cases} -u''(x) + u(x) = -1, & 0 < x < s, \\ u(0) = 1, \\ u(s) = u'(s) = 0 \end{cases}$$

has the exact solution: $u(x) = \frac{1}{2}(e^{x-s} + e^{-(x-s)} - 2)$ and $s = \log(2 + \sqrt{3}) \approx 1.31695789$.

EXAMPLE 3. The problem:

$$\begin{cases} -u''(x) + u(x) = -e^{x}, & 0 < x < s, \\ u(0) = 1, \\ u(s) = u'(s) = 0 \end{cases}$$

has the exact solution: $u(x) = (e^x/4)(2x - 2s - 1) + \frac{1}{4}e^{2s-x}$ and s is the unique root of the equation: $2s + 5 - e^{2s} = 0$, $s \approx 0.96842369$.

EXAMPLE 4. The problem:

$$\begin{cases} -u''(x) - u(x) = -e^{x}, & 0 < x < s, \\ u(0) = 1, \\ u(s) = u'(s) = 0 \end{cases}$$

has the following solution, satisfying $u(s) = u'(s) = 0$:

$$u(x) = \frac{e^x}{2} - \frac{1}{2}\sin x\, e^s\,(\cos s + \sin s) + \frac{1}{2}\cos x\, e^s\,(\sin s - \cos s).$$

The initial condition $u(0) = 1$ is satisfied for each $s > 0$ with $e^s(\sin s - \cos s) - 1 = 0$. But this equation has infinite solutions in $(0, +\infty)$, in particular, it has two in $(0, 2\pi]$. This result proves that the hypothesis $q(x) \geq 0$ in Theorem 4 is fundamental to ensure the uniqueness of the solution.

Table 3.
Method I
$n = 20,\ s_1 = 1,\ s_2 = 3$

x_i	$\tilde{u}_i$	*Error*
0.066	0.89020374$E+00$	0.95209094$E-06$
0.132	0.78860620$E+00$	0.17322030$E-05$
0.198	0.69476671$E+00$	0.23567897$E-05$
0.263	0.60827822$E+00$	0.28409226$E-05$
0.329	0.52876560$E+00$	0.31984108$E-05$
0.395	0.45588396$E+00$	0.34419111$E-05$
0.461	0.38931717$E+00$	0.35830313$E-05$
0.527	0.32877648$E+00$	0.36324263$E-05$
0.593	0.27399932$E+00$	0.35998881$E-05$
0.658	0.22474807$E+00$	0.34944294$E-05$
0.724	0.18080911$E+00$	0.33243625$E-05$
0.790	0.14199185$E+00$	0.30973737$E-05$
0.856	0.10812791$E+00$	0.28205929$E-05$
0.922	0.79070410$E-01$	0.25006597$E-05$
0.988	0.54693301$E-01$	0.21437863$E-05$
1.054	0.34890846$E-01$	0.17558175$E-05$
1.119	0.19577147$E-01$	0.13422879$E-05$
1.185	0.86857746$E-02$	0.90847740$E-06$
1.251	0.21694825$E-02$	0.45946467$E-06$
1.317	0.12449740$E-21$	0.17939132$E-09$

Table 4.
Method II
$n = 20,\ s_1 = 1,\ s_2 = 3,\ \gamma_1 = -1.5,\ \gamma_2 = -3.7$

x_i	$\tilde{u}_i$	*Error*
1.251	0.21680296$E-02$	0.35582383$E-06$
1.185	0.86830880$E-02$	0.71198135$E-06$
1.119	0.19573428$E-01$	0.10630605$E-05$
1.054	0.34886282$E-01$	0.14037413$E-05$
0.988	0.54688064$E-01$	0.17285936$E-05$
0.922	0.79064659$E-01$	0.20320235$E-05$
0.856	0.10812180$E+00$	0.23082187$E-05$
0.790	0.14198551$E+00$	0.25510925$E-05$
0.724	0.18080268$E+00$	0.27542256$E-05$
0.658	0.22474167$E+00$	0.29108061$E-05$
0.593	0.27399306$E+00$	0.30135662$E-05$
0.527	0.32877049$E+00$	0.30547168$E-05$
0.461	0.38931154$E+00$	0.30258772$E-05$
0.395	0.45587882$E+00$	0.29180025$E-05$
0.329	0.52876105$E+00$	0.27213049$E-05$
0.263	0.60827436$E+00$	0.24251713$E-05$
0.198	0.69476365$E+00$	0.20180747$E-05$
0.132	0.78860405$E+00$	0.14874799$E-05$
0.066	0.89020261$E+00$	0.81974242$E-06$
0.000	0.10000000$E+01$	0.18482542$E-20$

Table 5.
Method I
$n = 20,\ s_1 = 1,\ s_2 = 3$

x_i	$\tilde{u}_i$	*Error*
0.048	0.90701382$E+00$	0.21266752$E-07$
0.097	0.81861653$E+00$	0.35716555$E-07$
0.145	0.73472299$E+00$	0.67563440$E-07$
0.194	0.65526464$E+00$	0.11362558$E-06$
0.242	0.58018972$E+00$	0.17072955$E-06$
0.291	0.50946341$E+00$	0.23569232$E-06$
0.339	0.44306808$E+00$	0.30530298$E-06$
0.387	0.38100362$E+00$	0.37630419$E-06$
0.436	0.32328781$E+00$	0.44537316$E-06$
0.484	0.26995672$E+00$	0.50910225$E-06$
0.533	0.22106521$E+00$	0.56397886$E-06$
0.581	0.17668746$E+00$	0.60636484$E-06$
0.629	0.13691762$E+00$	0.63247506$E-06$
0.678	0.10187049$E+00$	0.63835520$E-06$
0.726	0.71682243$E-01$	0.61985861$E-06$
0.775	0.46511300$E-01$	0.57262218$E-06$
0.823	0.26539209$E-01$	0.49204110$E-06$
0.872	0.11971647$E-01$	0.37324232$E-06$
0.920	0.30394964$E-02$	0.21105685$E-06$
0.968	−.46748737$E-17$	0.95959661$E-11$

Table 6.
Method II
$n = 20,\ s_1 = 1,\ s_2 = 5,\ \gamma_1 = -1.5,\ \gamma_2 = -3.7$

x_i	$\tilde{u}_i$	*Error*
0.920	0.30383938$E-02$	0.26039215$E-06$
0.872	0.11969640$E-01$	0.46478701$E-06$
0.823	0.26536477$E-01$	0.61911289$E-06$
0.775	0.46508006$E-01$	0.72892227$E-06$
0.726	0.71678535$E-01$	0.79943214$E-06$
0.678	0.10186650$E+00$	0.83555379$E-06$
0.629	0.13691347$E+00$	0.84192092$E-06$
0.581	0.17668324$E+00$	0.82291655$E-06$
0.533	0.22106103$E+00$	0.78269850$E-06$
0.484	0.26995267$E+00$	0.72522383$E-06$
0.436	0.32328396$E+00$	0.65427221$E-06$
0.387	0.38100003$E+00$	0.57346828$E-06$
0.339	0.44306481$E+00$	0.48630324$E-06$
0.291	0.50946051$E+00$	0.39615556$E-06$
0.242	0.58018724$E+00$	0.30631114$E-06$
0.194	0.65526261$E+00$	0.21998275$E-06$
0.145	0.73472143$E+00$	0.14032905$E-06$
0.097	0.81861548$E+00$	0.70473127$E-07$
0.048	0.90701328$E+00$	0.13520672$E-07$
0.000	0.10000000$E+01$	0.27422097$E-07$

6. REMARKS AND CONCLUSIONS

1. We have found through our numerical experiments that the explicit Runge-Kutta-3 method (see [5, p. 120 (ii)]) behaves satisfactorily combined with the resolution of the secant method, used as iterative method.

2. Our methods solve models more complicated than Epperson's method.

3. The numerical analysis realized proves the stability of the methods proposed.

4. The convergence is conditioned by the election of the pivots. But our numerical analysis proves that our methods can choose the pivots in a wider interval. Epperson has to choose the pivots after some initial estimations. Epperson's method applied to Example 1 to compute the free-boundary $x = s$ is divergent using $s_1 = 3$ as pivot and $n = 20$ grid points. In Table 1, $s_1 = 3$ is one of our pivots and our method is convergent.

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