ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Applied Mathematical Modelling 28 (2004) 877–890
APPLIED MATHEMATICAL MODELLING
www.elsevier.com/locate/apm

A mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation

W.Q. Chen [a,b,*], C.F. Lü [a], Z.G. Bian [a]

[a] *Department of Civil Engineering, Zhejiang University, Hangzhou 310027, China*
[b] *State Key Lab of CAD and CG, Zhejiang University, Hangzhou 310027, China*

Received 2 October 2003; received in revised form 15 April 2004; accepted 16 April 2004
Available online 11 June 2004

Abstract

In this paper a mixed method, which combines the state space method and the differential quadrature method, is proposed for bending and free vibration of arbitrarily thick beams resting on a Pasternak elastic foundation. Based on the two-dimensional state equation of elasticity, the domain along the axial direction is discretized according to the principle of differential quadrature (DQ). As a result, the state equations about the variables at discrete points are established. With consideration of the end conditions and the upper and lower boundary conditions in the derived state equations, governing equations for bending and free vibration problems are formulated. Numerical results prove that the present approach is very efficient and reliable. The effects of Poisson's ratio and foundation parameters on the natural frequencies are discussed.

Keywords: State space method; Differential quadrature method; Pasternak elastic foundation; State equation; Bending and free vibration

1. Introduction

Beams resting on elastic foundations have wide application in modern engineering and pose great technical problems in structural design. As a result, numerous research reports involving the calculation and analysis approach for beams on elastic foundation have been presented.

* Corresponding author. Address: Department of Civil Engineering, Zhejiang University, Hangzhou 310027, China. Tel.: +86-571-879-52284; fax: +86-571-879-52165.
E-mail addresses: chenwq@zju.edu.cn, lvchaofeng@hotmail.com (W.Q. Chen).

0307-904X/$ - see front matter
doi:10.1016/j.apm.2004.04.001

As is known to all, the Winkler model of elastic foundation is the most preliminary in which the vertical displacement is assumed to be proportional to the contact pressure at an arbitrary point [1]. A variety of investigations on the free vibration, buckling and stability behavior of Winkler foundation beams have been conducted by many researchers. For instance, Eisenberger and Clastornik [2] studied the vibration and buckling of beams on a variable Winkler foundation and numerical results were obtained. Ding [3] investigated the free vibration of simply-supported beams resting on a variable Winkler elastic foundation. Eisenberger et al. [4] employed the finite element method to analyze the stability of elastic foundation beams. Au et al. [5] investigated the axial-loaded vibration and stability of non-uniform beams with abrupt changes of cross-section resting on a Winkler elastic foundation with arbitrary foundation stiffness, applying a unified method in which C^1 modified beams vibration functions were used. More reports on Winkler foundation beams may also be found in literature, such as Refs. [6] and [7].

Although the Winkler model is simple and widely necessary, the assumption that the foundation soil is composed of closely spaced, independent and linear elastic springs leads to the dis continuity of the soil which was proved to be absent of enough accuracy [8]. To overcome this sprawl, some researchers proposed various two-parameter foundation models, which may capture the real behavior state of the soil more precisely, such as the generalized foundation [9], Pasternak foundation [10] and Vlasov foundation [11]. Lee and Kes [12] conducted a study to determine the natural frequencies of non-uniform Euler beams resting on a non-uniform foundation with general elastic end restraints. Franciosi and Masi [13] employed a finite element method with the hypotheses of exact shape function to investigate the free vibration of Bernoulli beams on elastic foundations with two parameters. Wang et al. [14] presented an exact solution of Timoshenko beams resting on two-parameter elastic foundations using Green's functions, and performed the numerical calculation for bending, free vibration and buckling of several beams. De Rosa and Maurizi [15] investigated the influence of concentrated masses and Pasternak soil on free vibration of beams and gave exact solutions for Bernoulli–Euler beams based on the beam theory. Free and forced vibration of a general elastically end restrained non-uniform beam, resting on a non-homogeneous elastic foundation and subjected to axial tensile and transverse forces, were studied by Ho and Chen [16] using differential transform. Recently, Chen [17] developed a new approach called the differential quadrature element method (DQEM) for the free vibration analysis of shallow beams resting on elastic foundations. Analyses and calculating approaches as to elastic foundation beams may also be found in references [18–20]. Note that all the above-mentioned studies are based on various beam theories, in which more or less assumptions on deformation along the thickness direction are introduced.

The differential quadrature method (DQM) has been proved adequately proficient coupled with various beam or plate theories [21–25]. In this paper, a new approach using the DQ technique and based on the state space formulations is developed for bending and free vibration analysis of isotropic beams with arbitrary depth-to-length ratios (see Fig. 1) resting on a Pasternak elastic foundation. In this model, the normal stress $\sigma_y(x,0)$ and vertical displacement $v(x,0)$ at an arbitrary point of the lower boundary of a beam hold the following relation

$$\sigma_y(x,0) = K_w v(x,0) - K_p \frac{\partial^2 v(x,0)}{\partial x^2}, \tag{1}$$

where K_w and K_p are the foundation moduli.

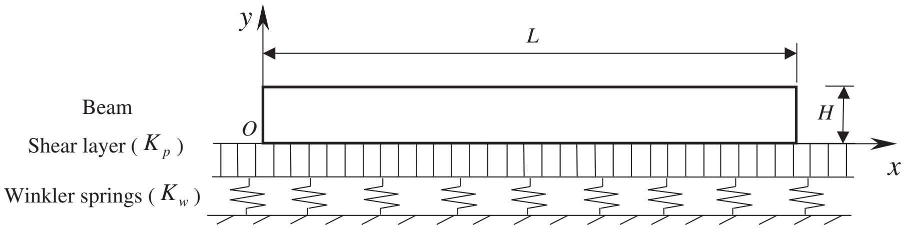

Fig. 1. Geometry of a beam on Pasternak foundation.

The main point of the present method is that the two-dimensional state equation is discretized in x-direction according to the principle of DQ so that the state equations with respect to the state variables at discrete points are established. End conditions are precisely incorporated and the general solution to the state equations is derived. Introduction of Eq. (1) and the upper as well as the lower boundary conditions into the derived general solution achieves the formulation of equilibrium equation for bending or frequency equation for free vibration of Pasternak foundation beams. Comparisons of mid-span deflection of uniformly loaded beams and natural frequencies with the published results indicate that the present mixed method is highly efficient and reliable. Influences of Poisson's ratio and foundation constants on the natural frequencies are discussed.

2. State equations and boundary conditions

Consider a straight beam of length L and depth H, having a rectangular cross-section of unit width. A Cartesian coordinate system (x, y) is defined as shown in Fig. 1, then the two-dimensional constitutive relations for the elastic body can be expressed as

$$\sigma_x = c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y}, \quad \sigma_y = c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y}, \quad \tau_{xy} = c_{66}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \tag{2}$$

where σ_x and σ_y are the normal stresses in x and y-directions respectively, τ_{xy} is the shear stress, u and v are the components of displacement in x and y-directions respectively, and the elastic constants c_{ij} are defined as

$$c_{11} = c_{22} = \frac{E}{1 - \nu^2}, \quad c_{12} = \frac{\nu E}{1 - \nu^2}, \quad c_{66} = \frac{E}{2(1 + \nu)}, \tag{3}$$

where E is elastic modulus, and ν is Poisson's ratio. In the absence of body forces, the differential equations of motion are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho\frac{\partial^2 u}{\partial t^2}, \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \rho\frac{\partial^2 v}{\partial t^2}, \tag{4}$$

where ρ is the mass density of material. Assuming that the isotropic elastic body undergoes free vibration with a circular frequency ω, utilization of Eq. (2) and Eq. (4) leads to the following state equation [26]

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}+\frac{1}{c_{66}}\tau_{xy},\quad \frac{\partial\sigma_y}{\partial y}=-\rho\omega^2 v-\frac{\partial\tau_{xy}}{\partial x},\quad \frac{\partial v}{\partial y}=-\frac{c_{12}}{c_{22}}\frac{\partial u}{\partial x}+\frac{1}{c_{22}}\sigma_y,$$
$$\frac{\partial\tau_{xy}}{\partial y}=\left(\frac{c_{12}^2}{c_{22}}-c_{11}\right)\frac{\partial^2 u}{\partial x^2}-\rho\omega^2 u-\frac{c_{12}}{c_{22}}\frac{\partial\sigma_y}{\partial x},\tag{5}$$

in which u, σ_y, v and τ are state variables, and the induced variable σ_x is determined by

$$\sigma_x=\left(c_{11}-\frac{c_{12}^2}{c_{22}}\right)\frac{\partial u}{\partial x}+\frac{c_{12}}{c_{22}}\sigma_y.\tag{6}$$

Note that for bending problems we can just set $\omega=0$ in Eq. (5).

According to the principle of DQ, the n^{th}-order partial derivative of a continuous function $f(x,y)$ with respect to x at a given point x_i can be approximated by the Lagrange polynomial:

$$\left.\frac{\partial^n f(x,y)}{\partial x^n}\right|_{x=x_i}=\sum_{k=1}^{N}W_{ik}^{(n)}f(x_k,y)\quad (i=1,2,\ldots,N;\ n=1,2,\ldots,N-1),\tag{7}$$

where N is the discrete points number, and $W_{ik}^{(n)}$ are the weight coefficients determined as follows [27]

$$W_{ik}^{(1)}=\frac{\prod_{j=1,j\neq i}^{N}(x_i-x_j)}{(x_i-x_k)\prod_{j=1,j\neq k}^{N}(x_k-x_j)},\tag{8}$$

$$W_{ik}^{(n)}=n\left[W_{ii}^{(n-1)}W_{ik}^{(1)}-\frac{W_{ik}^{(n-1)}}{x_i-x_k}\right]\quad (n=2,3,\ldots,N-1),\tag{9}$$

for $i,k=1,2,\ldots,N$, but $i\neq k$, while $W_{ii}^{(n)}$ are defined by

$$W_{ii}^{(n)}=-\sum_{k=1,k\neq i}^{N}W_{ik}^{(n)}\quad (i=1,2,\ldots,N,n=1,2,\ldots,N-1).\tag{10}$$

Applying the above-mentioned procedure to Eq. (5) by discretizing the domain of x, the following state equations at an arbitrary discrete point x_i are derived

$$\frac{\partial u_i}{\partial y}=-\sum_{k=1}^{N}W_{ik}^{(1)}v_k+\frac{1}{c_{66}}\tau_i,\quad \frac{\partial\sigma_{yi}}{\partial y}=-\rho\omega^2 v_i-\sum_{k=1}^{N}W_{ik}^{(1)}\tau_k,$$
$$\frac{\partial v_i}{\partial y}=-\frac{c_{12}}{c_{22}}\sum_{k=1}^{N}W_{ik}^{(1)}u_k+\frac{1}{c_{22}}\sigma_{yi},\quad \frac{\partial\tau_i}{\partial y}=\left(\frac{c_{12}^2}{c_{22}}-c_{11}\right)\sum_{k=1}^{N}W_{ik}^{(2)}u_k-\rho\omega^2 u_i-\frac{c_{12}}{c_{22}}\sum_{k=1}^{N}W_{ik}^{(1)}\sigma_{yk},\tag{11}$$

and Eq. (6) becomes

$$\sigma_{xi}=\left(c_{11}-\frac{c_{12}^2}{c_{22}}\right)\sum_{k=1}^{N}W_{ik}^{(1)}u_k+\frac{c_{12}}{c_{22}}\sigma_{yi},\tag{12}$$

in which $u_i = u(x_i, y)$, $v_i = v(x_i, y)$, $\sigma_{yi} = \sigma_y(x_i, y)$, $\tau_i = \tau_{xy}(x_i, y)$ and $\sigma_{xi} = \sigma(x_i, y)$, and $i = 1, 2, \ldots, N$.

For a special problem, it is necessary to take account of the end conditions in Eq. (11) to obtain the corresponding general solution. Three types of beams, i.e. simply supported–simply supported (S–S) beam, clamped–clamped (C–C) beam and clamped-free (C–F) beam, are investigated for examples in this paper with end conditions described as follows:

simply supported–simply supported (S-S) beam:

$$v_1 = \sigma_{x1} = 0, \quad \text{at } x = 0, \tag{13a}$$

$$v_N = \sigma_{xN} = 0, \quad \text{at } x = L. \tag{13b}$$

To involve all the end conditions in the state equations, Eq. (11), it is necessary to express the stress boundary conditions in Eq. (13) in terms of state variables. Hence, the following expression is obtained from Eq. (12),

$$-\frac{c_{12}}{c_{22}}\sigma_{yi} = \left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)\sum_{k=1}^{N} W_{ik}^{(1)} u_k \quad (i = 1, N), \tag{14}$$

clamped–clamped (C–C) beam:

$$u_1 = v_1 = 0, \quad \text{at } x = 0, \tag{15a}$$

$$u_N = v_N = 0, \quad \text{at } x = L, \tag{15b}$$

clamped-free (C–F) beam:

$$u_1 = v_1 = 0, \quad \text{at } x = 0, \tag{16}$$

$$\tau_N = 0, \quad -\frac{c_{12}}{c_{22}}\sigma_{yN} = \left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)\sum_{k=1}^{N} W_{Nk}^{(1)} u_k, \quad \text{at } x = L. \tag{17}$$

The second expression in Eq. (17) is derived in the same manner as the case of Eq. (14).

Substitution of the end conditions of each case into Eq. (11) respectively gives the following final state equations:

simply supported–simply supported (S–S) beam:

$$\frac{\partial u_i}{\partial y} = -\sum_{k=2}^{N-1} W_{ik}^{(1)} v_k + \frac{1}{c_{66}}\tau_i \quad (i = 1, \ldots, N), \quad \frac{\partial \sigma_{yi}}{\partial y} = -\rho\omega^2 v_i - \sum_{k=1}^{N} W_{ik}^{(1)}\tau_k \quad (i = 2, \ldots, N-1),$$

$$\frac{\partial v_i}{\partial y} = -\frac{c_{12}}{c_{22}}\sum_{k=1}^{N} W_{ik}^{(1)} u_k + \frac{1}{c_{22}}\sigma_{yi} \quad (i = 2, \ldots, N-1), \quad \frac{\partial \tau_i}{\partial y} = \left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)\sum_{k=1}^{N}\Big(W_{i1}^{(1)} W_{1k}^{(1)}$$

$$+ W_{iN}^{(1)} W_{Nk}^{(1)} - W_{ik}^{(2)}\Big) u_k - \rho\omega^2 u_i - \frac{c_{12}}{c_{22}}\sum_{k=2}^{N-1} W_{ik}^{(1)}\sigma_{yk} \quad (i = 1, \ldots, N). \tag{18}$$

clamped–clamped (C–C) beam:

$$\frac{\partial u_i}{\partial y} = -\sum_{k=2}^{N-1} W_{ik}^{(1)} v_k + \frac{1}{c_{66}}\tau_i \quad (i = 2, \ldots, N-1), \quad \frac{\partial \sigma_{yi}}{\partial y} = -\rho\omega^2 v_i - \sum_{k=1}^{N} W_{ik}^{(1)}\tau_k \quad (i = 1, \ldots, N),$$

$$\frac{\partial v_i}{\partial y} = -\frac{c_{12}}{c_{22}}\sum_{k=2}^{N-1} W_{ik}^{(1)} u_k + \frac{1}{c_{22}}\sigma_{yi} \quad (i = 2, \ldots, N-1),$$

$$\frac{\partial \tau_i}{\partial y} = \left(\frac{c_{12}^2}{c_{22}} - c_{11}\right)\sum_{k=2}^{N-1} W_{ik}^{(2)} u_k - \rho\omega^2 u_i - \frac{c_{12}}{c_{22}}\sum_{k=1}^{N} W_{ik}^{(1)}\sigma_{yk} \quad (i = 1, \ldots, N). \tag{19}$$

clamped–free (C–F) beam:

$$\frac{\partial u_i}{\partial y} = -\sum_{k=2}^{N} W_{ik}^{(1)} v_k + \frac{1}{c_{66}}\tau_i \quad (i = 2, \ldots, N), \quad \frac{\partial \sigma_{yi}}{\partial y} = -\rho\omega^2 v_i - \sum_{k=1}^{N-1} W_{ik}^{(1)}\tau_k \quad (i = 1, \ldots, N-1),$$

$$\frac{\partial v_i}{\partial y} = -\frac{c_{12}}{c_{22}}\sum_{k=2}^{N} W_{ik}^{(1)} u_k + \frac{1}{c_{22}}\sigma_{yi} \quad (i = 2, \ldots, N), \quad \frac{\partial \tau_i}{\partial y} = \left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)\sum_{k=2}^{N}\left(W_{i1}^{(1)} W_{1k}^{(1)} - W_{ik}^{(2)}\right) u_k - \rho\omega^2 u_i - \frac{c_{12}}{c_{22}}\sum_{k=1}^{N-1} W_{ik}^{(1)}\sigma_{yk} \quad (i = 1, \ldots, N-1). \tag{20}$$

3. Formulation of governing equations

For convenience, the final state Eqs. (18)–(20) can be written in a uniform matrix notation as

$$\frac{\partial}{\partial y}\{\delta\} = \mathbf{M}\{\delta\}, \tag{21}$$

where $\{\delta\} = \{\delta(y)\} = [\mathbf{u}^{\mathrm{T}}\boldsymbol{\sigma}_y^{\mathrm{T}}\mathbf{v}^{\mathrm{T}}\boldsymbol{\tau}^{\mathrm{T}}]^{\mathrm{T}}$, $\mathbf{u}$, $\mathbf{v}$ and $\boldsymbol{\sigma}_y$, $\boldsymbol{\tau}$ are column vectors which in consecutive manner consist of the unknown displacement and stress components for an arbitrary vertical coordinate y at all discrete points, and the coefficient matrix $\mathbf{M}$ can be obtained directly from Eqs. (18)–(20). The general solution to Eq. (21) can be expressed as

$$\{\delta(y)\} = e^{\mathbf{M}y}\{\delta(0)\}. \tag{22}$$

Setting $y = H$ in Eq. (22) yields

$$\{\delta(H)\} = \mathbf{S}\{\delta(0)\}, \tag{23}$$

where $\mathbf{S} = \exp(\mathbf{M}H)$. Eq. (23) establishes the transfer relationship of the state variable vectors at the upper and lower boundaries. As for the case of Pasternak foundation beams, Eq. (1) should be considered. To do so, Eq. (1) is also discretized by using the principle of DQ as

$$\sigma_{yi}(0) = K_w v_i(0) - K_p \sum_k W_{ik}^{(2)} v_k(0), \tag{24}$$

where the initial and terminal values of the subscript k depend on the specific end conditions. For example, we should take $k = 2, 3, \ldots, N-1$ for the cases of S–S and C–C beams, and

$k = 2, 3, \ldots, N$ for C–F beam. Substituting Eq. (24) into Eq. (23), with proper rearrangement, gives

$$\begin{Bmatrix} \mathbf{u}(H) \\ \boldsymbol{\sigma}_y(H) \\ \mathbf{v}(H) \\ \boldsymbol{\tau}(H) \end{Bmatrix} = \mathbf{T}\begin{Bmatrix} \mathbf{u}(0) \\ \mathbf{v}(0) \\ \boldsymbol{\tau}(0) \end{Bmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \\ \mathbf{T}_{41} & \mathbf{T}_{42} & \mathbf{T}_{43} \end{bmatrix}\begin{Bmatrix} \mathbf{u}(0) \\ \mathbf{v}(0) \\ \boldsymbol{\tau}(0) \end{Bmatrix}. \tag{25}$$

The formulation of Eq. (25) and the expression of matrix $\mathbf{T}$ are presented in Appendix A(taking S–S beam as an example). For free vibration, the upper and lower stress components are known as

$$\boldsymbol{\tau}(H) = \boldsymbol{\tau}(0) = \mathbf{0}, \quad \boldsymbol{\sigma}_y(H) = \mathbf{0}. \tag{26}$$

Introduction of Eq. (26) into Eq. (25) and elimination of $\mathbf{u}(H)$, $\mathbf{v}(H)$ and $\boldsymbol{\tau}(0)$ from Eq. (25) result in

$$\begin{bmatrix} \mathbf{T}_{21} & \mathbf{T}_{22} \\ \mathbf{T}_{41} & \mathbf{T}_{42} \end{bmatrix}\begin{Bmatrix} \mathbf{u}(0) \\ \mathbf{v}(0) \end{Bmatrix} = \mathbf{P}\begin{Bmatrix} \mathbf{u}(0) \\ \mathbf{v}(0) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{27}$$

For the non-trial solution of Eq. (27), it is essential that the determinant of matrix $\mathbf{P}$ be zero at the correct natural frequencies, i.e.

$$|\mathbf{P}| = 0. \tag{28}$$

It is worth noting that Eqs. (18)–(20) become the state equations for static analysis by setting $\omega = 0$. Assuming that the beam is subject to a distributed load $q(x)$ at the upper boundary, Eq. (26) becomes

$$\boldsymbol{\tau}(H) = \boldsymbol{\tau}(0) = \mathbf{0}, \quad \boldsymbol{\sigma}_y(H) = \mathbf{q}, \tag{29}$$

where $\mathbf{q}$ is a column vector consisting of $q(x_i)$ consecutively in which the subscript i is consistent with that in the second expression in Eqs. (18)–(20). Utilizing Eq. (29) and Eq. (25), we get

$$\begin{bmatrix} \mathbf{T}_{21} & \mathbf{T}_{22} \\ \mathbf{T}_{41} & \mathbf{T}_{42} \end{bmatrix}\begin{Bmatrix} \mathbf{u}(0) \\ \mathbf{v}(0) \end{Bmatrix} = \begin{Bmatrix} \mathbf{q} \\ \mathbf{0} \end{Bmatrix}. \tag{30}$$

By incorporating the displacement components of the lower boundary obtained from Eq. (30), as well as Eqs. (24) and (29) into Eq. (22), the state variable vector for an arbitrary vertical coordinate y can subsequently be derived.

4. Numerical results

In this paper, several examples are performed concerning bending and free vibration of beams with unit width rectangular cross section resting on an elastic foundation. Noting that the DQM has an excellent characteristic of convergence [28], the discrete points in the following examples are sampled with $N = 9$. Unequally spaced sampling points, the so-called Chebyshev–Gauss–Lobatto points, given by [29]

$$x_i = \frac{1 - \cos[(i-1)\pi/(N-1)]}{2}L \quad (i = 1, 2, \ldots, N), \tag{31}$$

are adopted for the calculation hereafter.

Tables 1 and 2 list the numerical results of the non-dimensional mid-span deflection $\bar{v}(x = L/2,\ y = H/2)$ for uniformly loaded S–S beams and C–C beams respectively, and Tables 3 and 4 tabulate the natural frequency parameters $\sqrt{\bar{\omega}}$ of the two beams, respectively. A comparison with the published results is also presented. The non-dimensional deflection $\bar{v}$, frequency parameter $\sqrt{\bar{\omega}}$ and the other parameters in the tables are defined as

$$\bar{v} = \frac{vEI}{qL^4},\quad \sqrt{\bar{\omega}} = \sqrt{\omega}\left(\frac{\rho AL^4}{EI}\right)^{\frac{1}{4}},\quad \lambda = \frac{H}{L},\quad \overline{K}_w = \frac{K_w L^4}{EI},\quad \overline{K}_p = \frac{K_p L^2}{EI},$$

where A is the cross-sectional area, and I is the second moment of the cross-sectional area. The Poisson's ratio is taken as $\nu = 0.3$ currently.

It is worth pointing out that, for S–S beams, the variations of state variables along x direction may be assumed as trigonometric functions to precisely satisfy the end conditions in Eq. (13), and then the exact solutions can be obtained based on Eq. (5). The procedure is very similar to the

Table 1
Mid-span deflection $\bar{v} \times 10^{-2}$ $(x = L/2,\ y = H/2)$ of uniformly loaded S–S beams

Foundation parameters		$\lambda = 1/120$			$\lambda = 1/15$		$\lambda = 1/5$	
$\overline{K}_w$	$\overline{K}_p$	Present	Exact	Ref. [14]	Present	Exact	Present	Exact
0	0	1.302290	1.302290	1.3033	1.315277	1.315271	1.420261	1. 420243
	10	0.644827	0.644827	0.6457	0.648347	0.648299	0.678202	0.674505
	25	0.366111	0.366111	0.3671	0.367416	0.367353	0.381703	0.376671
10	0	1.180567	1.180567	1.1814	1.191402	1.191335	1.282598	1.277311
	10	0.613325	0.613326	0.6141	0.616562	0.616485	0.646391	0.640247
	25	0.355668	0.355668	0.3566	0.356923	0.356843	0.372064	0.365680
100	0	0.640074	0.640074	0.6403	0.643767	0.643428	0.696100	0.668478
	10	0.425582	0.425582	0.4261	0.427409	0.427156	0.459267	0.438808
	25	0.282846	0.282846	0.2836	0.283799	0.283603	0.305161	0.289436

Table 2
Mid-span deflection $\bar{v} \times 10^{-2}$ $(x = L/2,\ y = H/2)$ of uniformly loaded C–C beams

Foundation parameters		$\lambda = 1/120$		$\lambda = 1/15$	$\lambda = 1/5$
$\overline{K}_w$	$\overline{K}_p$	Present	Ref. [14]	Present	Present
0	0	0.26064	0.2616	0.27493	0.38814
	10	0.20862	0.2095	0.21893	0.29426
	25	0.16081	0.1617	0.16811	0.21760
10	0	0.25547	0.2565	0.26921	0.37817
	10	0.20528	0.2062	0.21526	0.28874
	25	0.15880	0.1597	0.16593	0.21478
100	0	0.21670	0.2174	0.22662	0.30908
	10	0.17935	0.1800	0.18701	0.24823
	25	0.14273	0.1435	0.14853	0.19299

Table 3
Fundamental frequency parameter $\sqrt{\bar{\omega}}$ of S–S beams

Foundation parameters		$\lambda = 1/120$			$\lambda = 1/15$		$\lambda = 1/5$	
$\overline{K}_w$	$\overline{K}_p/\pi^2$	Present	Exact	Ref. [15]	Present	Exact	Present	Exact
0	0	3.141434	3.141417	3.1415	3.1302472	3.1302475	3.0479950	3.0479950
	0.5	3.476594	3.476589	3.4767	3.4667120	3.4667123	3.3945840	3.3945841
	1.0	3.735876	3.735859	3.7360	3.7265663	3.7265663	3.6580220	3.6580220
	2.5	4.296866	4.296879	4.2970	4.2880927	4.2880929	4.2183416	4.2183417
10^2	0	3.748233	3.748219	3.7483	3.7389476	3.7389477	3.6705002	3.6705003
	0.5	3.960677	3.960669	3.9608	3.9516805	3.9516807	3.8839761	3.8839762
	1.0	4.143563	4.143565	4.1437	4.1347186	4.1347188	4.0663636	4.0663637
	2.5	4.582266	4.582264	4.5824	4.5734720	4.5734720	4.4991384	4.4991384
10^4	0	10.02403	10.02404	10.024	9.9958218	9.9958219	7.3408114	7.3408115
	0.5	10.03610	10.03610	10.036	10.007782	10.007782	7.3408839	7.3408839
	1.0	10.04813	10.04813	10.048	10.019699	10.019699	7.3409553	7.3409553
	2.5	10.08394	10.08394	10.084	10.055193	10.055193	7.3411635	7.3411636
10^6	0	31.62172	31.62172	31.623	12.772265	12.772265	7.3508112	7.3508113
	0.5	31.62211	31.62211	31.623	12.772265	12.772265	7.3508112	7.3508113
	1.0	31.62249	31.62249	31.624	12.772265	12.772265	7.3508112	7.3508113
	2.5	31.62364	31.62365	31.625	12.772265	12.772265	7.3508112	7.3508113

analysis of a simply-supported plate (see Ref. [30]) for example, and is omitted here for brevity. From Tables 1 and 3, it is shown that the present numerical results have an excellent agreement with the exact solutions both for slender thin beams and short thick beams. It can also be seen from the comparison in Tables 1–4 that the mid-span deflections and natural frequencies of Bernoulli–Euler beams agree well with the results in Ref. [14] and Ref. [15] respectively.

Since the conventional beam theories can not involve the effect of Poisson's ratio, it is rather interesting to take a deep insight into it using the present mixed approach. Table 5 gives the variation of the first three natural frequency parameters ($\sqrt{\bar{\omega}}$) of C–C beams with the Poisson's ratio. It is shown that the natural frequency decreases gradually with the increasing of Poisson's ratio. We can see that the natural frequencies for $v = 0.5$ have an apparent derivation (between 7% and 8%) from that for $v = 0.1$. From this point of view, the Poisson's ratio is of great significance in structural design especially for composite material beams.

It is also interesting to investigate the variation of the deformation along the thickness direction, which can not be described exactly and completely by any beam theory. For example, the deflection of a beam at an arbitrary point x is usually assumed to be independent of the coordinate y in the conventional beam theories. Fig. 2 shows the curves of the non-dimensional mid-span deflection $\bar{v}(x = L/2)$ versus y/H of a C–C beam for different depth-to-length ratios. From Fig. 2, we can conclude that the deflections at different vertical coordinates y are almost the same for a slender thin beam, while for a short thick beam they are quite different from each other. Consequently, the present method is superior to any beam theory in investigating the variation of the deflection along the thickness direction.

Table 4
The first three natural frequency parameters $\sqrt{\varpi}$ of C–C beams

λ	$\overline{K}_w$	$\overline{K}_p/\pi^2$							
		0		0.5		1.0		2.5	
1/120	0	4.7314	(4.73)	4.8683	(4.869)	4.9938	(4.994)	5.3195	(5.32)
		7.8533	(7.854)	7.9680	(7.968)	8.0777	(8.078)	8.3812	(8.38)
		10.9908	(10.996)	11.0815	(11.086)	11.1700	(11.174)	11.4233	(11.43)
	100	4.9515	(4.95)	5.0718	(5.071)	5.1834	(5.182)	5.4783	(5.477)
		7.9044	(7.904)	8.0169	(8.017)	8.1247	(8.124)	8.4234	(8.423)
		11.0096	(11.014)	11.0998	(11.104)	11.1878	(11.192)	11.4400	(11.444)
	10 000	10.1227	(10.123)	10.1373	(10.137)	10.1517	(10.152)	10.1942	(10.194)
		10.8384	(10.839)	10.8827	(10.883)	10.9264	(10.927)	11.0539	(11.055)
		12.5216	(12.526)	12.5832	(12.588)	12.6439	(12.648)	12.8209	(12.825)
1/15	0		4.66554		4.80385		4.93027		5.25671
			7.61037		7.72927		7.84259		8.15441
			10.42711		10.52435		10.61889		10.88791
	100		4.89268		5.01352		5.12542		5.41981
			7.66521		7.78165		7.89277		8.19912
			10.44810		10.54476		10.63876		10.90635
	10 000		10.04899		10.06400		10.07881		10.12225
			10.70252		10.74610		10.78903		10.91414
			12.08187		12.14487		12.20684		12.38693
1/5	0		4.26343		4.41970		4.55951		4.91020
			6.46481		6.62631		6.77597		7.16627
			7.40127		7.40477		7.40820		7.42199
	100		4.54177		4.67208		4.79099		5.09741
			6.54716		6.70266		6.84714		7.22280
			7.40183		7.40543		7.40905		7.42612
	10 000		7.40541		7.40733		7.40909		7.41353
			8.54580		8.60124		8.64917		8.76160
			10.11249		10.15262		10.19198		10.30595

The results in parentheses were calculated by De Rosa and Maurizi [15].

Table 5
Effect of Poisson's ratios on the first three natural frequency parameters $\sqrt{\varpi}$ of C–C beams ($\overline{K}_w = 1000$, $\overline{K}_p/\pi^2 = 1.0$)

λ	Poisson's ratio (v)				
	0.1	0.2	0.3	0.4	0.5
1/120	6.6159	6.4748	6.3487	6.2352	6.1325
	8.8753	8.6879	8.5217	8.3738	8.2420
	11.8220	11.5698	11.3443	11.1412	10.9574
1/15	6.5768	6.3456	6.3092	6.1954	6.0924
	8.6587	8.4744	8.3106	8.1642	8.0330
	11.2652	11.0262	10.8124	10.6199	10.4457
1/5	6.2920	6.1469	6.0165	5.8992	5.7942
	7.6877	7.5117	7.3492	7.2002	7.0634
	7.7483	7.5955	7.4697	7.3638	7.2734

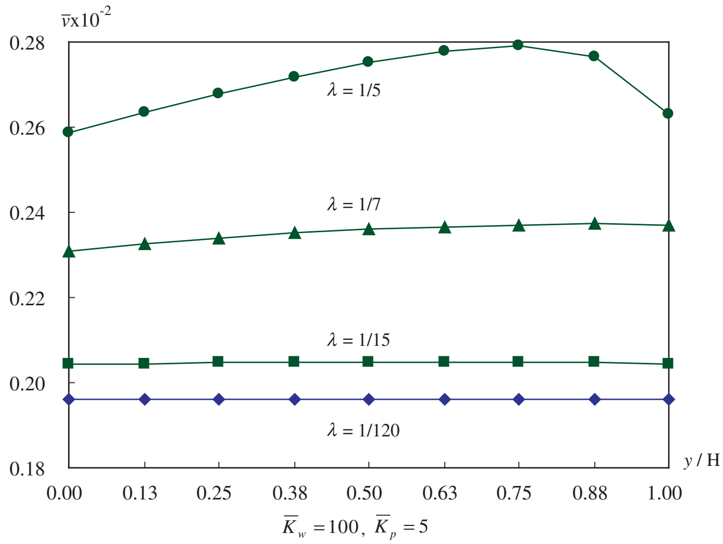

Fig. 2. Curve of $\bar{v}(x = L/2)$ versus y/H for a C–C beam.

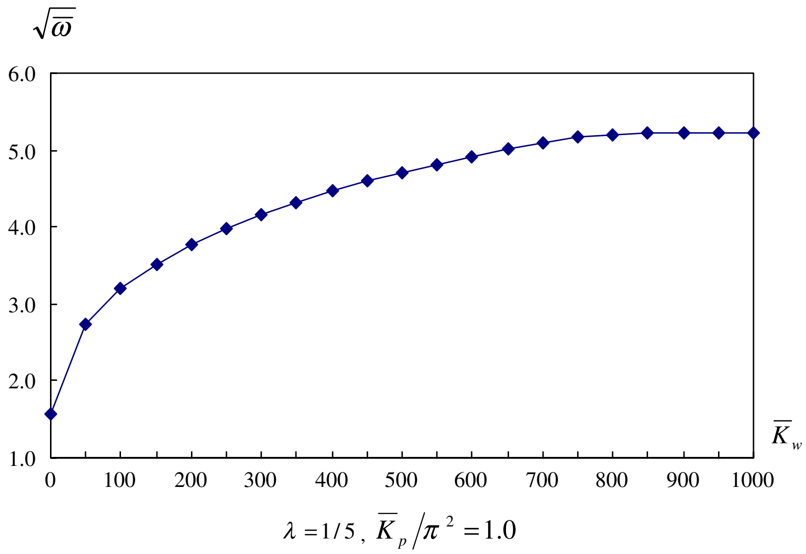

Fig. 3. Curve of $\sqrt{\varpi}$ versus $\overline{K}_w$ for a C–F beam.

The curve of the fundamental frequency parameter $\sqrt{\varpi}$ with the change of foundation parameter $\overline{K}_w$ is shown in Fig. 3 for a C–F beam. It can easily be seen that the increasing of $\sqrt{\varpi}$ is relatively apparent when $\overline{K}_w \leqslant 800$, whereas it becomes considerably slow when $\overline{K}_w > 800$. Further study indicates that it is also the case for S–S and C–C beams. This observation was also reported for a spherical shell embedded in an elastic foundation [31].

5. Conclusions

A two-dimensional elasticity approach is introduced in this paper for bending and free vibration analysis of beams resting on a Pasternak elastic foundation. On the basis of two-dimensional elasticity, the state space concept makes no hypothesis of deformation along the thickness direction and does not use the Saint-Venant principle when treating the end conditions by discretizing using the DQ technique. Thus, the current method can precisely analyze foundation beams with arbitrary depth-to-length ratio, and can deal with arbitrary end conditions.

As regards DQM, although equally spaced sampling points scheme is very convenient, it has been shown that the convergence characteristic is not satisfactory when disposing the bending of slender thin C–C beams. Hence, unequally spaced discrete points are generally preferred.

Numerical comparisons indicate that the current numerical results have a perfect agreement with that obtained from other methods. In fact, the present mixed method is more accurate than other numerical methods as compared to the exact elasticity solutions for S–S beams shown in Tables 1 and 3. Hence, the present approach can undoubtedly serve as a good reference for future numerical research. The present method also has advantages over any beam theories in describing the exact and complete variation of the beam deformations along the thickness direction. Investigation also shows that the Poisson's ratio has a great influence on the natural frequencies, which renders an important hint in selecting the materials for structural design.

Acknowledgement

The work was supported by the National Natural Science Foundation of China (No. 10002016).

Appendix A

Here, we take an S–S beam for an example to illustrate the derivation of Eq. (25) and the expression of matrix $\mathbf{T}$. Partitioning the matrix $\mathbf{S}$ into four sub-matrices, Eq. (23) can be written as

$$\{\delta(H)\} = \mathbf{S}\{\delta(0)\} = [\mathbf{S}_1 \quad \mathbf{S}_2 \quad \mathbf{S}_3 \quad \mathbf{S}_4]\{\delta(0)\} = \mathbf{S}_1\mathbf{u}(0) + \mathbf{S}_2\boldsymbol{\sigma}_y(0) + \mathbf{S}_3\mathbf{v}(0) + \mathbf{S}_4\boldsymbol{\tau}(0). \quad \text{(A.1)}$$

Similarly, $\mathbf{S}_2\boldsymbol{\sigma}_y(0)$ can be written in the partitioned matrix notation as

$$\mathbf{S}_2\boldsymbol{\sigma}_y(0) = [\mathbf{A}_2 \quad \mathbf{A}_3 \quad \cdots \quad \mathbf{A}_{N-1}]\left\{\begin{array}{c} \sigma_{y2}(0) \\ \sigma_{y3}(0) \\ \vdots \\ \sigma_{y(N-1)}(0) \end{array}\right\} = \sum_{i=2}^{N-1}\mathbf{A}_i\sigma_{yi}(0), \quad \text{(A.2)}$$

in which $\mathbf{A}_i$ is the column vector corresponding to $\sigma_{yi}(0)$. Elimination of $\sigma_{yi}(0)$ from Eq. (A.2) using Eq. (24) yields

$$\begin{aligned}\sum_{i=2}^{N-1}\mathbf{A}_i\sigma_{yi}(0) &= \sum_{i=2}^{N-1}\mathbf{A}_i\left[K_w v_i(0) - K_p\sum_{k=2}^{N-1}W_{ik}^{(2)}v_k(0)\right]\\ &= K_w\sum_{i=2}^{N-1}\mathbf{A}_i v_i(0) - K_p\sum_{i=2}^{N-1}\mathbf{A}_i\sum_{k=2}^{N=1}W_{ik}^{(2)}v_k(0)\\ &= K_w\mathbf{S}_2\mathbf{v}(0) - K_p\sum_{i=2}^{N-1}\mathbf{A}_i\mathbf{w}_i^{(2)}\mathbf{v}(0)\\ &= K_w\mathbf{S}_2\mathbf{v}(0) - K_p\mathbf{S}_2\mathbf{w}^{(2)}\mathbf{v}(0).\end{aligned} \tag{A.3}$$

Hence, we obtain

$$\mathbf{S}_2\boldsymbol{\sigma}_y(0) = \mathbf{S}_2(K_w\mathbf{I}_{N-1} - K_p\mathbf{w}^{(2)})\mathbf{v}(0), \tag{A.4}$$

in which $\mathbf{I}_{N-1}$ is the $(N-1)$th-order unit matrix, and $\mathbf{w}^{(2)}$ is determined by

$$\mathbf{w}^{(2)} = \begin{bmatrix} W_{22}^{(2)} & W_{23}^{(2)} & \cdots & W_{2(N-1)}^{(2)}\\ W_{32}^{(2)} & W_{33}^{(2)} & \cdots & W_{3(N-1)}^{(2)}\\ \vdots & \vdots & \ddots & \vdots\\ W_{(N-1)2}^{(2)} & W_{(N-1)3}^{(2)} & \cdots & W_{(N-1)(N-1)}^{(2)}\end{bmatrix} = \begin{bmatrix}\mathbf{w}_2^{(2)}\\ \mathbf{w}_3^{(2)}\\ \vdots\\ \mathbf{w}_{N-1}^{(2)}\end{bmatrix}. \tag{A.5}$$

Substitution of Eq. (A.4) into Eq. (A.1) gives

$$\{\delta(H)\} = \begin{Bmatrix}\mathbf{u}(H)\\ \boldsymbol{\sigma}_y(H)\\ \mathbf{v}(H)\\ \boldsymbol{\tau}(H)\end{Bmatrix} = \mathbf{T}\begin{Bmatrix}\mathbf{u}(0)\\ \mathbf{v}(0)\\ \boldsymbol{\tau}(0)\end{Bmatrix}, \tag{A.6}$$

where

$$\mathbf{T} = \begin{bmatrix}\mathbf{S}_1 & \mathbf{S}_2(K_w\mathbf{I}_{N-1} - K_p\mathbf{w}^{(2)}) & \mathbf{S}_4\end{bmatrix}. \tag{A.7}$$

References

[1] M. Hetenyi, A general solution for the bending of beams on an elastic foundation of arbitrary continuity, J. Appl. Phys. 21 (1950) 55–58.

[2] M. Eisenberger, J. Clastornik, Vibrations and buckling of a beam on a variable Winkler elastic foundation, J. Sound Vib. 115 (1987) 233–241.

[3] Z. Ding, A general solution to vibrations of beams on variable Winkler elastic foundation, Comput. Struct. 47 (1993) 83–90.

[4] M. Eisenberger, D.Z. Yankelevsky, J. Clastornik, Stability of beams on elastic foundations, Comput. Struct. 24 (1986) 135–140.

[5] F.T.K. Au, D.Y. Zheng, Y.K. Cheung, Vibration and stability of non-uniform beams with abrupt changes of cross-section by using C^1 modified beam vibration functions, Appl. Math. Model. 29 (1999) 19–34.

[6] J. Clastornik, M. Eisenberger, D.Z. Yankelevsky, et al., Beams on variable Winkler elastic foundation, J. Eng. Mech. ASCE 53 (1986) 925–928.

[7] S.H. Farghaly, K.M. Zeid, An exact frequency equation for an axially loaded beam-mass-spring system resting on a Winkler elastic foundation, J. Sound Vib. 185 (1995) 357–363.

[8] P.L. Pasternak, On a new method of analysis of an elastic foundation by means of two foundation constants, Gos. Izd. Lip. po Strait i Arkh. Moscow, 1954 (in Russian).
[9] M.A. De Rosa, Free vibrations of Timoshenko beams on two-parameter elastic foundation, Comput. Struct. 57 (1995) 151–156.
[10] N.R. Naidu, G.V. Rao, Vibrations of initially stressed uniform beams on two-parameter elastic foundation, Comput. Struct. 57 (1995) 941–943.
[11] Y. Ayvaz, K. Özgan, Application of modified Vlasov model to free vibration analysis of beams resting on elastic foundations, J. Sound Vib. 255 (2002) 111–127.
[12] S.Y. Lee, H.Y. Kes, Free vibrations of non-uniform beams resting on non-uniform elastic foundation with general elastic end restraints, Comput. Struct. 34 (1990) 421–429.
[13] C. Franciosi, A. Masi, Free vibrations of foundation beams on two-parameter elastic soil, J. Sound Vib. 47 (1993) 419–426.
[14] C.M. Wang, K.Y. Lam, X.Q. He, Exact solutions for Tiomoshenko beams on elastic foundations using Green's functions, Mech. Struct. Mach. 26 (1998) 101–113.
[15] M.A. De Rosa, M.J. Maurizi, The influence of concentrated masses and Pasternak soil on the free vibrations of Euler beams—exact solution, J. Sound Vib. 212 (1998) 573–581.
[16] S.H. Ho, C.K. Chen, Analysis of general elastically end restrained non-uniform beams using differential transform, Appl. Math. Model. 22 (1998) 219–234.
[17] C.N. Chen, Vibration of prismatic beam on an elastic foundation by the differential quadrature element method, Comput. Struct. 77 (2000) 1–9.
[18] J.M. Davies, An exact finite element for beam on elastic foundation problems, J. Struct. Mech. 14 (1986) 489–499.
[19] R.C. Kar, T. Sujata, Parametric instability of Timoshenko beam with thermal gradient resting on a variable Pasternak foundation, Comput. Struct. 36 (1990) 659–665.
[20] C.N. Chen, Solution of beam on elastic foundation by DQEM, J. Eng. Mech. ASCE 124 (1998) 1381–1384.
[21] X. Wang, C.W. Bert, A.G. Striz, Differential and quadrature analysis of deflection, buckling and free vibration of beams and rectangular plates, Comput. Struct. 48 (1993) 473–479.
[22] P.A.A. Laura, R.H. Gutierrez, Analysis of vibrating Timoshenko beams using the method of differential quadrature, Shock and Vibration 1 (1993) 89–93.
[23] Z.A. Siddiqi, A.R. Kukreti, Analysis of eccentrically stiffened plates with mixed boundary conditions using differential quadrature method, Appl. Math. Model. 22 (1998) 251–275.
[24] C.N. Chen, A derivation and solution of dynamic equilibrium equations of shear undeformable composite anisotropic beams using the DQEM, Appl. Math. Model. 26 (2002) 833–861.
[25] C.N. Chen, Buckling equilibrium equations of arbitrarily loaded nonprismatic composite beams and the DQEM buckling analysis using EDQ, Appl. Math. Model. 27 (2003) 27–46.
[26] A.I. Lur'e, Three-Dimensional Problems of Theory of Elasticity, Interscience Publishers, New York, 1964.
[27] C. Shu, B.E. Richards, Application of generalized differential quadrature to solve two-dimensional incompressible Navier–Stokes equations, Int. J. Num. Methods Fluids 15 (1992) 791–798.
[28] C.W. Bert, M. Malik, Differential quadrature method in computational mechanics: a review, Appl. Mech. Rev. 49 (1996) 1–28.
[29] A.N. Sherbourne, M.D. Pandey, Differential quadrature method in the buckling analysis of beams and composite plates, Comput. Struct. 40 (1991) 903–913.
[30] H.J. Ding, W.Q. Chen, R.Q. Xu, On the bending, vibration and stability of laminated rectangular plates with transversely isotropic layers, Appl. Math. Mech. 22 (2001) 16–22.
[31] H.J. Ding, W.Q. Chen, Nonaxisymmetric free vibrations of a spherically isotropic spherical shell embedded in an elastic medium, Int. J. Solids Struct. 33 (1996) 2575–2590.