Response regimes of narrow-band stochastic excited linear oscillator coupled to nonlinear energy sink

Xiong Huai, Kong Xianren*, Yang Zhengu, Liu Yuan

Research Center of Satellite Technology, Harbin Institute of Technology, Harbin 150001, China

Received 13 June 2014; revised 11 August 2014; accepted 22 December 2014
Available online 23 February 2015

KEYWORDS
Nonlinear energy sink; Path integration method; Stochastic excitation; Targeted energy transfer; Vibration absorption

Abstract This paper draws attention to the issue of the vibration absorption of nonlinear mechanical system coupled to nonlinear energy sink (NES) under the impact of the narrow band stochastic excitation. Firstly, based on the complex-averaging method and frequency detuning methodology, response regimes of oscillators have been researched under the linear impact of coupling a nonlinear attachment with less relativistic mass and an external sinusoidal forcing, of which results turn out that the quasi-periodicity response regime of system which occurs when the external excitation amplitude exceeds the critical values will be the precondition of the targeted energy transfer. Secondly, basing on the path integration method, vibration suppression of NES has been researched when it is affected by a main oscillator with a narrow band stochastic force in the form of trigonometric functions, of which results show that response regimes are affected by the amplitude of stochastic excitation and the disturbance strength. Finally, all these conclusions have been approved by the numerical verification and coincided with the theoretical analysis; meanwhile, after the comparing analysis with the optimal linear absorber, it turns out that the NES which is affected by the narrow band stochastic force could also suppress the vibration of system with a better effect.

1. Introduction

The research of adding a strong nonlinear light-weight attachment on the linear oscillator has attracted a lot of attention. A significant number of researches show that the system which uses the targeted energy transfer (TET) as mechanism could offer higher energy transfer efficiency.1–5 Meanwhile, the energy exchange phenomenon in the non-conservative systems could be understood or explained as the nonlinear normal modes (NNMs).1 There is huge energy exchange existing between different NNMs. Under a certain condition, the localization will occur in the energy pumping.6,7 Using this localization and irreversible transfer, the nonlinear energy sink (NES) realizes its high efficiency vibration absorber and vibration suppression.8–10

It is Gendelman et al. who firstly analyze the dynamic behavior of ideal Hamilton system and supply the numerical proof of the TET from the linear oscillator to the nonlinear
The system under consideration is depicted in Fig. 1. It consists of a linear oscillator coupled to a nonlinear attachment (pure cubic nonlinearity and linear damp). The primary structure (linear oscillator) is subjected to various different external disturbances such as impact loading, periodic or random forcing, and we only focus on the narrow-band stochastic excitation in this paper.

The dynamic motion of this system is given by the set of equations

\[
\begin{align*}
    m_1 \ddot{x}_1 + c_1 x_1 + k_1 x_1 + c_2 (x_1 - \ddot{x}_2) + k_2 (x_1 - x_2)^3 &= F(t) \\
    m_2 \ddot{x}_2 + c_2 (x_2 - \ddot{x}_1) + k_2 (x_2 - x_1)^3 &= 0
\end{align*}
\]  

(1)

After change of variables, the system can be described by the following equations:

\[
\begin{align*}
    \ddot{x}_1 + \varepsilon \dddot{x}_1 + \varepsilon^2 x_1 + \varepsilon \dddot{x}_2 (x_1 - \ddot{x}_2) + k_0 (x_1 - x_2)^3 &= f(t) \\
    \ddot{x}_2 + \varepsilon \dddot{x}_2 (x_2 - \ddot{x}_1) + k_0 (x_2 - x_1)^3 &= 0
\end{align*}
\]  

(2)

where \( m_1 \) and \( m_2 \) are the mass of primary oscillator and NES, \( k_1 \) and \( k_2 \) are the linear stiffness and nonlinear stiffness, \( c_1 \) and \( c_2 \) represent the damp of primary oscillator and NES, \( x_1 \) and \( x_2 \) refer to the displacement of two oscillators respectively, \( \varepsilon \ll 1 \) is a small parameter which represents the mass ratio of the linear oscillator and the attachment, \( \lambda_1 = c_1 / m_2 \) and \( \lambda_2 = c_2 / m_2 \) relate to damping coefficient, \( k_0 = k_2 / m_1 \) is the cubic stiffness and \( \varepsilon^2 = k_1 / m_1 \) is natural frequency of the primary structure and \( f(t) = F(t) / m_1 \) denotes the external force. In Eq. (2), all coefficients are adopted to be of order unity for the simplicity of the analytical treatment.

A change of state variables:

\[
\begin{align*}
    u_1 &= x_1 + \varepsilon x_2 \\
    u_2 &= x_1 - x_2
\end{align*}
\]  

(3)

By applying several simple algebraic simplification, rewriting Eq. (2) yields

\[
\begin{align*}
    \ddot{u}_1 + \varepsilon^2 u_1 + \frac{\varepsilon}{1 + \varepsilon} \dddot{u}_1 + \frac{\varepsilon^2}{1 + \varepsilon} \dddot{u}_2 (u_2 - u_1) &= f(t) \\
    \ddot{u}_2 + \varepsilon^2 u_2 + \frac{\varepsilon}{1 + \varepsilon} \dddot{u}_2 + \frac{\varepsilon^2}{1 + \varepsilon} \dddot{u}_1 (u_2 - u_1) &= f(t) \\
    + (1 + \varepsilon) \dddot{u}_2 + \frac{1 + \varepsilon}{\varepsilon} k_0 u_2^3 &= f(t)
\end{align*}
\]  

(4)

2.1. Analytical study for harmonic force with a tuning method

The form of narrow-band excitation is defined as Wedig,\textsuperscript{29} which is expressed by

\[ f(t) = \varepsilon A \cos(\Omega t + \sigma W(t)) \]  

(5)

where \( \varepsilon A \) is the amplitude of stochastic excitation, \( \Omega \) the center frequency, \( W(t) \) standard Wiener excitation. \( \sigma \) is a tuned parameter and represents the disturbance strength and bandwidth of the random excitation, and \( \sigma \) is a rather small value for narrow-band excitation. We analyze the response regimes of system under harmonic force with a tuning method, because the form of harmonic force is similar to random excitation.

Response regimes of system with NES under periodic forcing are studied in a large amount of literature.\textsuperscript{7,17,18} Therefore, we briefly discuss the performance of vibration mitigation of a linear oscillator coupled to a single degree of freedom (DOF)
Near the external resonance region the external forcing is defined as

\[ f(t) = \varepsilon A \cos(\omega_0 t + \omega t) \]  

(6)

We are interested in the motion or response of the system in the vicinity of 1:1 resonance. Complex variables are introduced according to the following relationship \(^1\):

\[
\begin{align*}
\varphi_1(t) \exp(i \omega_0 t) &= i \varphi_j + i \omega \varphi_j (j = 1, 2) \\
\varphi_1(t) \exp(-i \omega_0 t) &= i \varphi_j - i \omega \varphi_j
\end{align*}
\]

(7)

Certainly, we assume that \( \varphi_j(t) \) slowly varies compared to the frequency of external forcing. Eq. (4) can be rewritten as

\[
\begin{align*}
\varphi_1 + i \frac{\omega_0}{2(1 + \varepsilon)} (\varphi_1 - \varphi_2) + \frac{\varepsilon \lambda_1}{2(1 + \varepsilon)} \varphi_1 &= \frac{\varepsilon A}{2} \exp(i \omega t) \\
\varphi_2 + i \frac{\omega_0}{2(1 + \varepsilon)} (\varphi_2 - \varphi_1) + \frac{\varepsilon \lambda_1}{2(1 + \varepsilon)} \varphi_1 + \frac{(1 + \varepsilon) \lambda_2}{2} \varphi_2 &= \frac{\varepsilon A}{2} \exp(i \omega t) \\
- 3i \omega \left( 1 + 1/8 \omega_0^2 \right) |\varphi_j|^2 \varphi_j &= \frac{\varepsilon A}{2} \exp(i \omega t)
\end{align*}
\]

(8)

By several simple mathematical transformations and according to the following expansion, multiple scale analysis is introduced as

\[
\begin{align*}
\varphi_j &= \varphi_j(\tau_0, \tau_1, \cdots) \\
\tau_0 &= \varepsilon^4 (n = 0, 1, \cdots) \\
\frac{d}{dt} &= \frac{\partial}{\partial \tau_0} + \frac{\partial}{\partial \tau_1} + \cdots = D_0 + \varepsilon D_1 + \cdots
\end{align*}
\]

(9)

The “fast” evolution of the system is obtained, which is the order \( O(\varepsilon^0) \). Then Eq. (8) can be integrated and presented in the following form:

\[ D_0 \varphi_2 + \varepsilon \left( i \omega_0 \varphi_2 + \lambda_2 \varphi_2 - i \lambda_2 \varphi_2^* \varphi_2 \right) + \varepsilon^2 C_0(\tau_1, \cdots) = 0 \]  

(10)

In Eq. (10), \( C_0(\tau_1, \cdots) \) is the function about higher order time scales. This paper focuses on the “fast” and “slow” evolution of the averaged system, so higher order (such as \( \tau_2 \) or more higher) is not used in approximate discussion and we only use time scale \( \tau_1 \). Stationary point of Eq. (10) can be express by \( \Phi_j(\tau_1) \) (this function contains the complex variables):

\[ \lambda_2 |\varphi_2|^2 + |\varphi_2|^2 \left( \omega_0 - \lambda_2 \right) \varphi_2 - i \lambda_2 \varphi_2^* \varphi_2 = C(\tau_1) \]  

(11)

In order to investigate the evolution of the amplitude of system, it is convenient to split complex variables into modules and arguments in the following form:

\[ \Phi_j(\tau_1) = R(\tau_1) e^{i \theta(\tau_1)} \]  

(12)

where \( R(\tau_1) \) and \( \theta(\tau_1) \) are real values. Eq. (11) can be rewritten as

\[
\begin{align*}
\lambda_2 N + (\omega_0 - b N) N &= C \\
N &= R^2
\end{align*}
\]

(13)

The argument function \( \theta(\tau_1) \) can be obtained in the same way. In Eq. (13), \( \lambda_2 \) and \( b \) determine the number of solutions. There are two cases about the solutions of the above equations. One is that the function in the left side is monotone; \( C \) has no effect on the number of solutions. Other one is that the function has extreme values, \( C \) may lead to bifurcation. The roots of left equation can help us distinguish two different cases above. The roots are given by

\[ 3b^2 N^2 - 4\omega_0 b N + \omega_0^2 + \lambda_2^2 = 0 \]

\[ \Rightarrow N_{1,2} = \frac{2\omega_0 \pm \sqrt{\omega_0^2 - 3\lambda_2^2}}{3b} \]  

(14)

It is easy to get that \( \lambda_2 = \omega_0 / \sqrt{3} \) is a critical value, and two roots coincide at this time. Certainly, the roots will be increase at \( \lambda_2 < \omega_0 / \sqrt{3} \), and the system has bifurcation phenomenon. The solutions have stable nodes and unstable saddle. In order to discuss, we can use Fig. 2 to describe the function of Eq. (13).

In Eq. (12), the function of \( R \) represents indirectly the amplitude of relative displacement of two oscillators with respect to the slow time scale. We can know the system may give rise to “jump” as the change of \( C \) from Fig. 2. The “jump” between different branches is denoted by dotted and dashed line, because dashed lines in Fig. 2 are unstable. In fact this phenomenon may cause special response. For the sake of discussion, we investigate \( \varphi_2(\tau_1) \) and consider the \( i \) term of expansion in Eq. (8) and we can get easily:

\[ \frac{\partial}{\partial \tau_1} \left( \lambda_2 \varphi_2 + i \omega_0 \varphi_2 - b |\varphi_2|^2 \varphi_2 \right) + \frac{i \omega_0}{2} \left( \lambda_2 \varphi_2 - b |\varphi_2|^2 \varphi_2 - A e^{i \theta(\tau_1)} + \frac{\lambda_1}{1 + \varepsilon} \varphi_1 \right) = 0 \]  

(15)

Because we only have interest in the performance of vibration absorption about NES, we assume that the damp of primary structure is very small or zero, and Eq. (15) may be written as

\[
\begin{align*}
\left[ \lambda_2 + i \omega_0 - b |\varphi_2|^2 \right] \frac{\partial \varphi_2}{\partial \tau_1} - i \lambda_2 \frac{\partial \varphi_2}{\partial \tau_1} &= J(\Phi_2, \Phi_2^*) \\
J(\Phi_2, \Phi_2^*) &= - \frac{i \omega_0}{2} \left( \lambda_2 \varphi_2 - b |\varphi_2|^2 \varphi_2 - A e^{i \theta(\tau_1)} \right)
\end{align*}
\]

(16)

Take complex conjugate transformation of Eq. (16), we can obtain

\[ 3b^2 |\varphi_2|^4 - 4\omega_0 b |\varphi_2|^2 + \omega_0^2 + \lambda_2^2 \frac{\partial \varphi_2}{\partial \tau_1} = \left( \lambda_2 + 2ib |\varphi_2|^2 - i\omega_0 \right) J + ib \Phi_2^* J' \]  

(17)

![Fig. 2](Image 328x91 to 516x272)

**Fig. 2** Relationship of \( N \) and \( C \) with \( \lambda_2 = 0.25 \), \( b = 1. \)
The value in the left-hand (not including the derivative) is similar to Eq. (14) and it is not equal to zero only when ϕ₂(τ₁) denotes the fixed points. We can obtain equations of the reduced flow in polar coordinate through Eq. (17):

\[
\begin{align*}
\frac{\partial R}{\partial \tau_1} &= C_2 \quad (3b^2 R^2 - 4c_b R + c_0^2 + \lambda_2^2) \\
\frac{\partial \theta}{\partial \tau_1} &= C_2 (\omega_0 \sin \theta + 3b^2 R \sin \theta - \lambda_2^2 R + \omega_0 b R^3 - 3b^2 R^2)
\end{align*}
\]

(18)

The treatment in this paper includes the frequency of external excitation with a tuning method, which is not necessary for the system under an external sinusoidal forcing.26-28 But the frequency of system under a stochastic forcing is variable. Therefore, we must take into account the frequency in treatment and analyze the effect on system response. Equilibrium points of the system are found from Eq. (18) by setting both equations equal to zero, then providing

\[
\begin{align*}
A (\dot{\theta} + \omega_0 \cos \theta - b R \cos \theta) - \omega_0 \dot{\lambda}_2 R &= 0 \\
A (\dot{\lambda}_2 \cos \theta - \omega_0 \sin \theta + 3b^2 R \sin \theta) - \dot{\lambda}_2^2 R + \omega_0 b R^3 - 3b^2 R^2 &= 0
\end{align*}
\]

(19)

It is easy to obtain

\[
\begin{align*}
b R^2 + \lambda_2^2 R^2 &= A^2 \\
\tan \theta &= b R^2 / \lambda_2
\end{align*}
\]

(20)

In addition, we can get the following form by simple manipulations according to the first equation of Eq. (19):

\[
\begin{align*}
\cos(\theta + \theta_0) &= \pm \omega_0 \lambda_2 R_1 \sqrt{A^2 (\omega_0 - b R^2)^2 + \lambda_2^2} \\
\sin \theta_0 &= \lambda_2 / \sqrt{(\omega_0 - b R^2)^2 + \lambda_2^2}
\end{align*}
\]

(21)

When the fixed points exist on the fold lines, we can obtain the critical coordinates (θ₀, R₀). We also note that the absolute value of right hand side must be equal to or less than unity, yielding

\[
|\cos(\theta_0 + \theta_0)| = \frac{\omega_0 \lambda_2 R_{1,2}}{A \sqrt{(\omega_0 - b R_{1,2}^2)^2 + \lambda_2^2}} \leq 1
\]

(22)

From Eq. (22), it is easy to get two critical amplitudes of external forcing and one obtains:

\[
\begin{align*}
A_{1,2} &= \omega_0 \lambda_2 R_{1,2} / \sqrt{(\omega_0 - b R_{1,2}^2)^2 + \lambda_2^2} \\
A_{2,2} &= \omega_0 \lambda_2 R_{2,2} / \sqrt{(\omega_0 - b R_{2,2}^2)^2 + \lambda_2^2}
\end{align*}
\]

(23)

Meanwhile, this critical amplitude has no relationship with σ. In other words, response regimes of system do not depend on small values but it associates with the amplitude of external forcing. We can construct several phase portraits for the system with different forcing amplitudes.

In this case (A < A_{1,2}), all trajectories are finally attracted to the fixed points which are below the unstable region. Then response regimes of system are steady-state. As the amplitude increase, the fixed point will move up. In this case, the amplitude of system may occur to “jump”, and TET will happen. Meanwhile, NES has the ability to vibration absorption. It is strongly quasi-periodic response as Refs. 15,28 described. Certainly, it is possible to obtain the steady state response. The response regimes depend on the initial conditions.

As the amplitude of external forcing becomes larger (but still in the same region, A_{1,2} < A < A_{2,2}), phase portraits bring about change obviously. There are no fixed points that are located in the stable regions. The amplitude of relative displacement of the two oscillators, i.e. R may be probable to “jump”. But it depends on initial conditions whether the system realizes TET and has the ability to vibration suppression.

In summary, the response regimes of a harmonically excited linear oscillator coupled with NES depend on the amplitude of external forcing on condition that the values of σ are very small. The amplitude of forcing must be at the interval of \( A \in [A_{1,2}, A_{2,2}] \) in order to vibration absorption. Many simulations and comparisons are described in Refs. 11,16 to verify the conclusions of this section. We only do some brief numeric simulations in Section 3.

2.2. Analysis of dynamics in the case of narrow band stochastic excitation with PI method

Now, we turn to a study of response regimes of primary coupled to NES under narrow-band random excitation in depth. In the present subsection we conclude that the detuning parameter of external forcing has a little effect on response regimes. Response of the primary structure with a narrow band random force may be similar to the response of system subjected to a harmonic forcing as a result of the same form of two excitations. The main difference is that the variables are stochastic under random excitation. In order to verify our judgment, PI method will be used in this section to investigate the performance of vibration absorption of a random forced linear oscillator with an NES attachment.

Due to the fact that small damping is the pre-condition of TET of system and the order of narrow-band excitation is equal to ε which is assumed in Eq. (5), in paper the stochastic averaging method can be used to obtain Fokker–Planck–Kolmogorov (FPK) equations.

This PI method is used to solve the FPK equation and to study the nature of the stochastic response of the nonlinear system. The response constitutes a Markov vector process whose transition probability density function (PDF) is governed by the FPK equation. The solution of FPK equation will give directly the join PDF of the system state. To find the solution of the FPK equation of the system, the stochastic differential equations (SDEs) must be obtained at the first step according to continuous Markov process.

Introducing the new state variables to Eq. (2) under a narrow band stochastic excitation:

\[
\begin{align*}
q_1 &= x_1 \\
q_2 &= x_1 - x_2 \\
p_1 &= \dot{q}_1 \\
p_2 &= \dot{q}_2
\end{align*}
\]

(24)

By simple transformations, the nonlinear dynamical equation can be rewritten in the following state-space form:
\[
\begin{align*}
\dot{q}_1 &= p_1 \\
\dot{p}_1 &= -\alpha_0 \dot{q}_1 - k_0 q_2 - \epsilon (\dot{\lambda}_2 p_1 + \lambda_1 p_2) + \epsilon f(t) \\
\dot{q}_2 &= p_2 \\
\dot{p}_2 &= -\alpha_0 \dot{q}_1 - \frac{1 + \epsilon}{\epsilon} k_0 q_2^2 - \epsilon \left( \dot{\lambda}_2 p_1 + \frac{1 + \epsilon}{\epsilon} \dot{\lambda}_2 p_2 \right) + \epsilon f(t)
\end{align*}
\]

(25)

where \( f(t) = A \cos(\Omega t + \sigma W(t)) \). For the convenience, Eq. (25) is rewritten as

\[
\begin{align*}
\dot{p}_i &= -g_i(q_i) - \epsilon h_i(p_1, p_2, q_1, q_2) + \epsilon f(t) \quad (i = 1, 2)
\end{align*}
\]

(26)

where

\[
\begin{align*}
g_1(q_1) &= \omega_0^2 q_1 \\
g_2(q_2) &= 1 + \frac{\epsilon}{\epsilon} k_0 q_2^2 \\
h_1 &= \frac{\epsilon^2}{\epsilon} \dot{\lambda}_2 p_1 + \lambda_1 p_2 \\
h_2 &= \omega_0^2 q_1 + \lambda_1 p_1 + \frac{1 + \epsilon}{\epsilon} \dot{\lambda}_2 p_2 \\
V_1(q_1) &= \int_0^q g_1(x) \, dx \\
V_2(q_2) &= \int_0^q g_2(x) \, dx
\end{align*}
\]

We assume that the form of solutions can be described as

\[
\begin{align*}
n_i(t) &= A_i(t) \cos \theta_i(t) \\
p_i(t) &= -A_i v_i(A_i, \theta_i) \sin \theta_i(t) \quad (i = 1, 2) \\
\theta_i(t) &= \omega_i(t) + \beta_i(t)
\end{align*}
\]

(27)

In Eq. (27), the instantaneous frequency of the oscillators is written as

\[
v(A_i, \theta_i) = \frac{dx}{dt} = \sqrt{\frac{2[V'(A_i) - V(A_i \cos \theta_i)]}{A_i^2 \sin^2 \theta_i}}
\]

(28)

or

\[
v^{-1}(A_i, \theta_i) = C_0(A_i) + \sum_{n=1}^{\infty} C_n(A_i) \cos(n\theta_i)
\]

(29)

Integrating Eq. (29) yields

\[
t = C_0(A_i) \cdot \omega_i + \sum_{n=1}^{\infty} \frac{1}{n} C_n(A_i) \sin(n\theta_i)
\]

(30)

Further integrating Eq. (30) from 0 to 2\(\pi\) leads to the averaged period

\[
T(A_i) = 2\pi C_0(A_i)
\]

\[
o(A_i) = \frac{1}{C_0(A_i)}
\]

(31)

Thus, we can use the following approximate relation:

\[
\theta_i(t) = o(A_i) t + \beta_i
\]

(32)

Meanwhile we have

\[
V(A_i) = V_i(-A_i)
\]

(33)

The second equation is subtracted from the derivative of the first equation in Eq. (27):

\[
\dot{A}_i \cos \theta_i - A_i \dot{\beta}_i \sin \theta_i = 0
\]

(34)

In the same way, substitute the derivative of the second equation in Eq. (27) into Eq. (25):

\[
\dot{A}_i \left[ v_i \sin \theta_i + A_i \frac{\partial}{\partial A_i} (v_i \sin \theta_i) \right] + \dot{\beta}_i A_i \frac{\partial}{\partial \theta_i} (v_i \sin \theta_i) = h_i(A_i, \theta_i) + f(t)
\]

(35)

where \( A_i, \theta_i, \lambda_i \) and \( \beta_i \) are random processes. Eq. (27) can be regarded as a set of random Van der Pol transformations from \( p_i, q_i \) to \( A_i, \theta_i \). With the transformations accomplished, Eq. (25) becomes

\[
\begin{align*}
\frac{d\theta_i}{dt} &= \frac{\epsilon A_i v_i \sin \theta_i}{g_i(A_i)} [h_i(A_i, \theta_i) - A_i \sin(\omega_0 t + \sigma W(t))] \\
\frac{dA_i}{dt} &= \frac{ev_i \cos \theta_i}{g_i(A_i)} [h_i(A_i, \theta_i) - A_i \sin(\omega_0 t + \sigma W(t))]
\end{align*}
\]

(36)

We are interested in the motion and the response of the system in the vicinity of 1:1 resonance. Thus we can assume that the relation of averaging frequency and external forcing can be written as

\[
o(A_i) = 1 + \sigma_i
\]

(37)

where \( \sigma_i \) is a detuning parameter. Eqs. (30) and (37) can be used to get the following form:

\[
o(A_i) = o(A_i) + \sum_{n=1}^{\infty} \frac{1}{n} C_n(A_i) \sin(n\theta_i) + \sigma_i \omega_i - \beta_i
\]

(38)

Therefore

\[
o(A_i) + \sigma W(t) = \theta_i' + A_i
\]

(39)

where

\[
\theta_i' = \theta_i + \sum_{n=1}^{\infty} \frac{1}{n} C_n(A_i) \sin(n\theta_i)
\]

\[
A_i = \omega_i \omega_i - \beta_i + \sigma W(t)
\]

and \( A \) describes the difference phases of external forcing and response. Based on Eqs. (36) and (39), the SDEs can be written as follows:

\[
\begin{align*}
\frac{dA_i}{dt} &= \langle \frac{\epsilon A_i v_i \sin \theta_i}{g_i(A_i)} [h_i(A_i, \theta_i) - A_i \sin(\theta_i' + A_i)] \rangle_{A_i} \\
\frac{d\theta_i}{dt} &= \left[ \frac{\Omega}{o(A_i)} - 1 \right] v(A_i, \theta_i) - \frac{ev_i \cos \theta_i}{g_i(A_i)} [h_i(A_i, \theta_i) - A_i \sin(\theta_i' + A_i)]
\end{align*}
\]

(40)

In Eq. (40), \( \theta_i \) is “fast” variable, but \( A_i \) and \( A_i' \) are “slow” process. The equation of the limiting diffusion process is of the form:

\[
\begin{align*}
\frac{dA_i}{dt} &= m_1(A_i, A_i') dt \\
\frac{dA_i}{dt} &= m_2(A_i, A_i') dt + \sigma W(t)
\end{align*}
\]

(41)

where

\[
m_1 = \left\langle \frac{\epsilon A_i v_i \sin \theta_i}{g_i(A_i)} [h_i(A_i, \theta_i) - A_i \sin(\theta_i' + A_i)] \right\rangle_{A_i}.
\]

\[
m_2 = \left\langle \left[ \frac{\Omega}{o(A_i)} - 1 \right] v(A_i, \theta_i) - \frac{ev_i \cos \theta_i}{g_i(A_i)} [h_i(A_i, \theta_i) - A_i \sin(\theta_i' + A_i)] \right\rangle_{A_i}
\]

(42)

where \( \langle \cdot \rangle_{A_i} \) represents the averaging process. Eq. (40) can be represented in the following matrix form:

\[
dY = m(Y(t)) dt + \sigma dW(t)
\]
\[
\begin{align*}
Y &= [A_1, A_2, A_3]^	op \\
m &= [m_1, m_2, m_3]^	op \\
G &= [0 \sigma 0 \sigma]^	op
\end{align*}
\]

The averaged FPK equation associated with Ito Eq. (41) is of the form

\[
\frac{\partial p}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial y_i} [m_i(Y)p] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial y_i \partial y_j} [b_{ij}(Y)p]
\]

(43)

where \(b_{ij}\) is the \(ij\)th element of the matrix \(B\).\(Y(t_i) = GG^T\) and \(p(Y_i, y, t)\) is the transition PDF of \(Y(t)\) with the normalization condition

\[
\int_{\mathbb{R}} p(Y, t_i | Y^0, t_o) dY = 1
\]

(44)

The initial condition of FPK equation is

\[
p(Y, 0) = \delta(Y - Y^0)
\]

(45)

The stationary solution of the FPK equation is obtained by set \(\partial p/\partial t = 0\). The PI method is a general procedure for the solution of the FPK equation following the evolution in term of the conditional PDF of the Markov vector process from initial conditions. The evaluation of the PDF is obtained from the following equation:

\[
p(Y, t) = \int_{\mathbb{R}} p(Y, t | Y^0, t_o)p(Y^0, t_o) dY^0
\]

(46)

By dividing time interval \((t_o, t_f)\) into \(N\) small time intervals of length \(\Delta t = (t_f - t_o)/N\), Eq. (46) can be rewritten as

\[
p(Y, t) = \cdots \int \prod_{i=1}^{N} p(Y^i, t_i | Y^{i-1}, t_{i-1}) dY^{i-1} \cdots dY^0
\]

(47)

which is substituted into Eq. (46), yielding

\[
p(Y, t) = \int p(Y, t_i | Y^{i-1}, t_{i-1}) dY^{i-1} \times \cdots \times \int p(Y^1, t_1 | Y^{0}, t_0)p(Y^0, t_0 | Y^0) \]

(48)

This expression shows that a long evolution of the PDF can be expressed in a series of short evolutions. We can also express a typical step of probability evolution from

\[
p(Y^i, t_i) = \int_{\mathbb{R}} p(Y^i, t_i | Y^{i-1}, t_{i-1}) p(Y^{i-1}, t_{i-1}) dY^{i-1}
\]

(49)

where \(\mathbb{R}^d\) represents the reduced state range. For convenience, the essential features of the Gauss–Legendre-based modified PI algorithm will be illustrated by considering the system coupled to NES.

Let us take a one-dimensional system for instance, Eq. (49) can be discretized into the following composite Gaussian–Legendre quadrature form

\[
p(y^{(i)}, t) = \sum_{k=1}^{K} \delta_k \sum_{l=1}^{L_k} c_{kl} p(y^{(i-1)}, t_{i-1}) p(y^{(i-1)}, t_{i-1})
\]

(50)

where \(K\) is the number of sub-intervals, \(L_k\) the number of quadrature points in the subintervals \(K, \delta_k\) the length of sub-interval \(K, \) each \(y_{ki}\) the position of a Gauss quadrature point and \(c_{kl}\) its corresponding weight. Eq. (50) can be used to calculate the PDF at any point at step \(t_i\). In order to proceed to step \(t_{i+1}\), only the following PDF at the Gauss points are required:

\[
p(y^{(i)}, t) = \sum_{k=1}^{K} \delta_k \sum_{l=1}^{L_k} c_{kl} p(y^{(i-1)}, t_{i-1}) p(y^{(i-1)}, t_{i-1})
\]

(51)

where \(y_{mn}\) is Gauss points. Eq. (51) provides a scheme to calculate the evolution of a PDF step by step from a given initial PDF. Each transition probability \(p(y^{(i)}, t | y^{(i-1)}, t_{i-1})\) in Eq. (51) is assumed to be Gaussian, which only depends on the conditional mean and variance of \(y_{mn}(t)\). Certainly, the equations for the statistical moments can be derived from the original equations. With this we can get the conditional mean and variance at any time.

However, Gaussian transition PDF may significantly misrepresent the frequency of high response level, which is important in estimating the first passage failure probabilities, when structural behavior is strongly nonlinear. The PI method based on the adjustable non-Gaussian transition PDF is expressed as a product of a Gaussian PDF. Hermite polynomials have been widely used to approximate probability distributions by virtue of their orthogonal property with respect to the Gaussian PDFs. Hence, the transition PDF from \(y_{kl}\) at \(t_{i-1}\) to \(y_{mn}\) at \(t_i\) is assumed in the non-Gaussian form as

\[
p(y^{(i)}, t | y^{(i-1)}, t_{i-1}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} He_m\left(\frac{(y^{(i)} - m_l(t))}{2\sigma(t)}\right) \exp\left\{ -\left[\frac{y^{(i)} - m_l(t)}{2\sigma(t)}\right]^2\right\}
\]

(52)

where \(He_m(x)\) represents the Hermite polynomial and satisfies the relationship given by

\[
\frac{d}{dx} He_m(x) = nHe_{m-1}(x)
\]

(53)

The coefficients \(c_m\) in Eq. (52) can be evaluated as expectation of Hermite polynomials

\[
c_m = (-1)^n \int_{-\infty}^{\infty} He_m(x)p(x) dx
\]

(54)

And the PDF \(p(x)\) is a standardized density function which gives \(c_0 = 1\) and \(c_1 = c_2 = 0\) here. However, \(c_m\) will be more complicated at \(n > 4\), attention of us is commonly limited \(n = 4\) and \(m = 3\) in fact.

As well, PDF of the primary with an NES attachment can be calculated with the same procedures; the only difference is that the system is high-dimensional and the numerical results are more complicated. With \(p(Y, t)\), we can get the probability function of the system such as \(p(y_1, t)\) or \(p(y_3, t)\) at any time by the following relationships:

\[
p(y, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(Y, t) dy_1 dy_2 \cdots dy_j (j\neq i)
\]

(55)

3. Mathematical models of the system under consideration

In the present section, we will provide numerical confirmation for the analytically predicted phenomena in Section 2. As mentioned above, the system coupled with NES under the external excitation which has the similar form with trigonometric functions has many different responses. The response regimes of
system with a nonlinear attachment are connected with the amplitude of external force. No matter what the response regimes is, we only pay attentions to the performance of vibration absorption of NES. The amplitude of the system response is time-dependent. Therefore, we use average energies over period of time by at least an order of magnitude in order to assess the absorber efficiency.

The total average energy of system (corresponding to conservation part of system) is given by

$$E_{\text{total}} = \frac{1}{2} x_0^2 + \frac{1}{2} x_1^2 + \frac{1}{4} k_n(x_1 - x_2)^4 + \frac{1}{2} \varepsilon x_2^2$$  \hfill (56)

In this section, we compare the total average energy response of nonlinear absorber with a best tuned linear absorber or without any attachment. The comparison is made for the external amplitude which is variable and other parameters are constant.

Given a linear absorber, the equations of motion of system under the external excitation are given by

$$\begin{cases}
\ddot{x}_1 + \varepsilon \dot{x}_1 + x_0^2 x_1 + \varepsilon \lambda_2 (\dot{x}_1 - \dot{x}_2) + k_L (x_1 - x_2) = f(t) \\
\varepsilon \ddot{x}_2 + \varepsilon \lambda_2 (\dot{x}_2 - \dot{x}_1) + k_L (x_2 - x_1) = 0
\end{cases}$$  \hfill (57)

where $k_L$ is a linear stiffness. As we know, the optimal stiffness and damping of linear absorber can be represented as the following form\textsuperscript{17} based on the minimum of root mean square value under stochastic excitation.

$$\begin{cases}
k_{\text{opt}} = \varepsilon/(1 + \varepsilon)^2 \\
\lambda_{\text{opt}} = \sqrt{\varepsilon}/(1 + \varepsilon)^2
\end{cases}$$  \hfill (58)

In order to evaluate the performance of absorber, we use the following criteria to compare the current one with the optimal linear absorber

$$\eta = \left(1 - \frac{E_{\text{output}}}{E_{\text{input}}}\right) \times 100\%$$  \hfill (59)

where $E_{\text{input}}$ and $E_{\text{output}}$ represent the total energy of input and output, respectively. Input and output can be used as average total energy or root mean square values. We use the root mean square value to evaluate the performance of absorber when the primary oscillator is subjected to a random excitation.

In our particular work, we adjust the forcing amplitude for relatively low values of damping parameter to provide the comparable amplitudes of two oscillators.

Fig. 3 Time series of relative displacement of system coupled with an NES (system parameters are $A = 0.25, \varepsilon = 0.1, k_n = 0.15, \lambda_1 = 0.1, \lambda_2 = 0.15$).

Fig. 4 Total system energy and primary mass kinetic energy for nonlinear and linear absorbers (system parameters are $A = 0.25, \varepsilon = 0.1, k_n = 0.15, \lambda_1 = 0.1, \lambda_2 = 0.15$).
3.1. System response to a sinusoidal excitation

Because the response of system is steady-state response when the amplitude of external sinusoidal excitation is zero or out of the critical values decided by Eq. (23), NES is not able to absorb vibration and we will not do any numerical simulations about this case.

For the case of the external amplitude that is inside the two critical values (i.e. \( A_{1cr} < A < A_{2cr} \)), time series of relative displacement between the linear oscillator and the nonlinear attachment are illustrated in Fig. 3, and initial conditions of numerical simulation have been chosen out of unstable regions associated with the critical values.

The first simulation was performed for an amplitude value higher than the critical values \( A_{1cr} \) in Fig. 3, and the other parameters were set as: \( \varepsilon = 0.1, k_n = 0.15, \lambda_1 = 0.1, \lambda_2 = 0.25, A_{1cr} = 0.222 \) and \( A_{2cr} = 0.975 \). Time series of relative displacement of original system is plotted and the response of approximate system has also been plotted. We can note that the analytical model and numerical simulation is in a fair agreement with each other no matter what the value of detuning parameter is. Meanwhile the response regimes of system belong to steady-state response as depicted by phase portraits in the sections before. There is only one stable attractor for the set of parameters.

Fig. 4(a) shows the total average system energy plot of two absorbers and the linear stiffness has been selected as the optimum. Primary mass kinetic energy for linear and nonlinear vibration absorber is depicted in Fig. 4(b). We can note that the critical energy peaks (total average energy peak and kinetic energy peak) of nonlinear attachment are higher than the linear attachment, but the area occupied under the energy curve of nonlinear system is less than linear system (The total average energy of nonlinear absorber and linear absorber is 0.0323 and 0.0382 respectively. The kinetic energy of primary oscillators of nonlinear and linear absorber is 0.0129 and 0.0133 respectively). It means that NES is more efficient in a wide range of frequency than the optimal linear attachment.

As the previous discussion shows, the set of parameters cannot make the NES transfer the energy from the primary structure to the nonlinear attachment, i.e. the TET phenomenon does not occur. The reason why the nonlinear absorber has a very high efficiency to vibration suppression is that the nonlinear stiffness is so small that the NES cannot "jump". The action of nonlinear attachment is similar to the optimal linear attachment.

---

**Fig. 5** Time series of relative displacement of system coupled with an NES (system parameters are \( A = 0.50, \varepsilon = 0.1, k_n = 0.28, \lambda_1 = 0.1, \lambda_2 = 0.25 \)).

**Fig. 6** Total system energy and primary mass kinetic energy for nonlinear and linear absorbers (system parameters are \( A = 0.50, \varepsilon = 0.1, k_n = 0.28, \lambda_1 = 0.1, \lambda_2 = 0.25 \)).
The next simulation was performed for an amplitude value \((y_1 - y_2)\) belonging to the predicted threshold, but it is higher than the previous simulation and lower than the upper critical value (see Fig. 5). The rest system parameters are left the same. It is obvious that the response of relative displacement of two oscillators has a periodicity. As the analytical model depicted in the above section, the relative displacement of system has been jumped from one stable branch to another. Meanwhile the approximate system is in agreement with numerical simulation, in that the relative displacement has very strong modification and periodicity is able to mitigate efficiently. It can be concluded that the NES has the capability of vibration suppression when the relative displacement has a periodicity. This conclusion can help us discuss the performance of vibration absorption of system with an NES attachment under a random excitation.

As the amplitude of external excitation increases, the relative displacement of system has been jumped from one stable branch to another. The system coupled to an NES mitigate efficiently, because of TET. Figs. 6(a) and (b) provide the results for the total average energy of system and the kinetic energy of primary mass. The peaks of two curves of nonlinear attachment are almost equal to linear attachment, the effectiveness of the nonlinear absorber is more better because the linear attachment is optimum and the nonlinear attachment has not use any optimal technology. Certainly, the area occupied under the curves of nonlinear system is less than linear. It also demonstrates that NES has a good performance of vibration absorber than the linear attachment as the results of previous section concluded.

3.2. System response to a narrow-band stochastic excitation

As previous investigated, the system response regimes for the case of sinusoidal excitation amplitudes among the two thresholds have two different responses which the regimes are depended on the initial conditions. However the initial conditions have no impact on the system regimes of narrow-band stochastic excitation with an NES attachment; the amplitude and the disturbance intensity have a major role and the relation between performance of NES and amplitude of random force is similar to the NES attachment under sinusoidal excitation. In order to verify our conclusions, we conduct some numerical verification with two different external amplitudes; one is near the smaller critical values and the other is bigger one near the upper threshold.

![Fig. 7](image1)  
(a) \(\sigma=0.1\)

![Fig. 8](image2)  
(a) \(\sigma=0.1\)
The first numerical simulation for the lower narrow-band amplitude of $A = 0.25$ was performed. Time series of the response of system is presented in Figs. 7(a) and (b) that the disturbance intensity is 0.1 and 0.9, respectively. As becomes clear from the results, only similar steady state response may be observed time series of the amplitude varies very little as the time changes when the disturbance intensity is small in Fig. 7(a). As the disturbance intensity increases, the variation of the amplitude increases, too. But this variation is smooth as Fig. 7(b) illustrates. It means that the nonlinear of system has no effect on the energy transfer, or the amplitude of excitation is not enough to jump and bring about TET.

In the next numerical simulations, we are interested in studying the performance of vibration suppression of NES. The power spectrum density (PSD) $S(f)$ of the displacement of nonlinear absorber and linear absorber is presented in Figs. 8(a) and (b) by the numerical method that the disturbance intensity is 0.1 and 0.9, respectively. Certainly, the system parameters coupled to linear absorber are the optimal. We can note that the peaks of linear absorber are lower than the nonlinear absorber, but the frequency of primary oscillator does not changed and it is still near the natural frequency. It is clear that the area of the linear attachment occupied under the curve of PSD is lower than the nonlinear attachment. It means that the root mean square value of linear attachment is lower than the nonlinear system. We can know that the efficiency of absorbers is respectively based on Eq. (59) under the narrow-band excitation. Although the performance of NES is not better than the optimal linear absorber, the efficient of NES is still over when the amplitude of the narrow-band excitation is near the lower critical value.

In the next simulations we increase the value of the amplitude of narrow-band excitation to $A = 0.50$, time series of relative displacement of two oscillators is illustrated in Figs. 9(a) and (b). It is obvious that the variation of amplitude of response is similar to the response of oscillators under the sinusoidal excitation, because the system has brought about jumping. When the disturbance intensity increases, the variation of amplitude takes on the random force, but this change is very different from the low amplitude of previous simulations: it not only has the periodicity but also changes violently. It is clear from the results of Fig. 9(b) that the amplitude of narrow-band has brought about the jump. It means that the system can transfer energy from the primary oscillator to the NES. In other words, the NES could absorb the vibration from the primary.

The PSD of the primary oscillator with absorbers is depicted in Figs. 10(a) and (b) under the excitation amplitude.
of Figs. 9(a) and (b). The set of system parameter is the same as the previous simulations. We can note that the peaks of nonlinear absorber are nearly equal. And the frequency of primary oscillator has a little bit of change near the natural frequency; this variation will be obvious as the bandwidth of narrow-band excitation increases, because the relative displacement of two oscillators brings about the jump. And the energy of primary oscillators can transfer to the attachment as the previous analysis with the PI method. In other words, the higher excitation amplitude can bring about the TET, and the NES can absorb vibration from the primary oscillator.

In the same way, comparison with the area occupied under the curve of PSD can get the performance of vibration suppression. For this case of amplitude, the root mean square value of nonlinear absorber is higher than the linear absorber (Based on the Eq. (59), the efficiencies of nonlinear and linear absorbers are 95.2% and 92.3% respectively in Fig. 10(a). And the efficiencies are 94.4% and 93.2% respectively), it means that the efficiency of NES is better than the optimal linear attachment. Namely, the NES can not only absorb the energy of primary oscillators under the sinusoidal excitation but also the narrow-band stochastic excitation.

4. Conclusions

This paper simply sketches out the energy transfer excitation and the response regimes of the system coupled to an NES under the sine excitation force through the complex-averaging method and the detuning parameter theory. Combining with the conclusion of the relevant literature, vibration suppression of NES under the narrow band stochastic excitation has been researched on the basis of the PI method. Also its suppression efficiency has been tested. Results show that under the narrow band stochastic force in the form of trigonometric functions, the system will present the similar response regime excited by the sine force, which are the quasi-periodic regime and the steady-state regime. Although the stochastic jump is the energy transferring mechanism at this moment, the TET could also be realized. Compared with the numerical research of the optimal linear absorber, it turns out that without any parameter optimization, the suppression effect on the narrow band stochastic excitation of the NES is better than the optimal linear absorber.

This part summarizes the main achievements of the present researches as follows:

The amplitude and frequency of external excitation such as the sine or stochastic will affect the response regime of the coupling NES system. Response regimes of system depend on the external excitation amplitude and the initial condition of system. But the system affected by the narrow band stochastic excitation depends on the amplitude of external force and the disturbance strength. Under certain disturbance strength or amplitude, response regimes of system will obviously change. But the amplitude only matters the parameter setting of NES.

On the basis of the PI method, it shows that under the narrow band stochastic excitation, there are two different response regimes existing when coupling the linear main oscillator with the NES; one is the steady-state regime, the other is a similar quasi-periodic regime. Probably different from the system affected by the sine excitation, it is the stochastic jump mechanism which could lead to the similar quasi-periodic regime of TET, which could realize the vibration suppression. When the main oscillator is affected by a narrow band stochastic excitation, similar quasi-periodic regime of system could supply higher suppression efficiency than the steady-state regime. In addition, if the central frequency of the narrow band stochastic excitation is near to that of the system, and with certain nonlinear parameter, the suppression efficiency of the NES is better than the optimal linear absorber.

Acknowledgment

The authors thank the anonymous reviewers for their critical and constructive review of the manuscript. This study was supported by the National Natural Science Foundation of China (No. 51375109).

References


Xiong Huai is a Ph.D. student at School of Astronautics, Harbin Institute of Technology. He received his M.S. degree from Harbin Institute of Technology in 2013. His area of research includes nonlinear vibration and structural dynamics.

Kong Xianren is a professor and Ph.D. supervisor at research center of satellite, Harbin Institute of Technology, China. He received the Ph.D. degree from the same university. Her current research interests are structural dynamics, nonlinear dynamics and control, as well as thermal control of satellite.

Yang Zhenguo received the B.S. and M.S. degrees in aerospace design from Harbin Institute of Technology in 2010 and 2012 respectively, and then became a Ph.D. student there. His main research interests are structural dynamics and control.