On the mean value of $L$-functions with the weight of character sums

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Abstract

The main purpose of this paper is using the estimate for character sums and the analytic method to study the mean value of the Dirichlet $L$-functions with the weight of character sums, and give an interesting mean value theorem.

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1. Introduction

Let $q \geq 2$ be an integer, $\chi$ denotes a Dirichlet character modulo $q$, and $L(s, \chi)$ be the Dirichlet $L$-function corresponding to $\chi$. The main purpose of this paper is using the estimate for character sums and the analytic method to study the mean value properties of

$$\sum_{\chi \text{ mod } q \not\equiv \chi_0} \left| \sum_{n \leq N} \chi(n) \right|^2 \cdot |L(1, \chi)|^2,$$

and give an interesting mean value formula for it. About this problem, it seems that none had studied it yet, at least we have not seen such a paper before. However, some people studied the

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related contents, and obtained many interesting results. For example, the author, Yi Yuan and He Xiaoli [4] obtained an asymptotic formula for the 2kth power mean of the Dirichlet L-function with the weight of general Kloosterman sums. The author and Yi Yuan [3] studied the hybrid mean value of inversion of L-function and Gauss sums. Xu Zhefeng and the author [2] studied the 2kth power mean of the character sums over quarter intervals.

In this paper, we shall prove the following:

**Theorem.** Let \( q > 2 \) be an integer. Then for any real number \( N \) with \( 1 < N < \sqrt{q} \), we have the asymptotic formula

\[
\sum_{\chi \mod q} \left| \sum_{n \leq N} \chi(n) \right|^2 \cdot |L(1, \chi)|^2 = \frac{\pi^2}{6} \cdot \frac{\phi^2(q)}{q} N \prod_{p|q} \left(1 - \frac{1}{p^2}\right) + O(\phi(q)2^{\omega(q)} \ln^2 q)
\]

\[
+ \frac{\pi^2}{3} \cdot \frac{\phi^2(q)}{q} N \prod_{p|q} \left(1 - \frac{1}{p^2 + p + 1}\right)
\]

\[
\times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)} + O(N^3 \ln^2 q),
\]

where \( \sum_{\chi \neq \chi_0} \) denotes the summation over all non-principal characters modulo \( q \), \( \phi(q) \) is the Euler function, \( \prod_{p|q} \) denotes the product over all different prime divisors of \( q \), \( \sum_{m=1}^{\infty} \) denotes the summation over all \( m \) such that \( (m, q) = 1 \), and \( \omega(q) \) denotes the number of all different prime divisors of \( q \).

It is clear that

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)}
\]

is a constant depending only on mod \( q \). If \( q = p \) be a prime, then from our theorem we may immediately deduce the following:

**Corollary 1.** Let \( p > 2 \) be a prime. Then for any real number \( N \) with \( 1 < N < \sqrt{p} \), we have the asymptotic formula

\[
\sum_{\chi \mod p} \left| \sum_{n \leq N} \chi(n) \right|^2 \cdot |L(1, \chi)|^2 = \frac{\pi^2}{6} \cdot p \cdot N \cdot \left[ 1 + \frac{2}{\zeta(3)} \sum_{m=1}^{\infty} \frac{1}{m^2} \left( \sum_{k=1}^{m} \frac{1}{k} \right) \right]
\]

\[+ O(p \ln^2 p + N^3 \ln^2 p),\]

where \( \zeta(s) \) is the Riemann zeta-function.
Corollary 2. Let $p > 2$ be a prime, $\varepsilon$ be any fixed positive number. Then for any real number $N$ with $p^{\varepsilon} < N < p^{\frac{1}{2} - \varepsilon}$, we have the asymptotic formula

$$\frac{1}{p} \sum_{\substack{\chi \text{ mod } p \\ \chi \neq \chi_0}} \left| \sum_{n \leq N} \chi(n) \right|^2 \cdot |L(1, \chi)|^2 \sim \frac{\pi^2}{6} \cdot N \cdot \left[ 1 + \frac{2}{\zeta(3)} \sum_{m=1}^{\infty} \frac{1}{m^2} \left( \sum_{k=1}^{m} \frac{1}{k} \right) \right], \quad p \to +\infty.$$ 

Using our method we can also give a similar asymptotic formula for the general $2k$th $(k \geq 2)$ power mean

$$\sum_{\substack{\chi \text{ mod } q \\ \chi \neq \chi_0}} \left| \sum_{n \leq N} \chi(n) \right|^2 \cdot |L(1, \chi)|^{2k},$$

but the constant is very complicate. So we have not given general conclusion in this paper.

2. Several lemmas

In order to complete the proof of the theorems, we need following several lemmas. First we have

Lemma 1. Let $q > 2$ be a positive integer, $a$ be any integer with $1 \leq a \leq q$ and $(a, q) = 1$. Then we have the asymptotic formula

$$\sum_{\chi \text{ mod } q} \chi(a) |L(1, \chi)|^2 = \sum_{\chi \text{ mod } q} \chi(a) \left| \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} \right|^2 + O\left( \frac{\ln^2 q}{\sqrt{q}} \right).$$

Proof. For convenience, firstly we let

$$A(\chi, y) = \sum_{q^2 < n \leq y} \chi(n).$$

Then applying Abel’s identity (see Theorem 4.2 of [1]) we have

$$L(1, \chi) = \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} + \int_{q^2}^{+\infty} \frac{A(\chi, y)}{y^2} dy.$$

From (2) we may get
\[
\sum_{\chi \neq \chi_0} \chi(a) \left| L(1, \chi) \right|^2 = \sum_{\chi \neq \chi_0} \chi(a) \left| \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} + \int_{q^2}^{+\infty} \frac{A(\chi, y)}{y^2} \, dy \right|^2
\]

\[
= \sum_{\chi \neq \chi_0} \chi(a) \left| \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} \right|^2
\]

\[
+ \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} \right) \left( \int_{q^2}^{+\infty} \frac{A(\overline{\chi}, y)}{y^2} \, dy \right)
\]

\[
+ \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{1 \leq n \leq q^2} \frac{\overline{\chi}(n)}{n} \right) \left( \int_{q^2}^{+\infty} \frac{A(\chi, y)}{y^2} \, dy \right)
\]

\[
+ \sum_{\chi \neq \chi_0} \chi(a) \left| \int_{q^2}^{+\infty} \frac{A(\chi, y)}{y^2} \, dy \right|^2.
\]

Using the Cauchy inequality and the estimate \( |\sum_{m \leq n \leq M} \chi(n)| \ll \sqrt{q} \ln q \) we have the estimates

\[
\left| \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} \right) \left( \int_{q^2}^{+\infty} \frac{A(\overline{\chi}, y)}{y^2} \, dy \right) \right| \ll \left( \sum_{\chi \neq \chi_0} \left| \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} \right|^2 \right)^{1/2} \left( \sum_{\chi \neq \chi_0} \left| \int_{q^2}^{+\infty} \frac{A(\chi, y)}{y^2} \, dy \right|^2 \right)^{1/2} \ll \frac{\ln q}{\sqrt{q}},
\]

\[
\left( \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{1 \leq n \leq q^2} \frac{\overline{\chi}(n)}{n} \right) \left( \int_{q^2}^{+\infty} \frac{A(\chi, y)}{y^2} \, dy \right) \right) \ll \frac{\ln q}{\sqrt{q}}
\]

and

\[
\left| \sum_{\chi \neq \chi_0} \chi(a) \int_{q^2}^{+\infty} \frac{A(\chi, y)}{y^2} \, dy \right|^2 \ll \frac{\ln^2 q}{q^2}.
\]

Combining (3)–(5) and (6) we may immediately get the asymptotic formula

\[
\sum_{\chi \mod q \atop \chi \neq \chi_0} \chi(a) \left| L(1, \chi) \right|^2 = \sum_{\chi \mod q \atop \chi \neq \chi_0} \chi(a) \left| \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} \right|^2 + O\left( \frac{\ln^2 q}{\sqrt{q}} \right).
\]

This proves Lemma 1. \( \square \)
Lemma 2. Let $p > 2$ be a prime. Then we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)} = \frac{1}{\zeta(3)} \sum_{m=1}^{\infty} \frac{1}{m^2} \left( \sum_{k=1}^{m} \frac{1}{k} \right) + O\left( \frac{1}{p} \right),$$

where $\sum_{m=1}^{\infty}$ denotes the summation over all $m$ such that $(m, p) = 1$.

Proof. Since $p$ be a prime, so for any integer $m$, $(p, m) = 1$ or $p$. Note that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^3} = \frac{1}{\zeta(3)}$, we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)} + O\left( \sum_{1 \leq n \leq p-1} \frac{1}{p(p-n)n} \right) + O\left( \frac{1}{p} \right).$$

This proves Lemma 2. \qed

3. Proof of the theorem

In this section, we shall complete the proof of the theorem. First from Lemma 1 and the orthogonality relation for character sums modulo $q$ we have

$$\sum_{\chi \neq \chi_0} \left| \sum_{n \leq N} \chi(n) \right|^2 \cdot |L(1, \chi)|^2 = \sum_{\chi \neq \chi_0} \sum_{m \leq N} \sum_{n \leq N}^{'} \chi(nm) |L(1, \chi)|^2$$

$$= \sum_{m \leq N} \sum_{n \leq N}^{'} \sum_{\chi \neq \chi_0} \chi(nm) |L(1, \chi)|^2$$

$$= \sum_{\chi \neq \chi_0} \left| \sum_{n \leq N} \chi(n) \right|^2 \cdot \left| \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} \right|^2 + O\left( \frac{N^2 \ln^2 q}{\sqrt{q}} \right)$$

$$= \sum_{\chi \mod q} \left| \sum_{n \leq N} \chi(n) \right|^2 \cdot \left| \sum_{1 \leq n \leq q^2} \frac{\chi(n)}{n} \right|^2 + O\left( N^2 \ln^2 q \right)$$
\[
\phi(q) \sum_{m \leq N} \sum_{n \leq N} \sum_{1 \leq u \leq q^2} \sum_{1 \leq v \leq q^2} \frac{1}{uv} + O(N^2 \ln^2 q) = \phi(q) \sum_{m \leq N} \sum_{n \leq N} \sum_{1 \leq u \leq q^2} \sum_{1 \leq v \leq q^2} \frac{1}{uv} + O(N^2 \ln^2 q) \\
+ \phi(q) \sum_{m \leq N} \sum_{n \leq N} \sum_{1 \leq u \leq q^2} \sum_{1 \leq v \leq q^2} \frac{1}{uv} + O(N^2 \ln^2 q) \\
\equiv M_1 + M_2, \quad (7)
\]

where \( \sum'_{m \leq N} \) denotes the summation over all \( 1 \leq m \leq N \) with \( (m, q) = 1 \).

Now we estimate \( M_1 \) and \( M_2 \) respectively in (7). We have

\[
M_2 = \phi(q) \sum_{m \leq N} \sum_{n \leq N} \sum_{1 \leq u \leq q^2} \sum_{1 \leq v \leq q^2} \frac{1}{uv} \\
= 2\phi(q) \sum_{m \leq N} \sum_{n \leq N} \sum_{1 \leq u \leq q^2} \sum_{1 \leq v \leq q^2} \frac{m}{muv} \\
\ll \phi(q) \sum_{m \leq N} \sum_{n \leq N} \sum_{1 \leq k \leq q^2} \sum_{1 \leq v \leq q^2} \frac{m}{(kq + nv)v} \\
\ll N^3 \ln^2 q. \quad (8)
\]

Note that \( \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} \prod_p \left( 1 - \frac{1}{p^2} \right) \), \( \sum'_{1 \leq u \leq N} 1 = \frac{\phi(q)}{q} N + O(2^{\omega(q)}) \), we have

\[
M_1 = \phi(q) \sum_{m \leq N} \sum_{n \leq N} \sum_{1 \leq u \leq q^2} \sum_{1 \leq v \leq q^2} \frac{1}{uv} \\
= \phi(q) \sum_{d \leq N} \sum_{m \leq \frac{N}{d}} \sum_{n \leq \frac{N}{d}} \sum_{1 \leq v \leq \min\left\{ \frac{q^2}{m}, \frac{q^2}{n} \right\}} \frac{1}{mnv^2} \\
= \phi(q) \sum_{m \leq N} \sum_{n \leq N} \sum_{d \leq \min\left\{ \frac{N}{m}, \frac{N}{n} \right\}} \sum_{1 \leq v \leq \min\left\{ \frac{q^2}{m}, \frac{q^2}{n} \right\}} \frac{1}{mnv^2} \\
= \frac{\pi^2}{6} \frac{\phi^2(q)}{q} N \prod_{p|q} \left( 1 - \frac{1}{p^2} \right) + O(\phi(q)2^{\omega(q)} \ln^2 q)
\]
\[ + \frac{\pi^2}{3} \frac{\phi^2(q)}{q} N \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) \sum' \sum' \frac{1}{nm^2} \]

\[ = \frac{\pi^2}{6} \frac{\phi^2(q)}{q} N \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) + O \left( \phi(q) 2^\omega(q) \ln^2 q \right) \]

\[ + \frac{\pi^2}{3} \frac{\phi^2(q)}{q} N \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) \sum' \sum' \frac{1}{nm^2} \]

\[ = \frac{\pi^2}{6} \frac{\phi^2(q)}{q} N \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) + O \left( \phi(q) 2^\omega(q) \ln^2 q \right) \]

\[ + \frac{\pi^2}{3} \frac{\phi^2(q)}{q} N \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) \sum' \sum' \frac{1}{mn(m+n)} \]

\[ = \frac{\pi^2}{6} \frac{\phi^2(q)}{q} N \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) + O \left( \phi(q) 2^\omega(q) \ln^2 q \right) \]

\[ + \frac{\pi^2}{3} \frac{\phi^2(q)}{q} N \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) \sum' \sum' \frac{1}{mn(m+n)} \]

\[ = \frac{\pi^2}{6} \frac{\phi^2(q)}{q} N \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) + O \left( \phi(q) 2^\omega(q) \ln^2 q \right) \]

\[ + \frac{\pi^2}{3} \frac{\phi^2(q)}{q} \frac{N}{\zeta(3)} \prod_{p \mid q} \left( 1 - \frac{1}{p^2} + \frac{1}{p^2 + p + 1} \right) \sum' \sum' \frac{1}{mn(m+n)} \]

(9)

Combining (7), (8) and (9) we may immediately get

\[ \sum_{\chi \neq \chi_0} \left| \sum_{n \leq N} \chi(n) \right|^2 \cdot |L(1, \chi)|^2 = \frac{\pi^2}{6} \frac{\phi^2(q)}{q} N \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) + O \left( \phi(q) 2^\omega(q) \ln^2 q \right) \]

\[ + \frac{\pi^2}{3} \frac{\phi^2(q)}{q} \frac{N}{\zeta(3)} \prod_{p \mid q} \left( 1 - \frac{1}{p^2 + p + 1} \right) \]

\[ \times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)} + O \left( N^3 \ln^2 q \right). \]

This completes the proof of the theorem.

Corollary 1 follows from our theorem and Lemma 2.
References