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Elementary transition systems

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0. Introduction

Transition systems are a simple and powerful formalism for explaining the operational behaviour of models of concurrency. They provide a common framework for investigating the interrelationships between different approaches to the study of distributed systems. Hence an important question to be answered is: which subclass of transition systems corresponds to a particular model of distribted systems? In this paper we provide an answer to this question for elementary net systems. Within net theory, which is one well-established theory of distributed systems, elementary net systems constitute a basic systems model. Using this model, fundamental concepts such as causality, concurrency, conflict and confusion can be clearly defined and separated from each other (see [15]). Much is known about the behavioural aspects of elementary net systems in terms of trace theory, nonsequential processes and event structures as shown in [10]. Trace theory was initiated by Mazurkiewicz [7] (see also [1]). The theory of nonsequential processes originates from the work of Petri [12]; see also [2]. Event structures arose out of the work of Nielsen, Plotkin and Winskel [9] and they now possess a rich theory mainly due to the efforts of Winskel [18]. Elementary net systems also have a strong relationship to transition systems. More precisely, there is a natural way of associating a transition system with each elementary net system in order to explain the operational behaviour of elementary net systems in purely sequential terms. Hence the question arises as to which transition systems correspond to elementary net systems.

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A complete answer to this question was provided by Ehrenfeucht and Rozenberg [4] by identifying—what seems to be—a basic notion called regions for transition systems in general. The class of transition systems identified by this work will be called *elementary transition systems* here. The main aim of this paper is to show that this link established between elementary transition systems and elementary net systems can be lifted to respect appropriately chosen behaviour preserving transformations between elementary transition systems on the one hand and between elementary net systems on the other hand. More precisely, we shall show that the notion of a G-morphism between a pair of elementary transition systems corresponds in a standard fashion to the notion of an N-morphism between the associated pair of elementary net systems. G-morphisms were identified in [4] where they were referred to as uniform morphisms. At the level of transition systems, these morphisms were also identified by Winskel [17]. N-morphisms are a modified form of the morphisms between net systems defined and studied by Winskel [18]. As a consequence of our main result one can go back and forth between a category of elementary transition systems and a category of elementary net systems.

In the process of working out the main result we run into a number of interesting observations. In particular, it turns out that elementary net systems admit canonical representations. These canonical representations are in some sense "maximal" objects in terms of the N-morphisms that are supported by an elementary net system.

Finally, based on the work reported here, we can now look forward to defining operations on elementary net systems based on their transition systems representations. This is rather important because by taking this route one is defining operations on the *behaviour* of net systems whereas operations on net systems themselves invariably boil down to "cutting and pasting" the underlying nets and rearranging the initial markings where necessary; one is still left with the task of computing the behaviour of the resulting object in terms of the behaviour of its constituents.

In the next section we introduce the category $\mathcal{ETS}$ whose objects are elementary transition systems and whose arrows are G-morphisms. In Section 3 we introduce the category $\mathcal{ENS}$ whose objects are elementary net systems and whose arrows are N-morphisms. We then construct in the subsequent two sections a functor H going from $\mathcal{ENS}$ to $\mathcal{ETS}$ and a functor J going from $\mathcal{ETS}$ to $\mathcal{ENS}$. The main result of this paper, namely, J and H constitute an adjunction (in fact, a coreflection) with J as a left adjoint and H as a right adjoint is proved in Section 5. In the concluding section we tie-up some loose ends. We also discuss briefly two other kinds of net morphisms in the literature due to Petri [11] and Meseguer and Montanari [8].

1. Elementary transition systems

A *transition system* is a quadruple $TS = (S, E, T, s_{in})$, where

- S is a *non-empty* set of *states*,
- E is a set of *events*,

- $T \subseteq S \times E \times S$ is the *transition relation*,
- $s_{in} \in S$ is the *initial state*.

Usually transition systems are based on actions which may be viewed as labelled events. Here we are mainly interested in relating transition systems to (unlabelled) net systems and hence the minor departure from conventional practice. If $(s, e, s') \in T$ then the idea is that TS can go from s to s' as a result of the event e occurring at s.

We will often pictorially represent a transition system as a rooted edge-labelled directed graph. An example of a transition system is shown in Fig. 1 where the initial state is decorated as indicated. We will follow this convention through the rest of the paper.

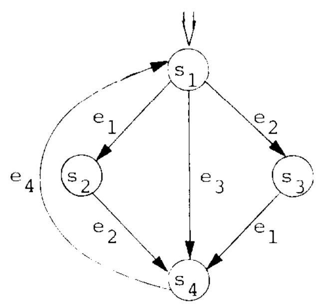

Fig. 1.

Let $TS = (S, E, T, s_{in})$ be a transition system. When TS is clear from the context we will often write $s \xrightarrow{e} s'$ instead of $(s, e, s') \in T$. We say that the event e is enabled at the state s (denoted $s \xrightarrow{e}$) if there exists a state s' such that $s \xrightarrow{e} s'$. Given our aims it is necessary to restrict ourselves to transition systems which fulfill some additional requirements.

From now on, unless stated otherwise, every transition system $TS = (S, E, T, s_{in})$ that we encounter will be assumed to (or will be proved to!) satisfy the following axioms:

(A1) $\forall (s, e, s') \in T.\ s \neq s'$;

(A2) $\forall (s, e_1, s_1), (s, e_2, s_2) \in T.\ [s_1 = s_2 \Rightarrow e_1 = e_2]$;

(A3) $\forall e \in E.\ \exists (s, e, s') \in T$;

(A4) $\forall s \in S - \{s_{in}\} \exists e_0, e_1, \ldots, e_{n-1} \in E$ and $\exists s_1, s_2, \ldots, s_n \in S$ such that $s_0 = s_{in}$, $s_n = s$ and $(s_i, e_i, s_{i+1}) \in T$ for $0 \leqslant i < n$.

(A1) forbids self-loops and (A2) forbids multiple arcs between a pair of states. (A3) demands that every event should have an occurrence and (A4) demands that every state should be reachable from the initial state through a finite sequence of event occurrences. The relative importance of these axioms will become clear at appropriate places in the rest of the paper. We also have some additional remarks concerning these restrictions in the concluding section.

The notion of a *region* will play a central role in this paper.

Definition 1.1. Let $TS=(S, E, T, s_{in})$ be a transition system. Then $r \subseteq S$ is a region of TS iff the following two conditions are satisfied:

(i) $(s, e, s') \in T \wedge s \in r \wedge s' \notin r \Rightarrow \forall (s_1, e, s'_1) \in T. [s_1 \in r \wedge s'_1 \notin r]$

(ii) $(s, e, s') \in T \wedge s \notin r \wedge s' \in r \Rightarrow \forall (s_1, e, s'_1) \in T. [s_1 \notin r \wedge s'_1 \in r]$.

So a region is a subset of states with which *all* occurrences of an event have the same "crossing" relationship. Regions were first identified in [4] where they were used—among other things—to characterize the transition systems that "correspond" to elementary net systems. A region is supposed to model a "local" state where each (global) state consists of a *set* of local states. This will become clear in Section 3. For the transition system shown in Fig. 1, $\{s_1, s_2\}$, $\{s_2, s_4\}$ and $\{s_1, s_2, s_3, s_4\}$ are regions. $r=\{s_1, s_4\}$ is *not* a region because we have $s_1 \xrightarrow{e_1} s_2$ with $s_1 \in r$ and $s_2 \notin r$ while we also have $s_3 \xrightarrow{e_1} s_4$ with $s_3 \notin r$ and $s_4 \in r$. $r'=\{s_1, s_2, s_4\}$ is *not* a region because we have $s_1 \xrightarrow{e_2} s_3$ with $s_1 \in r'$ and $s_3 \notin r'$ while we also have $s_2 \xrightarrow{e_2} s_4$ with $s_2 \in r'$ and $s_4 \in r'$.

It will be convenient to adopt some notations concerning regions. These notations will be extensively used in the sequel.

Let $TS=(S, E, T, s_{in})$ be a transition system. Then it is easy to see that both S and $\emptyset$ are regions. They will be called the *trivial* regions. R_{TS} will denote the set of *non-trivial regions* of TS. For each $s \in S$ we let R_s denote the set of *non-trivial regions containing* s. More precisely,

$$\forall s \in S.\ R_s \stackrel{\text{def}}{=} \{r \mid s \in r \in R_{TS}\}.$$

The set of pre-regions and post-regions of an event will also play an important role.

$$\forall e \in E.\ {}^\circ e \stackrel{\text{def}}{=} \{r \mid r \in R_{TS} \wedge \exists (s, e, s') \in T. [s \in r \wedge s' \notin r]\}$$

(The pre-regions of e),

$$e^\circ \stackrel{\text{def}}{=} \{r \mid r \in R_{TS} \wedge \exists (s, e, s') \in T. [s \notin r \wedge s' \in r]\}$$

(The post-regions of e).

For the transition system shown in Fig. 1,

$$R_{s_1}=\{\{s_1, s_2\}, \{s_1, s_3\}\}, \qquad {}^\circ e_1=\{\{s_1, s_3\}\},$$

$$e_2^\circ=\{\{s_3, s_4\}\} \quad \text{and} \quad {}^\circ e_4=\{\{s_2, s_4\}, \{s_3, s_4\}\}.$$

Here are some useful properties of regions.

Proposition 1.2. *Let $TS=(S, E, T, s_{in})$ be a transition system.*

(i) *$r \subseteq S$ is a region iff $\bar{r} \stackrel{\text{def}}{=} S-r$ is a region.*

(ii) *Suppose $e \in E$. Then $e^\circ=\{\bar{r} \mid r \in {}^\circ e\}$.*

(iii) *Suppose $s \xrightarrow{e} s'$. Then $R_s - R_{s'} = {}^\circ e$ and $R_{s'} - R_s = e^\circ$. Consequently*

$${}^\circ e \subseteq R_s \text{ and } e^\circ \cap R_s = \emptyset \text{ and } R_{s'} = (R_s - {}^\circ e) \cup e^\circ.$$

Proof. (i) follows at once from the definition of a region.

To prove (ii), consider $r \in {}^\circ e$. Then there exists a transition $s \xrightarrow{e} s'$ such that $s \in r$ and $s' \notin r$. This implies that $\bar{r} \in e^\circ$ which in turn implies that $\{\bar{r} \mid r \in {}^\circ e\} \subseteq e^\circ$. To show containment in the other direction is equally easy.

To prove (iii), consider $r \in {}^\circ e$. Then there exists a transition $s_1 \xrightarrow{e} s_1'$ such that $s_1 \in r$ and $s_1' \notin r$. Since r is a region and $s \xrightarrow{e} s'$, we must have $s \in r$ and $s' \notin r$. Clearly r is a non-trivial region. Hence $r \in R_s$ and it is also clear that $r \notin R_{s'}$. Hence ${}^\circ e \subseteq R_s - R_{s'}$. Now consider $r \in R_s - R_{s'}$. Since $s \in r$ and $s' \notin r$ and $s \xrightarrow{e} s'$ we get $r \in {}^\circ e$. Hence $R_s - R_{s'} \subseteq {}^\circ e$. By a symmetric argument we can show that $R_{s'} - R_s = e^\circ$. The rest of (iii) follows now immediately. □

We can now identify the class of elementary transition systems.

Definition 1.3. The transition system $TS = (S, E, T, s_{in})$ is said to be *elementary* if it satisfies (in addition to (A1)-(A4)) the two regional axioms:

(A5) $\forall s, s' \in S. [R_s = R_{s'} \Rightarrow s = s']$.

(State Separation Property)

(A6) $\forall s \in S\ \forall e \in E. [{}^\circ e \subseteq R_s \Rightarrow s \xrightarrow{e}]$.

(Forward Closure Property)

The transition system shown in Fig. 1 is elementary. The two transition systems shown in Fig. 2 are not elementary.

The system shown in Fig. 2(a) is not elementary because $R_{s_1} = R_{s_2} = \{\{s_1, s_2, s_3\}\}$ thus violating (A5). The system shown in Fig. 2(b) is not elementary because ${}^\circ e_1 = \{\{s_1, s_3\}\}$ and hence ${}^\circ e_1 \subseteq R_{s_3} = \{\{s_1, s_3\}, \{s_3, s_4\}\}$. But e_1 is not enabled at s_3 and this violates (A6).

We can now define "behaviour-preserving" morphisms between transition systems.

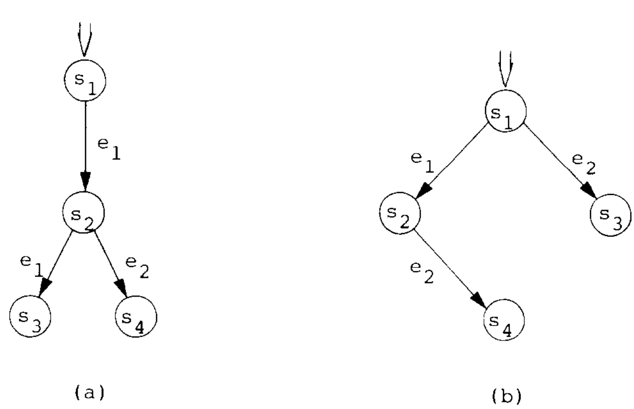

Fig. 2.

Definition 1.4. Let $TS_i = (S_i, E_i, T_i, s^i_{in})$ for $i = 1, 2$ be a pair of transition systems. A *G-morphism* from TS_1 to TS_2 is a function $f: S_1 \to S_2$ which satisfies:

(i) $f(s^1_{in}) = s^2_{in}$,

(ii) $\forall (s, e, s') \in T_1$ [either $f(s) = f(s')$ or there exists $e_2 \in E_2$ such that $(f(s), e_2, f(s')) \in T_2$],

(iii) If $(s, e, s') \in T_1$ and $(f(s), e_2, f(s')) \in T_2$ then $(f(s_1), e_2, f(s_1')) \in T_2$ for every $(s_1, e, s_1') \in T_1$.

The idea is that TS_2 is capable of "partially simulating" TS_1 as specified by f. If $(s, e, s') \in T_1$ and $f(s) = f(s')$ then under f this occurrence of e is "internal" to TS_1 and will not be "seen" by TS_2. The motivations underlying the first two requirements should now be obvious. In [4] maps which satisfy just these two conditions are called morphisms. The third condition demands that all the occurrences of an event should be simulated in a uniform manner. For transition systems that model distributed computations this is a crucial requirement. In TS_1, and event e (in some context) might occur concurrently with a number of other events. This will be reflected in TS_1 by a number of transitions all labelled with e. Clearly the choice of which event in TS_2 simulates e should not depend on a particular transition involving e. In [4] a fourth condition is added to capture the fact that "internal" occurrences are also handled in a uniform fashion. Such morphisms are then called uniform morphisms. We first show that the three conditions imposed in Definition 1.4 suffice to guarantee that internal transitions will also be handled in a uniform way.

Proposition 1.5. *Let $f: TS_1 \to TS_2$ be a G-morphism from TS_1 to TS_2 where $TS_i = (S_i, E_i, T_i, s^i_{in})$ for $i = 1, 2$. Suppose $(s, e, s') \in T_1$ and $f(s) = f(s')$. Then $f(s_1) = f(s_1')$ for every $(s_1, e, s_1') \in T_1$.*

Proof. Suppose $(s_1, e, s_1') \in T_1$. If $f(s_1) \neq f(s_1')$ then by part (ii) of Definition 1.4, there exists $e_2 \in E_2$ such that $(f(s_1), e_2, f(s_1')) \in T_2$. But then by part (iii) of Definition 1.4, this leads to $(f(s), e_2, f(s')) \in T_2$. Since $f(s) = f(s')$ by hypothesis, we now have the contradiction that TS_2 violates the axiom (A1). $\square$

Two examples of G-morphisms are shown in Fig. 3.

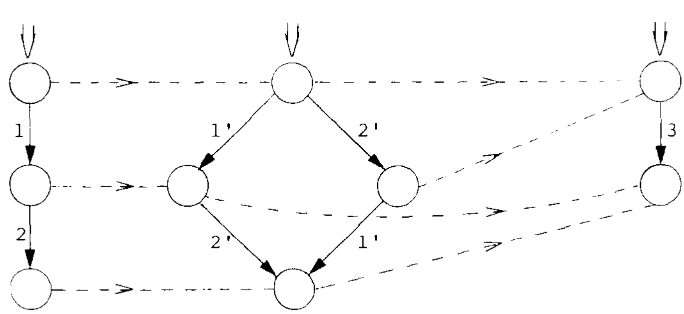

Fig. 3.

It follows that for transition systems satisfying (A2) above, each G-morphism $f: TS_1 \to TS_2$ uniquely determines a partial function $\eta_f : E_1 \to_* E_2$ defined by

$$\eta_f(e_1) = \begin{cases} e_2 & \text{if } (f(s), e_2, f(s')) \in T_2 \text{ for some } (s, e_1, s') \in T_1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In the sequel we will often appeal to this partial function (denoted η_f) determined by the G-morphism f. It turns out that for *deterministic* transition systems, this associated partial function also determines the G-morphism uniquely.

The transition system $TS = (S, E, T, s_{\text{in}})$ is said to be *deterministic* if

$$\forall (s, e_1, s_1), (s, e_2, s_2) \in T. [e_1 = e_2 \Rightarrow s_1 = s_2].$$

Note that in case TS is elementary then TS is deterministic. This follows from the fact that if $(s, e_1, s_1), (s, e_1, s_2) \in T$ then by Proposition 1.2,

$$R_{s_1} = (R_s - {}^\circ e_1) \cup e_1^\circ = R_{s_2}$$

and hence by (A5), $s_1 = s_2$.

Proposition 1.6. *Let TS_1 and TS_2 be two deterministic transition systems and f and g two G-morphisms from TS_1 to TS_2 such that $\eta_f = \eta_g$. Then $f = g$.*

Proof. We prove that $f(s) = g(s)$ for every $s \in S_1$ by induction on the number of event occurrences it takes to reach s from s_{in}^1 (recall axiom (A4)).

Clearly $f(s_{\text{in}}^1) = s_{\text{in}}^2 = g(s_{\text{in}}^1)$ by definition of a G-morphism. Assume $f(s) = g(s)$ for $s \in S_1$ and that $(s, e, s') \in T_1$. If $f(s) = f(s')$ then $\eta_f(e)$ is undefined and hence $\eta_g(e)$ is undefined. But this implies that $g(s) = g(s')$. Thus $f(s') = g(s')$.

If $f(s) \neq f(s')$ then $(f(s), e_2, f(s')) \in T_2$ for some $e_2 \in E_2$. This implies that $\eta_f(e) = e_2$ and so $\eta_g(e) = e_2$. Since g is a G-morphism, from $(s, e, s') \in T_1$ we can now infer that $(g(s), e_2, g(s')) \in T_2$. Since $f(s) = g(s)$ by the induction hypothesis, we have $f(s') = g(s')$ from the fact that TS_2 is deterministic. $\square$

From our observation above it follows that for elementary transition systems, the notion of a G-morphism is exactly the notion of a morphism introduced by Winskel in [17].

A basic property of G-morphisms is that they preserve regions in the following sense.

Proposition 1.7. *Let $f: TS_1 \to TS_2$ be a G-morphism where $TS_i = (S_i, E_i, T_i, S_{in}^i)$ for $i = 1, 2$. Suppose $r \subseteq S_2$ is a region in TS_2. Then $f^{-1}(r)$ is a region in TS_1. Furthermore, for every $e_1 \in E_1$, $f^{-1}(r) \in {}^\circ e_1$ (e_1°) iff $\eta_f(e_1)$ is defined and $r \in {}^\circ\eta_f(e_1)$ ($\eta_f(e_1)^\circ$, respectively).*

Proof. Set $f^{-1}(r) = r'$. Suppose that $(s_1, e, s_2), (s_3, e, s_4) \in T_1$ are such that $s_1 \in r'$ and $s_2 \notin r'$. We must show that $s_3 \in r'$ and $s_4 \notin r'$. Since $s_1 \in r'$ and $s_2 \notin r'$, we must have $f(s_1) \in r$ and $f(s_2) \notin r$. Hence $f(s_1) \neq f(s_2)$. This implies that $\eta_f(e)$ is defined and $(f(s_1), e', f(s_2)) \in T_2$ where $\eta_f(e) = e'$. By the definition of a G-morphism this implies that $(f(s_3), e', f(s_4)) \in T_2$. Since r is a region and $f(s_1) \in r$ and $f(s_2) \notin r$ we can conclude $f(s_3) \in r$ and $f(s_4) \notin r$. Hence $s_3 \in r'$ and $s_4 \notin r'$.

Similar arguments apply in case of events entering r' and hence r' is indeed a region. Furthermore it follows immediately from the above argument that for $e_1 \in E_1$, $f^{-1}(r) \in {}^\circ e_1$ implies that $\eta_f(e_1)$ is defined and $r \in {}^\circ\eta_f(e_1)$. On the other hand, assume that $\eta_f(e_1)$ is defined and that $r \in {}^\circ e_2$ where $\eta_f(e_1) = e_2$. Take any $(s, e_1, s') \in T_1$ (there must be at least one by axiom (A3)). We know that $(f(s), e_2, f(s')) \in T_2$. From $r \in {}^\circ e_2$ we know that $f(s) \in r$ and $f(s') \notin r$. Hence $s \in r'$ and $s' \notin r'$. This shows that $r' \in {}^\circ e_1$.

Symmetric arguments apply to post-regions. □

It follows easily that elementary transition systems with G-morphisms form a category with normal functional composition and the identity function as identity. Let us denote this category as $\mathcal{ETS}$. Our aim now is to form an appropriate category of elementary net systems and relate it to $\mathcal{ETS}$.

2. Elementary net systems

Elementary net systems are a basic system model of net theory. Here we shall give only a brief introduction to this model. For more details and background material the reader is referred to [14], [15] and [10].

In net theory, models of distributed systems are based on objects called nets which specify the local states and local transition and the fixed neighbourhood relationships between them.

Definition 2.1. A net is a triple $N = (S, T, F)$ where:

(i) S is a set of *S-elements*.
(ii) T is a set of *T-elements*.
(iii) $S \cap T = \emptyset$.
(iv) $F \subseteq (S \times T) \cup (T \times S)$ is the *flow relation*.
(v) $\forall x \in S \cup T.\ \exists y \in S \cup T.\ [(x, y) \in F \vee (y, x) \in F]$.

The last condition states that we do not permit isolated elements in our nets. In this paper the S-elements will be used to denote local states called *conditions* and the T-elements will be used local transitions called *events*. Following usual practice we use B to denote the set of conditions and E to denote the set of events; consequently we use the triple (B, E, F) to denote a net.

Let $N = (B, E, F)$ be a net and $x \in B \cup E$.

$${}^{\bullet}x \stackrel{\text{def}}{=} \{y \mid (y, x) \in F\} \quad \text{(the } \textit{pre-elements} \text{ of } x\text{)},$$

$$x^{\bullet} \stackrel{\text{def}}{=} \{y \mid (x, y) \in F\} \quad \text{(the } \textit{post-elements} \text{ of } x\text{)}.$$

In case $e \in E$, ${}^{\bullet}e$ will be called the set of *pre-conditions* and $e^{\bullet}$ will be called the set of *post-conditions of e*. In this paper elementary net systems will be based on a subclass of nets called simple nets.

The net $N = (B, E, F)$ is said to be *simple* iff

$$\forall x, y \in B \cup E.\, [{}^{\bullet}x = {}^{\bullet}y \wedge x^{\bullet} = y^{\bullet} \;\Rightarrow\; x = y].$$

Definition 2.2. An *elementary net system* is a quadrupule $\mathbf{N} = (B, E, F, c_{\text{in}})$ where

(i) (B, E, F) is a *simple* net called the *underlying net* of $\mathbf{N}$ (and denoted in the sequel as $N_{\mathbf{N}}$.)

(ii) $c_{\text{in}} \subseteq B$ is the *initial case.*

In diagrams the conditions will be drawn as circles, the events as boxes and members of the flow relations as directed arcs. The initial case will be indicated by marking (with small darkened dots) the members of the initial case. An example of an elementary net system is shown in Fig. 4.

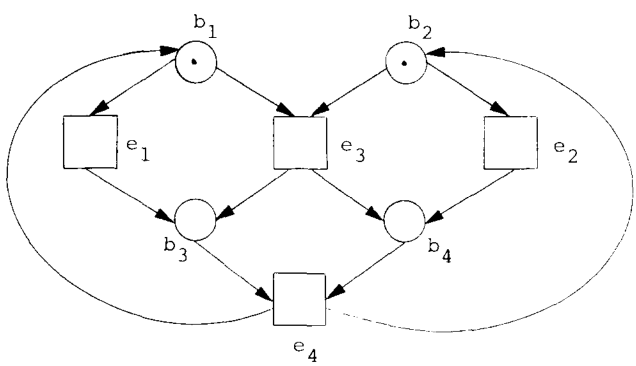

Fig. 4.

From now on we will often refer to elementary net systems as just net systems. The dynamics of a net system are straightforward. The states of a net system are called *cases.* A case consists of a set of conditions that hold concurrently. The system can go from a case to a case through the occurrence of an event. An event can occur at a case iff all its pre-conditions hold and none of its post-conditions hold at the case. When an event occurs, all its pre-conditions cease to hold and all its post-conditions begin to hold. It will be convenient to formalize these ideas as follows.

Let $N = (B, E, F)$ be a net. Then $\rightarrow_N \subseteq 2^B \times E \times 2^B$ is the *transition relation* of N and is given by

$$\rightarrow_N = \{(c, e, c') \mid c - c' = {}^{\bullet}e \wedge c' - c = e^{\bullet}\}.$$

Definition 2.3. Let $\boldsymbol{N} = (B, E, F, c_{\text{in}})$ be a net system.

(i) $C_{\boldsymbol{N}}$ is the *state space* of $\boldsymbol{N}$ (also called the set of *reachable cases* of $\boldsymbol{N}$) and it is the least subset of 2^B containing c_{in} which satisfies: $(c, e, c') \in \rightarrow_{N_{\boldsymbol{N}}} \wedge c \in C_{\boldsymbol{N}} \Rightarrow c' \in C_{\boldsymbol{N}}$. (Recall that $N_{\boldsymbol{N}} = (B, E, F)$.)

(ii) $\rightarrow_{\boldsymbol{N}}$ is the *transition relation* of $\boldsymbol{N}$ and it is $\rightarrow_{N_{\boldsymbol{N}}}$ restricted to $C_{\boldsymbol{N}} \times E \times C_{\boldsymbol{N}}$.

(iii) $E_{\boldsymbol{N}}$ is the set of *active events* of $\boldsymbol{N}$ and it is the subset of E given by $E_{\boldsymbol{N}} = \{e \mid \exists (c, e, c') \in \rightarrow_{\boldsymbol{N}}\}$.

The state space of the net system shown in Fig. 4 is $\{\{b_1, b_2\}, \{b_1, b_4\}, \{b_3, b_2\}, \{b_3, b_4\}\}$. Note that, in general, the set of active events of a net system is a proper subset of the set of events of the underlying net. We can now associate a transition system (often called the case *graph* and sometimes the sequential case graph) with a net system to explain its operational behaviour.

Definition 2.4. Let $\boldsymbol{N} = (B, E, F, c_{\text{in}})$ be a net system. Then the transition system $TS_{\boldsymbol{N}} \stackrel{\text{def}}{=} (C_{\boldsymbol{N}}, E_{\boldsymbol{N}}, \rightarrow_{\boldsymbol{N}}, c_{\text{in}})$ is called the *transition system associated with* $\boldsymbol{N}$.

Normally one uses E instead of $E_{\boldsymbol{N}}$ to specify $TS_{\boldsymbol{N}}$. Here we have tightened up the definition of $TS_{\boldsymbol{N}}$ because of the axioms imposed on transition systems in the previous section and the anticipated results of a later section.

It is easy to check that $TS_{\boldsymbol{N}}$ as specified in Definition 2.4 is indeed a transition system. In fact we will prove this, and more, in the next section. It is also easy to check that the transition system associated with the net system of Fig. 4 is the transition system shown in Fig. 1 (provided of course the obvious identifications of cases with states are made).

We can now point out that $TS_{\boldsymbol{N}}$ captures the informal explanation of the dynamics of the net system $\boldsymbol{N}$ stated previously.

Proposition 2.5. *Let* $\boldsymbol{N} = (B, E, F, c_{\text{in}})$ *and let* $TS_{\boldsymbol{N}}$ *be the transition system associated with* $\boldsymbol{N}$.

(i) $\forall c \in C_{\boldsymbol{N}}.\ \forall e \in E.\ [c \xrightarrow{e} \Leftrightarrow {}^{\bullet}e \subseteq c \wedge e^{\bullet} \cap c = \emptyset]$.

(ii) $\forall (c, e, c') \in \rightarrow_{\boldsymbol{N}}.\ [c' = (c - {}^{\bullet}e) \cup e^{\bullet}]$.

(iii) $\forall (c_1, e, c_2), (c_3, e, c_4) \in \rightarrow_{\boldsymbol{N}}.\ [c_1 - c_2 = c_3 - c_4 \wedge c_2 - c_1 = c_4 - c_3]$.

(iv) $(c, e, c_1), (c, e, c_2) \in \rightarrow_{\boldsymbol{N}} \Rightarrow c_1 = c_2$.

Proof. Follows easily from the definitions. □

(i) says that an event is enabled to occur at a case iff all its pre-conditions hold and none of its post-conditions hold at the case. (ii) says that when an event occurs all its pre-conditions cease to hold and all its post-conditions begin to hold. (iii) guarantees that an event produces the same change whenever it occurs. (iv) says that the transition system associated with an (unlabelled!) net system is deterministic.

Fundamental concepts concerning distributed systems such as causality, concurrency, conflict and confusion can be easily defined and separated from each other with the help of net systems. A variety of behavioural tools also exist to study the non-sequential behaviour of net systems. The interested reader is referred to [14, 15, 10] for more details.

Here we shall make use of a primitive behavioural tool called firing sequences. Let $\mathbf{N}=(B, E, F, c_{\text{in}})$ be a net system. Then $FS_{\mathbf{N}} \subseteq E^*$, the set of *firing sequences* of $\mathbf{N}$, is the least subset of E^* (the free monoid generated by E) defined inductively as follows. In doing so, it will be convenient to simultaneously build up the relation $[\ \rangle \subseteq \{c_{\text{in}}\} \times E^* \times C_{\mathbf{N}}$. We use Λ to denote the empty sequence.

(i) $\Lambda \in FS_{\mathbf{N}}$ and $c_{\text{in}}[\Lambda\rangle c_{\text{in}}$

(ii) Suppose $\rho \in FS_{\mathbf{N}}$ and $c_{\text{in}}[\rho\rangle c$ and $(c, e, c') \in \rightarrow_{\mathbf{N}}$. Then $\rho e \in FS_{\mathbf{N}}$ and $c_{\text{in}}[\rho e\rangle c'$.

For the net system shown in Fig. 4, $e_1e_2e_4$, $e_2e_1e_4$, e_3e_4 are some of its firing sequences. With $c_{\text{in}}=\{b_1, b_2\}$ and $c=\{b_2, b_3\}$ we have $c_{\text{in}}[e_2e_1e_4e_1\rangle c$ and $c_{\text{in}}[e_1\rangle c$.

Before introducing morphisms between net systems it will be convenient to adopt some notations concerning binary relations and partial functions. If B_1 and B_2 are sets and $\beta \subseteq B_1 \times B_2$ then β^{-1} is the binary relation

$$\beta^{-1} \stackrel{\text{def}}{=} \{(b_2, b_1) \mid (b_1, b_2) \in \beta\}.$$

For $B \subseteq B_1$,

$$\beta(B) \stackrel{\text{def}}{=} \{b_2 \mid \exists b_1 \in B.\, (b_1, b_2) \in \beta\}.$$

If $\beta_1 \subseteq B_1 \times B_2$ and $\beta_2 \subseteq B_2 \times B_3$ the *composition* of β_1 and β_2 is denoted as $\beta_2 \circ \beta_1$ and it is the subset of $B_1 \times B_3$ given by

$$\beta_2 \circ \beta_1 \stackrel{\text{def}}{=} \{(b_1, b_3) \mid \exists b_2.\, (b_1, b_2) \in \beta_1 \text{ and } (b_2, b_3) \in \beta_2\}.$$

A partial function η from set E_1 to the set E_2 will be indicated by $\eta : E_1 \rightarrow_* E_2$. (We have already followed this convention in the previous section.) If $\eta_1 : E_1 \rightarrow_* E_2$ and $\eta_2 : E_2 \rightarrow_* E_3$ are two partial functions then the *composition* of η_1 and η_2, denoted by $\eta_2 \circ \eta_1$, is the partial function from E_1 to E_3 given by:

$$\forall e_1 \in E_1,\ \eta_2 \circ \eta_1(e_1) = \begin{cases} e_3, & \text{if } \exists e_2 \in E_2.\ \eta_1(e_1) = e_2 \text{ and } \eta_2(e_2) = e_3 \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Definition 2.6. Let $\mathbf{N}_i = (B_i, E_i, F_i, c^i_{\text{in}})$, $i = 1, 2$ be a pair of net systems. Then an *N-morphism* from $\mathbf{N}_1$ to $\mathbf{N}_2$ is an ordered pair (β, η) where $\beta \subseteq B_1 \times B_2$ is a binary relation and $\eta : E_1 \rightarrow_* E_2$ is a partial function such that:

(i) β^{-1} is a partial function from B_2 to B_1.

(ii) $\forall (b_1, b_2) \in \beta.\, [b_1 \in c^1_{\text{in}} \Leftrightarrow b_2 \in c^2_{\text{in}}]$.

(iii) $\forall e_1 \in E_1$ if $\eta(e_1)$ is undefined, then $\beta({}^\bullet e_1) = \emptyset = \beta(e_1^\bullet)$.

(iv) $\forall e_1 \in E_1$ if $\eta(e_1) = e_2$ then $\beta({}^\bullet e_1) = {}^\bullet e_2$ and $\beta(e_1^\bullet) = e_2^\bullet$.

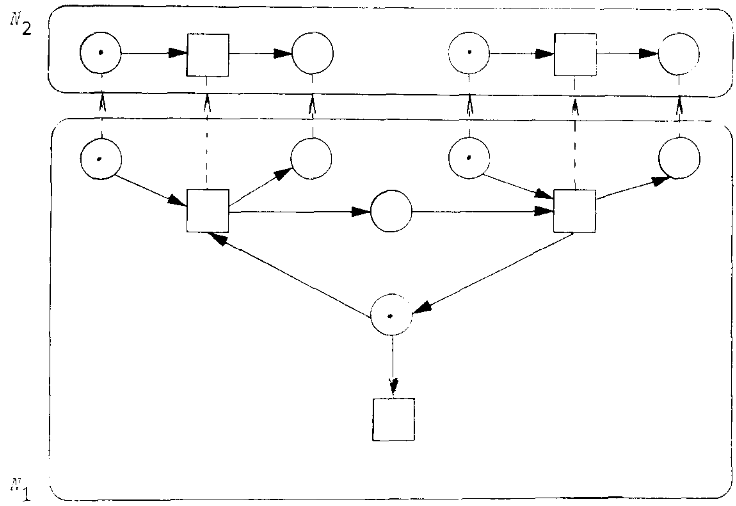

Fig. 5.

An example of an N-morphism (from $\boldsymbol{N}_1$ to $\boldsymbol{N}_2$) is shown in Fig. 5.

N-morphisms are a modified form of net morphisms advocated by Winskel [18]. For convenience we shall refer to Winskel's net morphisms as W-morphisms. An N-morphism differs from a W-morphism in three respects. Firstly we require β^{-1} to be (globally) a partial function. In the case of a W-morphism, β^{-1} is required to be a total function when restricted to $c_{\text{in}}^2 \times c_{\text{in}}^1$ and when restricted to $\beta({}^{\bullet}e) \times {}^{\bullet}e$ and $\beta(e^{\bullet}) \times e^{\bullet}$ whenever $\eta(e)$ is defined. In fact Winskel [16] also started with the notion of a morphism in which β^{-1} was required to be a (global) partial function. Later this requirement was weakened in order to support the notion of unfoldings for net systems. The second difference is that for a W-morphism, one demands $\beta(c_{\text{in}}^1) = c_{\text{in}}^2$. We have weakened this assumption since we do not wish to permit isolated elements in our nets. The third difference is that we do not require our net systems to be contact-free whereas W-morphisms (when specialized to net systems) "work" in the sense of Proposition 2.7 to be proved soon, only for contact-free net systems.

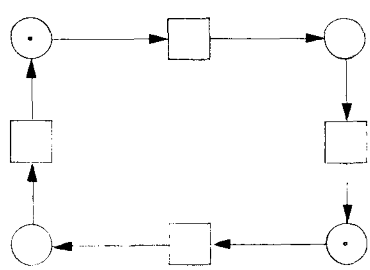

Fig. 6.

The net system $N=(B, E, F, c_{\text{in}})$ is said to be *contact-free* iff

$$\forall c \in C_N \forall e \in E.\, {}^\bullet e \subseteq c \;\Rightarrow\; e^\bullet \cap c = \emptyset.$$

An example of a net system which has contact is shown in Fig. 6.

We will discuss in greater depth the differences between N-morphisms and W-morphisms in Section 6. Turning now to the study of N-morphisms, we first show that N-morphisms are "behaviour-preserving" in the following sense.

Proposition 2.7. *Let* $(\beta, \eta): N_1 \to N_2$ *be an N-morphism, where* $N_i = (B_i, E_i, F_i, c^i_{\text{in}})$, $i=1,2$. *Let* $f_\beta : C_{N_1} \to 2^{B_2}$ *be given by*

$$\forall c \in C_{N_1}.\, f_\beta(c) = \beta(c) \cup (c^2_{\text{in}} - \beta(c^1_{\text{in}})).$$

Then

(i) $\forall c \in C_{N_1}.\, f_\beta(c) \in C_{N_2}$.

(ii) *Suppose* $(c, e, c') \in \to_{N_1}$. *Then* $f_\beta(c) = f_\beta(c')$ *in case* $\eta(e)$ *is undefined. Otherwise,* $(f_\beta(c), \eta(e), f_\beta(c')) \in \to_{N_2}$.

Proof. Let $c \in C_{N_1}$ and ρ be a firing sequence of N_1 such that $c_{\text{in}}[\rho\rangle c$. We first prove (i) by induction on $k = |\rho|$.

(a) $k=0$. Then $c = c^1_{\text{in}}$ and $f_\beta(c^1_{\text{in}}) = c^2_{\text{in}}$.

(b) $k>0$. Let $\rho = \rho' e$ and $c^1_{\text{in}}[\rho'\rangle c'$ in N_1. By the induction hypothesis, $f_\beta(c') \in C_{N_2}$. Let $f_\beta(c') = c''$. We have $(c', e, c) \in \to_{N_1}$. Hence $c = (c' - {}^\bullet e) \cup e^\bullet$.

Suppose $\eta(e)$ is undefined. Then $\beta({}^\bullet e) = \emptyset = \beta(e^\bullet)$. It is now easy to verify that $f_\beta(c) = f_\beta(c')$. Hence $f_\beta(c) \in C_{N_2}$ as required.

Suppose $\eta(e) = e_2$. We wish to argue that e_2 is enabled at c''. Now e is enabled at c' and hence ${}^\bullet e \subseteq c'$ and $e^\bullet \cap c' = \emptyset$. From ${}^\bullet e \subseteq c'$ we get $\beta({}^\bullet e) = {}^\bullet e_2 \subseteq \beta(c') \subseteq f_\beta(c')$. Now suppose $e_2^\bullet \cap c'' \neq \emptyset$. Then there exists $b_2 \in e_2^\bullet$ such that $b_2 \in \beta(c')$ or $b_2 \in c^2_{\text{in}} - \beta(c^1_{\text{in}})$. But $b_2 \in e_2^\bullet$ together with $\beta(e^\bullet) = e_2^\bullet$ implies that there exists $b_1 \in e^\bullet$ such that $(b_1, b_2) \in \beta$. If $b_2 \in \beta(c')$ then we get $b_1 \in c'$ (because β^{-1} is a partial function), which contradicts $e^\bullet \cap c' = \emptyset$. If $b_2 \in c^2_{\text{in}} - \beta(c^1_{\text{in}})$ then by the definition of an N-morphism we get $b_1 \in c^1_{\text{in}}$. But this at once leads to the contradiction that $b_2 \in \beta(c^1_{\text{in}})$. Thus e_2 is enabled at c''. Let $c_2 = (c'' - {}^\bullet e_2) \cup e_2^\bullet$. Then from $\beta({}^\bullet e) = {}^\bullet e_2$ and $\beta(e^\bullet) = e_2^\bullet$ it is easy to verify that $f_\beta(c) = c_2$. Clearly $c_2 \in C_{N_2}$.

From the proof above, the second part of the proposition now follows easily. □

It will be convenient to establish the "net version" of Proposition 1.6.

Proposition 2.8. *Let* (β_1, η_1) *and* (β_2, η_2) *be a pair of N-morphisms from* N_1 *to* N_2 *where* $N_i = (B_i, E_i, F_i, c^i_{\text{in}})$, $i=1,2$ *are two net systems. If* $\eta_1 = \eta_2$ *then* $\beta_1 = \beta_2$.

Proof. Assume that $\eta_1 = \eta_2$. Suppose that $(b_1, b_2) \in \beta_1$. Since we do not permit isolated elements in our net, there exists $e_1 \in {}^\bullet b_1 \cup b_1^\bullet$ in N_1. Assume that $e_1 \in {}^\bullet b_1$. Then by the definition of an N-morphism, $\eta_1(e_1)$ is defined. Moreover, $\beta_1(e_1^\bullet) = \eta_1(e_1)^\bullet$ and hence $b_2 \in \eta_1(e_1)^\bullet$. But then $\eta_1 = \eta_2$ implies that $b_2 \in \eta_2(e_1)^\bullet$ as well.

Hence $\exists b_1' \in e_1^\bullet$ such that $(b_1', b_2) \in \beta_2$ and $e_1 \in {}^\bullet b_1'$. We now claim that ${}^\bullet b_1 = {}^\bullet b_1'$ and $b_1^\bullet = (b_1')^\bullet$ in $\boldsymbol{N}_1$ which will then lead to $b_1 = b_1'$ due to the simplicity of the underlying net of $\boldsymbol{N}_1$.

So suppose that $e \in {}^\bullet b_1\ (b_1^\bullet)$. Then $(b_1, b_2) \in \beta_1$ implies that $\eta_1(e)$ is defined and moreover $\eta_1(e) \in {}^\bullet b_2\ (b_2^\bullet)$ in $\boldsymbol{N}_2$. But then $\eta_1(e) = \eta_2(e)$ and hence $\eta_2(e) \in {}^\bullet b_2(b_2^\bullet)$ as well. Since $(b_1', b_2) \in \beta_2$ we now have $b_1' \in e^\bullet({}^\bullet e)$ and hence $e \in {}^\bullet b_1'((b_1')^\bullet)$. $\square$

Let $\mathscr{ENS}$ denote the category whose objects are net systems and whose arrows are N-morphisms. For each object $\boldsymbol{N} = (B, E, F, c_{\text{in}})$ let $1_{\boldsymbol{N}} = (\text{id}_B, \text{id}_E)$ be the identity morphism where $\text{id}_B : B \to B$ and $\text{id}_E : E \to E$ are the (total) identity functions. For $(\beta_1, \eta_1) : \boldsymbol{N}_1 \to \boldsymbol{N}_2$ and $(\beta_2, \eta_2) : \boldsymbol{N}_2 \to \boldsymbol{N}_3$ define the composition of these two N-morphisms (denoted $(\beta_2, \eta_2) \circ (\beta_1, \eta_1)$) as $(\beta_2 \circ \beta_1, \eta_2 \circ \eta_1)$. It follows easily that $\mathscr{ENS}$ is indeed a category. We can now begin to relate $\mathscr{ETS}$ and $\mathscr{ENS}$ to each other.

3. From $\mathscr{ENS}$ to $\mathscr{ETS}$

We wish to construct a functor from $\mathscr{ENS}$ to $\mathscr{ETS}$.

Definition 3.1. Let H be a map which assigns to each object $\boldsymbol{N}$ in $\mathscr{ENS}$, the transition system $TS_{\boldsymbol{N}}$ associated with $\boldsymbol{N}$ which we recall is $(C_{\boldsymbol{N}}, E_{\boldsymbol{N}}, \to_{\boldsymbol{N}}, c_{\text{in}})$. Furthermore, H assigns to each arrow $(\beta, \eta) : \boldsymbol{N}_1 \to \boldsymbol{N}_2$ (with $\boldsymbol{N}_i = (B_i, E_i, F_i, c_{\text{in}}^i)$, $i = 1, 2$) in $\mathscr{ENS}$, the function $f_\beta : C_{\boldsymbol{N}_1} \to C_{\boldsymbol{N}_2}$ given by:

$$\forall c_1 \in C_{\boldsymbol{N}_1}.\ f_\beta(c_1) = \beta(c_1) \cup (c_{\text{in}}^2 - \beta(c_{\text{in}}^1)).$$

Note that f_β as defined above is indeed a function from $C_{\boldsymbol{N}_1}$ to $C_{\boldsymbol{N}_2}$. This follows at once from Proposition 2.7. The main result of this section is that H is a functor.

In what follows, given a net system $\boldsymbol{N} = (B, E, F, c_{\text{in}})$ and $b \in B$ we let r_b stand for the cases of $\boldsymbol{N}$ in which b holds. More precisely,

$$r_b \stackrel{\text{def}}{=} \{c \mid c \in C_{\boldsymbol{N}} \wedge b \in c\}.$$

Proposition 3.2. *Let $\boldsymbol{N} = (B, E, F, c_{\text{in}})$ be a net system.*

(i) *$H(\boldsymbol{N})$ is an elementary transition system.*

(ii) *$\forall b \in B$. r_b is a region of $H(\boldsymbol{N})$.*

Proof. Recall the $H(\boldsymbol{N}) = (C_{\boldsymbol{N}}, E_{\boldsymbol{N}}, \to_{\boldsymbol{N}}, c_{\text{in}})$. We will first verify that $H(\boldsymbol{N})$ satisfies the axioms (A1) through (A4).

Suppose $(c, e, c') \in \to_{\boldsymbol{N}}$. Then by the definition of $\to_{\boldsymbol{N}}$ we have $c - c' = {}^\bullet e$ and $c' - c = e^\bullet$. If $c = c'$ then this would imply that ${}^\bullet e = \emptyset$ and $e^\bullet = \emptyset$ so that ${}^\bullet e \cup e^\bullet = \emptyset$. But this violates the definition of a net. Hence $H(\boldsymbol{N})$ satisfies (A1).

Suppose (c, e_1, c_1) and (c, e_2, c_2) are both members of $\to_{\boldsymbol{N}}$. Then $c_1 = (c - {}^\bullet e_1) \cup e_1^\bullet$ and $c_2 = (c - {}^\bullet e_2) \cup e_2^\bullet$ by Proposition 2.5. If $c_1 = c_2$ then ${}^\bullet e_1 = c - c_1 = {}^\bullet e_2$ and $e_1^\bullet = c_1 - c = e_2^\bullet$. Since the underlying net of $\boldsymbol{N}$ is simple we obtain $e_1 = e_2$. Thus $H(\boldsymbol{N})$ fulfills (A2).

The facts that $H(\mathbf{N})$ satisfies (A3) and (A4) follow at once from the definitions of $E_{\mathbf{N}}$ and $C_{\mathbf{N}}$.

The second part of the result follows at once from the definitions.

Now consider $c_1, c_2 \in C_{\mathbf{N}}$ such that $c_1 \neq c_2$. Without loss of generality assume that $b \in c_1$ and $b \notin c_2$. Then $c_1 \in r_b$ and $c_2 \notin r_b$. Hence $r_b \in R_{c_1}$ and $r_b \notin R_{c_2}$. This verifies that $H(\mathbf{N})$ satisfies (A5).

Suppose $c \in C_{\mathbf{N}}$ and $e \in E_{\mathbf{N}}$ such that ${}^\circ e \subseteq R_c$. Consider any $b \in {}^\bullet e$ in $\mathbf{N}$. Since $e \in E_{\mathbf{N}}$ there exists $(c_1, e, c_2) \in \rightarrow_{\mathbf{N}}$. Hence $b \in c_1$ and $b \notin c_2$. Clearly $c_1 \in r_b$ and $c_2 \notin r_b$. Consequently $r_b \in {}^\circ e$. From ${}^\circ e \subseteq R_c$ we then obtain $r_b \in R_c$ which means that $c \in r_b$ which in turn means $b \in c$. We now have established that ${}^\bullet e \subseteq c$. By Proposition 1.2, $e^\circ \cap R_c = \emptyset$. Using a symmetric version of the preceding argument we can show $e^\bullet \cap c = \emptyset$. Hence by Proposition 2.5, e is enabled at c. Hence $H(\mathbf{N})$ satisfies (A6). □

Next we wish to show that H maps N-morphisms to G-morphisms.

Lemma 3.3. *Let $\mathbf{N}_i = (B_i, E_i, F_i, c_{\text{in}}^i)$, $i = 1, 2$ be a pair of net systems and (β, η) an N-morphism from $\mathbf{N}_1$ to $\mathbf{N}_2$. Then $H((\beta, \eta)) = f_\beta$ is a G-morphism from $H(\mathbf{N}_1) = TS_1$ to $H(\mathbf{N}_2) = TS_2$, such that $\eta_{f_\beta} = \eta$.*

Proof. Recall that $f_\beta(c) = \beta(c) \cup (c_{\text{in}}^2 - \beta(c_{\text{in}}^1))$ for every $c \in C_1$. Set $c_{\text{in}}^2 - \beta(c_{\text{in}}^1) = c_{21}$. First note that $f_\beta(c_{\text{in}}^1) = \beta(c_{\text{in}}^1) \cup (c_{\text{in}}^2 - \beta(c_{\text{in}}^1)) = c_{\text{in}}^2$ as required.

Now suppose that $c_1 \xrightarrow{e_1} c_1'$ in TS_1. If $\eta(e_1)$ is undefined then from this it follows from Proposition 2.7 that $f_\beta(c_1) = f_\beta(c_1')$. If on the other hand, $\eta(e_1) = e_2$ then it follows—once again from Proposition 2.7—that $f_\beta(c_1) \xrightarrow{e_2} f_\beta(c_1')$ in TS_2.

Now suppose $c_1 \xrightarrow{e_1} c_1'$ in TS_1 and $f_\beta(c_1) \xrightarrow{e_2} f_\beta(c_1')$ in TS_2 and $c_3 \xrightarrow{e_1} c_4$ in TS_1. It follows at once—yet again from Proposition 2.7—that $f_\beta(c_3) \xrightarrow{e_2} f_\beta(c_4)$ in TS_2. Hence f_β is a G-morphism from TS_1 to TS_2 with the required property. □

Theorem 3.4. *$H : \mathscr{ENS} \to \mathscr{ETS}$ is a functor.*

Proof. Clearly H preserves identities. So assume that $(\beta_1, \eta_1) : \mathbf{N}_1 \to \mathbf{N}_2$ and $(\beta_2, \eta_2) : \mathbf{N}_2 \to \mathbf{N}_3$ are two N-morphisms with $\mathbf{N}_i = (B_i, E_i, F_i, c_{\text{in}}^i)$, $i = 1, 2, 3$. We must show that $f_{\beta_2 \circ \beta_1} = f_{\beta_2} \circ f_{\beta_1}$.

Let $f_{\beta_2 \circ \beta_1} = f$, $f_{\beta_1} = f_1$ and $f_{\beta_2} = f_2$. Then from Lemma 3.3 and the fact that $\eta_{f_2 \circ f_1} = \eta_{f_2} \circ \eta_{f_1}$ (easy to prove) it follows that $\eta_f = \eta_{f_2 \circ f_1} : E_{\mathbf{N}_1} \rightarrow_* E_{\mathbf{N}_3}$. The required result now follows at once from Proposition 1.6. □

4. From $\mathscr{ETS}$ to $\mathscr{ENS}$

We now wish to construct a functor from $\mathscr{ETS}$ to $\mathscr{ENS}$.

Definition 4.1. Let J be a map which assigns to each object $TS = (S, E, T, s_{\text{in}})$ in $\mathscr{ETS}$ the structure $\mathbf{N}_{TS} = (\mathbf{R}_{TS}, E, F_{TS}, R_{s_{\text{in}}})$ where

$$F_{TS} \overset{\text{def}}{=} \{(r, e) \mid r \in \mathbf{R}_{TS} \wedge e \in E \wedge r \in {}^\circ e\} \cup \{(e, r) \mid r \in \mathbf{R}_{TS} \wedge e \in E \wedge r \in e^\circ\}.$$

Furthermore, let J assign to each G-morphism $f: TS_1 \to TS_2$ (with $TS_i = (S_i, E_i, T_i, s_{in}^i)$, $i = 1, 2$) the pair (β_f, η_f) where η_f is the partial function $\eta_f: E_1 \to_* E_2$ determined by f (as specified in Section 1) and $\beta_f \subseteq R_{TS_1} \times R_{TS_2}$ given by:

$$(r_1, r_2) \in \beta_f \Leftrightarrow f^{-1}(r_2) = r_1. \qquad \square$$

By Proposition 1.7, β_f as specified above is well-defined. The main result of this section is that J is a functor. The first result we need is that $J(TS)$ is an elementary net system for every object in $\mathscr{ETS}$. As things stand, this can fail to be the case for a "silly" reason. It might be that $R_{TS} \cap E$ (where $TS = (S, E, T, s_{in})$) is nonempty and as a result (R_{TS}, E, F_{TS}) will not be a net! Hence one must define the map J on the objects of $\mathscr{ETS}$ more carefully. For instance, for each TS in $\mathscr{ETS}$, we could set $J(TS) = N_{TS}$ to be the structure $(R_{TS} \times \{0\},\ E \times \{1\},\ F_{TS},\ R_{S_{in}} \times \{0\})$ with the definition of F_{TS} modified in the obvious way. In what follows we ignore this complication in order to simplify the notations.

Proposition 4.2. *Let $TS = (S, E, T, s_{in})$ be an elementary transition system. Then $\mathbf{N}_{TS} = (R_{TS}, E, F_{TS}, R_{S_{in}})$ is an elementary net system.*

Proof. It suffices to prove that $N = (R_{TS}, E, F_{TS})$ is a simple net. By the remarks made above, we may assume that $R_{TS} \cap E = \emptyset$. Clearly

$$F_{TS} \subseteq (R_{TS} \times E) \cup (E \times R_{TS})$$

and hence N is a net.

Suppose $e_1, e_2 \in E$ such that ${}^\circ e_1 = {}^\circ e_2$ and $e_1^\circ = e_2^\circ$. By (A3) there exists $s \xrightarrow{e_1} s'$ in TS. By Proposition 1.2, ${}^\circ e_1 \subseteq R_s$ which implies ${}^\circ e_2 \subseteq R_s$. Since TS is elementary, e_2 must be enabled at R_s. Let $s \xrightarrow{e_2} s''$. Then once again by Proposition 1.2, $R_{s''} = (R_s - {}^\circ e_2) \cup e_2^\circ$. But this implies that $R_{s''} = R_{s'}$ since ${}^\circ e_1 = {}^\circ e_2$ and $e_1^\circ = e_2^\circ$ and by Proposition 1.2 (again!) $R_{s'} = (R_s - {}^\circ e_1) \cup e_1^\circ$. Since TS is elementary, we then get $s' = s''$. We now have $s \xrightarrow{e_1} s'$ and $s \xrightarrow{e_2} s'$. Hence by (A2) we can conclude that $e_1 = e_2$.

Next suppose that $r_1, r_2 \in R_{TS}$ such that ${}^\circ r_1 = {}^\circ r_2$ and $r_1^\circ = r_2^\circ$ where for every $r \in R_{TS}$,

$${}^\circ r \stackrel{\text{def}}{=} \{e \mid e \in E \wedge r \in e^\circ\}$$

and

$$r^\circ \stackrel{\text{def}}{=} \{e \mid e \in E \wedge r \in {}^\circ e\}.$$

We must show that $r_1 = r_2$. We will first show that $s_{in} \in r_1$ iff $s_{in} \in r_2$. To this end, assume that $s_{in} \in r_1$. Suppose $s_{in} \notin r_2$. We know that $r_2 \neq \emptyset$. Let $s \in r_2$ and $s_0, s_1, s_2, \ldots, s_n \in S$ and $e_0, e_1, \ldots, e_{n-1} \in E$ such that $s_{in} = s_0$, $s_n = s$ and $(s_i, e_i, s_{i+1}) \in T$ for $0 \leq i < n$. By (A4) such a sequence of states and events must exist. Let j be the least integer in $\{0, 1, \ldots, n-1\}$ such that $s_j \notin r_2$ and $s_{j+1} \in r_2$. Since we have assumed that $s_0 = s_{in} \notin r_2$ and $s_n = s \in r_2$ such a j must exist. Clearly $j \leq n-1$.

From $(s_j, e_j, s_{j+1}) \in T$ we can deduce that $r_2 \in e_j^\circ$ and consequently $e_j \in {}^\circ r_2$. But then ${}^\circ r_2 = {}^\circ r_1$ leads to $e_j \in {}^\circ r_1$ as well. But this implies that $r_1 \in e_j^\circ$ and hence $s_j \notin r_1$ and $s_{j+1} \in r_1$. We started with $s_0 = s_{\text{in}} \in r_1$ and now we have $s_j \notin r_1$. Hence there exists $k \in \{0, 1, \ldots, j-1\}$ such that $s_k \in r_1$ and $s_{k+1} \notin r_1$. Clearly $k \leq j-1$ and hence $k < j$. From $(s_k, e_k, s_{k+1}) \in T$ we can now conclude that $r_1 \in {}^\circ e_k$ and hence $e_k \in r_1^\circ = r_2^\circ$. But this implies that $s_k \in r_2$ which contradicts the definition of j. Hence $s_{\text{in}} \in r_2$ as well. Since the argument we developed to show was symmetric with respect to r_1 and r_2 we can in fact conclude that $s_{\text{in}} \in r_1$ iff $s_{\text{in}} \in r_2$.

A simple induction on the "distance" of a state from the initial state based on the above argument will easily show that $\forall s \in S.\ s \in r_1$ iff $s \in r_2$. Thus $r_1 = r_2$ as required. □

In Fig. 7 we have shown four elementary transition systems and in Fig. 8 their corresponding J-images. For convenience we have suppressed some names of states in Fig. 7 and some names of conditions in Fig. 8.

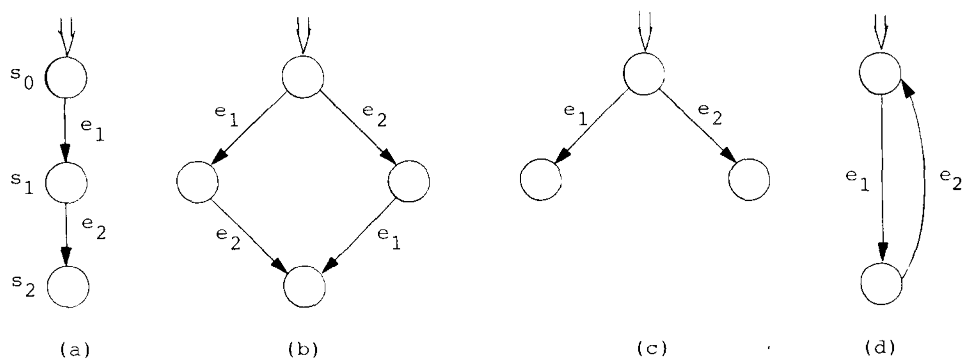

Fig. 7.

Net systems which are J-images of elementary transition systems are very pleasant objects to work with. We will explore some of their properties in the next section (see also [4]). Here we just note that $J(TS)$ is a contact-free net system for every elementary transition system $TS = (S, E, T, s_{\text{in}})$. This is so because by Proposition 1.2, $e^\circ = \{\bar{r} \mid r \in {}^\circ e\}$ for every $e \in E$. Hence if ${}^\circ e \subseteq R_s$ we at once get $e^\circ \cap R_s = \emptyset$ for every $s \in S$.

Proposition 4.3. *Let $f: TS_1 \to TS_2$ be a G-morphism in $\mathscr{ETS}$ with $TS_i = (S_i, E_i, T_i, s_{\text{in}}^i)$, $i = 1, 2$. Let $\beta_f \subseteq R_{TS_1} \times R_{TS_2}$ be as specified in Definition* 4.1. *Then (β_f, η_f) is an N-morphism from $J(TS_1)$ to $J(TS_2)$.*

Proof. Recall that $N_{TS_i} = (R_{TS_i}, E_i, F_{TS_i}, R^i_{s_{\text{in}}})$ for $i = 1, 2$. η_f is by definition a partial function from E_1 to E_2. As observed earlier, β_f is also well defined.

Clearly β_f^{-1} is a partial function from R_{TS_2} to R_{TS_1} because $(r_1, r_2), (r_1', r_2) \in \beta_f$ implies that $r_1 = f^{-1}(r_2) = r_1'$.

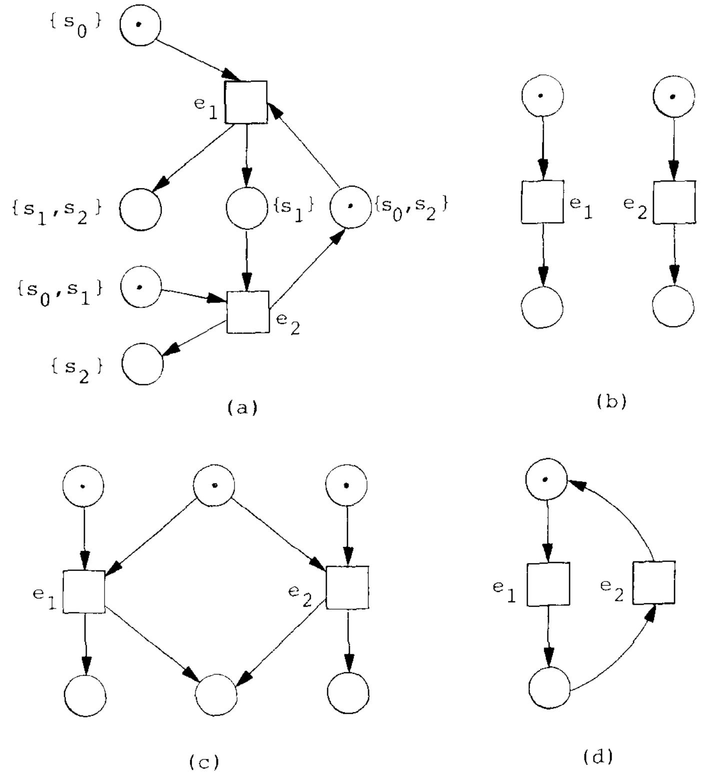

Fig. 8.

Next suppose that $(r_1, r_2) \in \beta_f$. We must show that $r_1 \in R_{s^1_{in}}$ iff $r_2 \in R_{s^2_{in}}$. f being a G-morphism, we know that $f(s^1_{in}) = s^2_{in}$. Hence $s^2_{in} \in r_2$ iff $s^1_{in} \in r_1$ since $f^{-1}(r_2) = r_1$. But $s^i_{in} \in r_i$ iff $r_i \in R_{s^i_{in}}$ for $i = 1, 2$.

Next suppose that $e_1 \in E_1$ such that $\eta_f(e_1)$ is undefined. We must show $\beta_f({}^\bullet e_1) = \emptyset = \beta_f(e_1^\bullet)$. Suppose that $r_1 \in {}^\circ e_1$ in TS_1 and $(r_1, r_2) \in \beta_f$. Then by (the latter part of) Proposition 1.7, we get the contradiction that $\eta_f(e_1)$ is defined. By a symmetric argument we can show that $\beta_f(e_1^\bullet) = \emptyset$ as well.

Assume next $\eta_f(e_1) = e_2$ for some $e_1 \in E_1$. From Proposition 1.7 it follows that $r_2 \in {}^\circ e_2$ iff $f^{-1}(r_2) \in {}^\circ e_1$ and $r_2 \in e_2^\circ$ iff $f^{-1}(r_2) \in e_1^\circ$. Now suppose that $r_2 \in \beta_f({}^\bullet e_1)$. Then there exists $r_1 \in {}^\circ e_1$ in TS_1 such that $(r_1, r_2) \in \beta_f$. But this implies, by the definition of β_f (i.e. $f^{-1}(r_2) = r_1$) that $r_2 \in {}^\circ e_2$ in TS_2 and hence $r_2 \in {}^\bullet e_2$ in $J(TS_2)$. Next assume that $r_2 \in {}^\bullet e_2$ in $J(TS_2)$ so that $r_2 \in {}^\circ e_2$ in TS_2. Then $f^{-1}(r_2) = r_1 \in {}^\circ e_1$ in TS_1. Clearly $r_1 \in {}^\bullet e_1$ in TS_1 and $(r_1, r_2) \in \beta_f$. Hence $r_2 \in \beta_f({}^\bullet e_1)$. We have now shown that $\beta_f({}^\bullet e_1) = {}^\bullet e_2$. In a similar fashion, we can show that $\beta_f(e_1^\bullet) = e_2^\bullet$. □

As an example the G-morphism shown in Fig. 9 from TS_1 to TS_2 is translated by J into the N-morphism shown in Fig. 10. Once again, we have suppressed all unnecessary details.

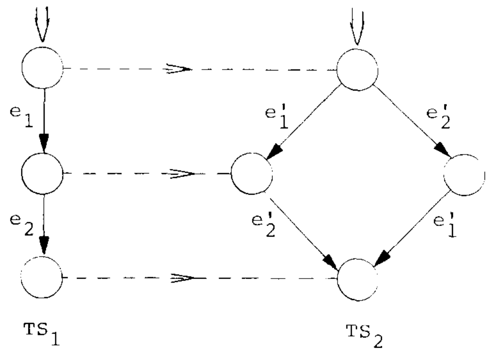

Fig. 9.

Theorem 4.4. *J is a functor.*

Proof. As before it is easy to verify that J preserves identities. Hence assume that $f_1: TS_1 \to TS_2$ and $f_2: TS_2 \to TS_3$ are two G-morphisms. We must prove that $(\beta_{f_2 \circ f_1}, \eta_{f_2 \circ f_1}) = (\beta_{f_2} \circ \beta_{f_1}, \eta_{f_2} \circ \eta_{f_1})$. From the definitions, it follows that $\eta_{f_2 \circ f_1} = \eta_{f_2} \circ \eta_{f_1}$. The required result now follows easily from Proposition 2.8. □

5. Back and forth between $\mathscr{E}\mathscr{T}\mathscr{S}$ and $\mathscr{E}\mathscr{N}\mathscr{S}$

The main result of this section is that the functors J and H form an adjunction with J a left adjoint. In the process of establishing this we will prove a number of intermediate results which are of independent interest.

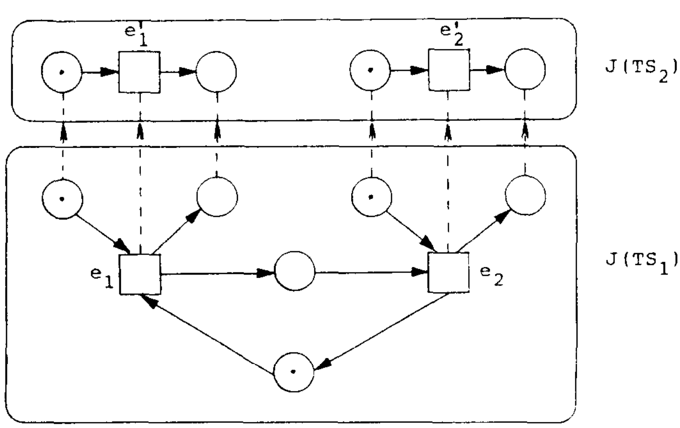

Fig. 10.

Lemma 5.1. *Let $TS = (S, E, T, s_{in})$ be an elementary transition system and $N = J(TS)$. Then*

(i) $C_N = \{R_s \mid s \in S\}$,
(ii) $\rightarrow_N = \{(R_s, e, R_{s'}) \mid (s, e, s') \in T\}$,
(iii) $E_N = E$.

Proof. To prove (i), consider $c \in C_N$. Let ρ be a firing sequence in N such that $c_{in}[\rho\rangle c$ where c_{in} is the initial case of N. We proceed by induction on $k = |\rho|$.

$k = 0$. By definition, $c_{in} = R_{s_{in}}$.

$k > 0$. Let $\rho = \rho' e$ with $e \in E$ and let $c_{in}[\rho'\rangle c'$. By the induction hypothesis there exists $s' \in S$ such that $c' = R_{s'}$. We know that $(c', e, c) \in \rightarrow_N$. Hence ${}^{\bullet}e \subseteq c'$. This means that ${}^{\circ}e \subseteq R_{s'}$ in TS by the definition of the flow relation of the underlying net of N. But then TS is elementary. Hence e is enabled at s' in TS. Let $(s', e, s) \in TS$. Then $R_s = (R_{s'} - {}^{\circ}e) \cup e^{\circ}$ by Proposition 1.2. At the same time, $c = (c' - {}^{\bullet}e) \cup e^{\bullet}$ in N. Since ${}^{\circ}e = {}^{\bullet}e$ and $e^{\circ} = e^{\bullet}$ we get $c = R_s$.

To show containment in the other direction first note that $R_{s_{in}}$ is the initial case of N. Hence $R_{s_{in}} \in C_N$. Now consider $s \in S - \{s_{in}\}$. By (A4), there exist $s_0, s_1, \ldots, s_n \in S$ and $e_1, e_2, \ldots, e_n \in E$ such that $s_{in} = s_0$, $s_n = s$ and $(s_i, e_{i+1}, s_{i+1}) \in T$ for $0 \leq i < n$. We proceed by induction on n.

$n = 1$. Then $s_{in} \xrightarrow{e_1} s$ in TS. By Proposition 1.2, ${}^{\circ}e_1 \subseteq R_{s_{in}}$ and $e_1^{\circ} \cap R_{s_{in}} = \emptyset$. Since ${}^{\circ}e_1 = {}^{\bullet}e_1$ and $e_1^{\circ} = e_1^{\bullet}$ (here ${}^{\bullet}e_1$ and $e_1^{\bullet}$ refer to the pre- and post-conditions of e_1 in N_N) we have that e_1 is enabled at $R_{s_{in}} (\in C_N)$ in N. This implies that $(R_{s_{in}} - {}^{\circ}e_1) \cup e_1^{\circ} \in C_N$. By Proposition 1.2 again, $R_s = (R_{s_{in}} - {}^{\circ}e_1) \cup e_1^{\circ}$.

$n > 1$. The fact that $R_s \in C_N$ follows from the induction hypothesis and the proof of the basis step.

The proofs of (ii) and (iii) are now easy. □

Our next result expresses the fact that for a given elementary transition system, TS, the case graph of $J(TS)$ is TS itself, up to isomorphism.

Theorem 5.2. *Let $TS = (S, E, T, s_{in})$ be an elementary transition system and $J(TS) = N$. Then the (regional) map $u: S \rightarrow C_N$ given by $u(s) = R_s$ for every $s \in S$ is a G-isomorphism from TS to $H \circ J(TS)$ $(= H(N))$.*

Proof. From the previous lemma it follows at once that u is a G-morphism with η_u as the (total) identify function. Now consider the map $u': C_N \rightarrow S$ given by $\forall c \in C_N.\ u'(c) = s$ where $s \in S$ is such that $c = R_s$. Since TS is elementary, $u(s_1) = u(s_2)$ implies $s_1 = s_2$ for every $s_1, s_2 \in S$. On the other hand by part (i) of the previous lemma, u is an onto function from S to C_N. Hence u' is well defined. Let $H(N) = TS'$. From part (ii) of the previous lemma it follows at once that u' is a G-morphism from TS' to TS. It is now routine to check that $u \circ u' = 1_{TS'}$ and $u' \circ u = 1_{TS}$. □

Thus our means of translating elementary transition systems into net systems is "sound" in the following sense. The operational behaviours of $J(TS)$—when viewed as a transition system—agrees completely with that of TS. As already hinted at previously, there is more to $J(TS)$ than that. It is a very special kind of a net system. Before we go into this in more detail, we shall indulge in a pleasant digression.

Corollary 5.3. *Let $TS=(S, E, T, s_{in})$ be a transition system (which fulfils the axioms* (A1) *through* (A4)). *Then TS is elementary iff there exists an elementary net system $\mathbf{N}$ such that TS and $H(\mathbf{N})$ are G-isomorphic to each other.*

Proof. One half of this result is Theorem 5.2. So assume that $\mathbf{N}=(B, E', F, c_{in})$ is an elementary net system such that there is a G-isomorphism f from TS to $H(\mathbf{N})$. We must verify that TS fulfils (A5) and (A6).

Let $s_1, s_2 \in S$ such that $s_1 \neq s_2$. We will show that $R_{s_1} \neq R_{s_2}$. Note first that $f(s_1) \neq f(s_2)$ because f is a G-isomorphism. Let $f(s_1)=c_1$ and $f(s_2)=c_2$. Then we can assume, without loss of generality, that there exists $b \in B$ such that $b \in c_1$ and $b \notin c_2$. Recall now that $r_b=\{c \mid c \in C_{\mathbf{N}}$ and $b \in c\}$ is a region of $H(\mathbf{N})$. We must have clearly $c_1 \in r_b$ and $c_2 \notin r_b$. Hence $s_1 \in f^{-1}(r_b)$ and $s_2 \notin f^{-1}(r_b)$. But then $r=f^{-1}(r_b)$ is a region of TS by Proposition 1.7. Consequently $r \in R_{s_1}$ and $r \notin R_{s_2}$.

Suppose $s \in S$ and $e \in E$ such that ${}^\circ e \subseteq R_s$. We must prove that e is enabled at s in TS. It is easy to check that $f: S \to C_{\mathbf{N}}$ is a bijection. Next, by (A3), we know that there exists $(s_1, e, s_1') \in T$. Let $f(s_1)=c_1$ and $f(s_1')=c_1'$. Since f is injective, we must have $c_1 \neq c_1'$. Now since f is a G-morphism there exists $e' \in E'$ such that $(c_1, e', c_1') \in \to_{\mathbf{N}}$.

Set $f(s)=c$. We claim that e' is enabled at c in $H(\mathbf{N})$. To see this, consider $b \in {}^\bullet e'$. Then $b \in c_1$ and $b \notin c_1'$. Hence $c_1 \in r_b$ and $c_1' \notin r_b$. Consequently $r_b \in {}^\circ e'$ in $H(\mathbf{N})$. Since $\eta_f(e)=e'$ we can now conclude that $f^{-1}(r_b) \in {}^\circ e$ in TS. Let $r=f^{-1}(r_b)$. We have assumed that ${}^\circ e \subseteq R_s$. Hence $s \in r$. But then $f(f^{-1}(r_b))=r_b$. Hence $f(s)=c \in r_b$. Consequently $b \in c$. We have shown that ${}^\bullet e' \subseteq c$.

Consider now $b \in (e')^\bullet$. Then $b \notin c_1$ and $b \in c_1'$. Hence $c_1 \notin r_b$ and $c_1' \in r_b$. Consequently $r_b \in (e')^\circ$ in $H(\mathbf{N})$. By Proposition 1.7, $f^{-1}(r_b) \in e^\circ$ in TS. Let $r=f^{-1}(r_b)$. By Proposition 1.2, $e^\circ=\{\bar{r} \mid r \in {}^\circ e\}$. We have assumed that ${}^\circ e \subseteq R_s$. Hence $e^\circ \cap R_s=\emptyset$ and consequently $r \notin R_s$. Thus $s \notin r$, and this implies $c \notin r_b$. We now have $b \notin c$ which leads to $(e')^\bullet \cap c=\emptyset$.

Thus the claim is proved and e' is enabled at c. From the fact that f is a G-isomorphism it follows easily that e is enabled at s. □

This result was proved—in a different setting—in [4]. It seems reasonable to call net systems which are N-isomorphic to J-images of elementary transition systems *saturated* net systems. The system is saturated with conditions. In other words, as shown in [4], no new conditions can be added without violating the simplicity of the underlying net or without altering the operational behaviour of the net system. For *finite* transition systems this boils down to the following.

Suppose $TS=(S, E, T, s_{in})$ is an elementary transition system so that $J(TS)=(R_{TS}, E, F_{TS}, R_{s_{in}})$. Suppose $J(TS)$ is a proper subsystem of a net system $\mathbf{N}'=(B, E, F, c_{in})$ with the same set of events. In other words, $R_{TS} \subset B$ and $R_{s_{in}} \subseteq c_{in}$ and

F_{TS} is F restricted to $(R_{TS} \times E) \cup (E \times R_{TS})$. Then there can be *no* G-isomorphism from TS to $H(\boldsymbol{N}')$.

Unfortunately the result holds only for finite transition system. Consider the infinite elementary transitions system shown in Fig. 11.

Let $TS = (\{s_i \mid i \in \mathbb{N}_0\}, \mathbb{N}, T, s_0)$ be this transition system where $\mathbb{N}_0$ is the set of non-negative integers, $\mathbb{N}$ is the set of positive integers and $T = \{(s_0, i, s_i) \mid i \in \mathbb{N}\}$. Let $\boldsymbol{N}' = (R_{TS} \cup \{b\}, \mathbb{N}, F_{TS} \cup \{(b, 1)\}, R_{s_0})$. Then it is easy to see that $\boldsymbol{N}'$ is an elementary net system. $H(\boldsymbol{N}')$ will then be as shown in Fig. 12.

It is easy to construct a G-isomorphism going from TS to $H(\boldsymbol{N}')$.

However, it may be proved that a modified version of the above result holds for arbitrary elementary transition systems (the conclusion in the result modified to: Then there can be *no* G-isomorphism f from TS to $H(\boldsymbol{N}')$ such that $\eta_f = \mathrm{id}: E \to E$).

Before proceeding further it is worthwhile to nail down the notion of a saturated net system. We will say that the net system $\boldsymbol{N}$ is *saturated* if there exists an elementary transition system TS such that there exists an N-isomorphism from $J(TS)$ to $\boldsymbol{N}$. It is not difficult to prove that the net system $\boldsymbol{N}$ is saturated iff it is N-isomorphic to $J \circ H(\boldsymbol{N})$. A different characterization of saturated net systems is provided in [4].

The saturated net systems are "maximal" objects w.r.t. N-morphisms in the sense that given a pair of net systems $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ in general there will be "many more" interesting N-morphisms from $J \circ H(\boldsymbol{N}_1)$ to $\boldsymbol{N}_2$ than from $\boldsymbol{N}_1$ to $\boldsymbol{N}_2$. To bring this out consider the pair of net systems shown in Fig. 13.

It is not difficult to check that there can be no N-morphism $(\beta, \eta): \boldsymbol{N}_1 \to \boldsymbol{N}_2$ in which η_f is a total function. It turns out that that this is the case even if we use

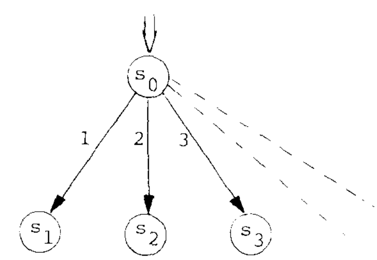

Fig. 11.

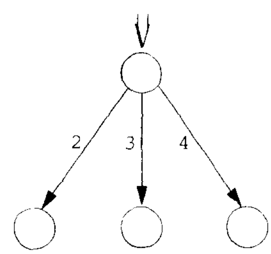

Fig. 12.

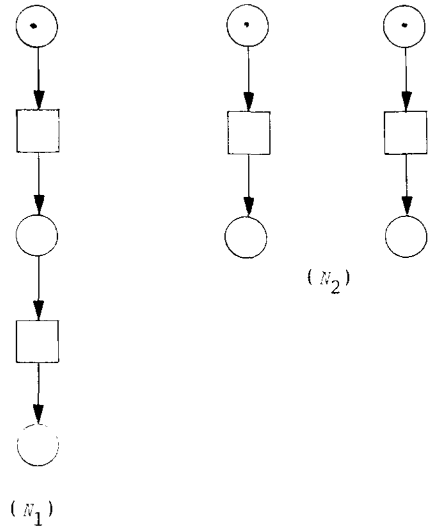

Fig. 13.

W-morphisms instead of *N*-morphisms. However there is such an *N*-morphism from $J \circ H(\mathbf{N}_1)$ to $\mathbf{N}_2$ as shown in Fig. 10. It seems reasonable to call $J \circ H(\mathbf{N})$ the *canonical representation* of the net system $\mathbf{N}$ for this and other reasons. We will however not explore this any further here. It is time now to establish the main result of this section.

Theorem 5.4. *$J: \mathscr{ETS} \to \mathscr{ENS}$ and $H: \mathscr{ENS} \to \mathscr{ETS}$ form an adjunction (coreflection) with J as left adjoint and $u: TS \to H \circ J(TS)$ as defined in the statement of Theorem 5.2 as unit.*

Proof. From [6] it follows that one way to prove this result is to show that for any objects TS_1 in $\mathscr{ETS}$ and any object $\mathbf{N}_2$ in $\mathscr{ENS}$ if there is a *G*-morphism *f* from TS_1 to $H(\mathbf{N}_2)$ then there is a unique *N*-morphism (β, η) from $J(TS_1)$ to $\mathbf{N}_2$ such that the diagram shown in Fig. 14 commutes.

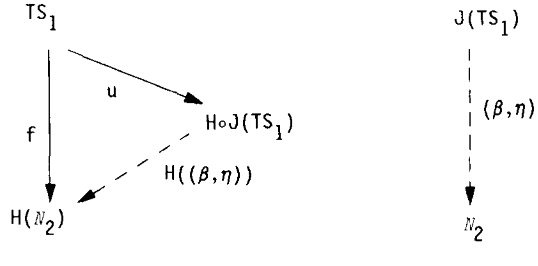

Fig. 14.

Let us denote,

$$TS_1=(S_1,E_1,T_1,s_{\rm in}^1), \qquad J(TS_1)=(R_1,E_1,F_1,c_{\rm in}^1),$$

$$\mathbf{N}_2=(B_2,E_2,F_2,c_{\rm in}^2) \qquad H(\mathbf{N}_2)=(C_2,E_2',T_2,s_{\rm in}^2).$$

Note that $R_1=R_{TS_1}$, $c_{\rm in}^1=R_{s_{\rm in}^1}$, $E_2'=E_{\mathbf{N}_2}$ and $s_{\rm in}^2=c_{\rm in}^2$, following the definitions of J and H. We propose the following (β,η).

$$\eta=\eta_f,$$

$\beta\subseteq R_1\times B_2$ given by

$$\beta=\{(r,b)\mid r=f^{-1}(\{c\in C_2\mid b\in c\})\}.$$

Clearly β and η are well-defined (remember that $\{c\in C_2\mid b\in c\}$ is a region of $H(\mathbf{N}_2)$ by Proposition 3.2(ii)).

We need to prove:

(i) (β,η) is an N-morphism from $J(TS_1)$ to $\mathbf{N}_2$,

(ii) $H((\beta,\eta))\circ u=f$, and

(iii) (β,η) is unique satisfying (i) and (ii) (among all the N-morphisms from $J(TS_1)$ to $\mathbf{N}_2$).

Proof of (i). Clearly η is a partial function and β^{-1} is a partial function from B_2 to R_1.

Suppose $(r,b)\in\beta$. Then $r\in c_{\rm in}^1$ iff $r\in R_{s_{\rm in}^1}$ iff $s_{\rm in}^1\in r$. But $(r,b)\in\beta$ implies that $r=f^{-1}(\{c\in C_2\mid b\in c\})$ and f being a G-morphism, we have $f(s_{\rm in}^1)=s_{\rm in}^2$ and $s_{\rm in}^2=c_{\rm in}^2$. Thus $s_{\rm in}^1\in r$ iff $c_{\rm in}^2\in\{c\in C_2\mid b\in c\}$ and hence $r\in c_{\rm in}^1$ iff $b\in c_{\rm in}^2$.

Assume that $\eta(e)$ is undefined. We have to prove that $\beta({}^\bullet e)=\emptyset=\beta(e^\bullet)$. Suppose that $r\in{}^\circ e(e^\circ)$ and $(r,b)\in\beta$ for some $b\in B_2$. Then from the fact that $r'=\{c\in C_{\mathbf{N}_2}\mid b\in c\}$ is a region in $H(\mathbf{N}_2)$ and that $f^{-1}(r')=r$ (by the definition of β) it follows from Proposition 1.7 that $\eta(e)$ is defined which is a contradiction.

Next assume that $\eta(e)$ is defined. We must prove that $\beta({}^\bullet e)={}^\bullet\eta(e)$ and $\beta(e^\bullet)=\eta(e)^\bullet$. Now for $b\in{}^\bullet\eta(e)$, let $r=f^{-1}(\{c\in C_{\mathbf{N}_2}\mid b\in c\})$. From Proposition 1.7 it follows at once that $r\in{}^\bullet e$ in $J(TS_1)$ and hence $(r,b)\in\beta$. This shows that ${}^\bullet\eta(e)\subseteq\beta({}^\bullet e)$. On the other hand $r\in{}^\circ e$ implies once again by Proposition 1.7 that $b\in{}^\bullet\eta(e)$ in case $(r,b)\in\beta$. This shows that $\beta({}^\bullet e)\subseteq{}^\bullet\eta(e)$. By symmetric arguments we get $\beta(e^\bullet)=\eta(e)^\bullet$. We have now shown that (β,η) is an N-morphism from $J(TS_1)$ to $\mathbf{N}_2$.

Proof of (ii). Let $H((\beta,\eta))=f'$. From Lemma 3.3 it follows that $\eta_{f'}=\eta_f$. But from Theorem 5.2 it follows that η_u is the (total) identity function from E_1 to E_1. Hence $\eta_{f''}=\eta_f$ where $f''=H((\beta,\eta))\circ u$. Consequently $f''=f$ by Proposition 1.6.

Proof of (iii). To prove the uniqueness of (β,η) we first note that $f=H((\beta,\eta))\circ u$ implies that $\eta_f=\eta_{H((\beta,\eta))}\circ\eta_u$ which in turn implies that $\eta_f=\eta_{H((\beta,\eta))}$, and hence from Lemma 3.3 $\eta(e)=\eta_f(e)$ for all $e\in E_1$. Now suppose (β',η') is any other N-morphism from $J(TS_1)$ to $\mathbf{N}_2$ which fulfils (i) and (ii), then by the above argument we must have $\eta'=\eta$. From Proposition 2.8 we then get $\beta'=\beta$.

Since u is a G-isomorphism by Theorem 5.2 we get that J and H form a co-reflection (in the sense of [5]). □

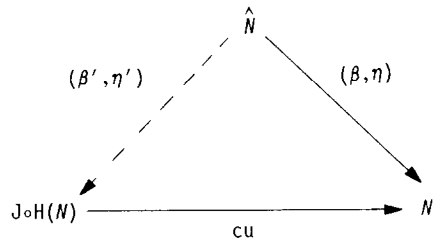

Fig. 15.

This theorem has an interesting corollary which throws some light on the construction of saturated net systems.

Corollary 5.5. *The co-unit cu of the coreflection established in Theorem* 5.4 *defines for each net system* $\boldsymbol{N} = (B, E, F, c_{in})$ *an N-morphism* $cu = (\beta, \eta): J \circ H(\boldsymbol{N}) \to \boldsymbol{N}$ *where*

(i) $\eta = \{(e, e) \mid e \in E_{\boldsymbol{N}}\}$.
(ii) $\beta \subseteq R_{H(\boldsymbol{N})} \times B$ *is defined by*

$$(r, b) \in \beta \Leftrightarrow r = f^{-1}(\{c \in C_{\boldsymbol{N}} \mid b \in c\}).$$

Proof. Follows easily from a simple computation following [6]. □

It is now easy to see that a net system $\boldsymbol{N}$ is saturated iff $cu: J \circ H(\boldsymbol{N}) \to \boldsymbol{N}$ is an N-isomorphism.

Corollary 5.6. *For every net system* $\boldsymbol{N}$, $J \circ H(\boldsymbol{N})$ *and* $cu: J \circ H(\boldsymbol{N}) \to \boldsymbol{N}$ *are cofree in the sense that for any saturated net system* $\hat{\boldsymbol{N}}$ *and G-morphism* $(\beta, \eta): \hat{\boldsymbol{N}} \to \boldsymbol{N}$ *there exists a unique G-morphism* $(\beta', \eta'): \hat{\boldsymbol{N}} \to J \circ H(\boldsymbol{N})$ *such that the diagram shown in Fig.* 15 *commutes.*

Proof. Immediate consequence of Theorem 5.4 following [6]. □

Yet another interesting consequence of our main result is the following.

Corollary 5.7. *Let* $\mathcal{SENS}$ *denote the category of saturated net systems with N-morphisms. Then* $J: \mathcal{ETS} \to \mathcal{SENS}$ *is an equivalence with adjoint H.*

Proof. Follows once again immediately from Theorem 5.4 (see [6]). □

We shall conclude this section by showing that $\mathcal{ETS}$ has products, coproducts and null objects.

Definition 5.8. Let $TS_i = (S_i, E_i, T_i, s_{in}^i)$ for $i = 1, 2$ be two elementary transition systems. Then the *product* $TS_1 \times TS_2$ is the structure $TS = (S, E, T, s_{in})$ where

- $S = S_1 \times S_2$.

- $E = (E_1 \times \{*\}) \cup (\{*\} \times E_2) \cup (E_1 \times E_2)$ where $*$ is not in $E_1 \cup E_2$.
- With $\pi_1 : S \to S_1$ and $\pi_2 : S \to S_2$ as the projections given by $\pi_i((x, y)) = x$ if $i = 1$ and y if $i = 2$, let

$$\begin{aligned} T = &\{(s, (e_1, *), s') \mid (\pi_1(s), e_1, \pi_1(s')) \in T_1 \text{ and } \pi_2(s) = \pi_2(s')\} \\ &\cup \{(s, (*, e_2), s') \mid (\pi_2(s), e_2, \pi_2(s')) \in T_2 \text{ and } \pi_1(s) = \pi_1(s')\} \\ &\cup \{(s, (e_1, e_2), s') \mid (\pi_1(s), e_1, \pi_1(s')) \in T_1 \text{ and } (\pi_2(s), e_2, \pi_2(s')) \in T_2\}. \end{aligned}$$

- $s_{in} = (s^1_{in}, s^2_{in})$.

From [17], it follows that in order to prove that $TS_1 \times TS_2$ is indeed the product of TS_1 and TS_2 in $\mathscr{ETS}$ it suffices to prove that $TS_1 \times TS_2$ is elementary. It is easy to check that $TS_1 \times TS_2$ is a transition system which fulfils the axioms (A1)–(A4). The following lemmas will be useful for verifying the regional axioms (A5) and (A6).

Lemma 5.9. *Let TS_1, TS_2 and $TS_1 \times TS_2$ be as in Definition 5.8. Let r_1 be a region in TS_1. Then $r_1 \times S_2$ is a region of $TS_1 \times TS_2$. Furthermore, if r_1 is a pre-region (post-region) of some $e_1 \in E_1$ then $r_1 \times S_2$ is a pre-region (post-region) of every event of the form (e_1, x) in $TS_1 \times TS_2$, $x \in E_2 \cup \{*\}$.*

Proof. Assume that $(s_1, s_2) \xrightarrow{(x,y)} (s_1', s_2')$ and $(v_1, v_2) \xrightarrow{(x,y)} (v_1', v_2')$ in $TS_1 \times TS_2$. Suppose that $(s_1, s_2) \in r_1 \times S_2$ and $(s_1', s_2') \notin r_1 \times S_2$. We must prove that $(v_1, v_2) \in r_1 \times S_2$ and $(v_1', v_2') \notin r_1 \times S_2$. We have two cases.

Case 1. Suppose $(x, y) = (e_1, *)$ or $(x, y) = (e_1, e_2)$ for some $e_1 \in E_1$ and $e_2 \in E_2$. Then $s_1 \xrightarrow{e_1} s_1'$ and $v_1 \xrightarrow{e_1} v_1'$ in TS_1. Moreover $s_1 \in r_1$ and $s_1' \notin r_1$. Since r_1 is a region we then get $v_1 \in r_1$ and $v_1' \notin r_1$. Hence $(v_1, v_2) \in r_1 \times S_2$ and $(v_1', v_2') \notin r_1 \times S_2$.

Case 2. Suppose $(x, y) = (*, e_2)$ for some $e_2 \in E_2$. This case is impossible because by the definition of $TS_1 \times TS_2$ we must have $s_1 = s_1'$ which contradicts $(s_1, s_2) \in r_1 \times S_2$ and $(s_1', s_2') \notin r_1 \times S_2$.

Symmetric arguments apply in case $(s_1, s_2) \notin r_1 \times S_2$ and $(s_1', s_2') \in r_1 \times S_2$. Thus we know that $r_1 \times S_2$ is a region of $TS_1 \times TS_2$. Now suppose that $r_1 \in {}^\circ e_1$ in TS_1 for some $e_1 \in E_1$. Consider any transition of the form $(s_1, s_2) \xrightarrow{(e_1,x)} (s_1', s_2')$ in $TS_1 \times TS_2$. Then $s_1 \in r_1$ and $s_1' \notin r_1$ and consequently $(s_1, s_2) \in r_1 \times S_2$ and $(s_1', s_2') \notin r_1 \times S_2$. Thus $r_1 \times S_2 \in {}^\circ(e_1, x)$ in $TS_1 \times TS_2$. Symmetric arguments apply for post-regions. □

Lemma 5.10. *Let TS_1, TS_2 and $TS_1 \times TS_2$ be as in Definition 5.8. Let r_2 be a region of TS_2. Then $S_1 \times r_2$ is a region of $TS_1 \times TS_2$. Furthermore, if r_2 is a pre-region (post-region) of some $e_2 \in E_2$ then $S_1 \times r_2$ is a pre-region (post-region) of every event of the form (x, e_2) in $TS_1 \times TS_2$, $x \in E_1 \cup \{*\}$.*

Proof. Very similar to the proof of Lemma 5.9. □

Theorem 5.11. *Let TS_1, TS_2 and $TS_1 \times TS_2$ be as in Definition 5.8. Then $TS_1 \times TS_2$ is the product of TS_1 and TS_2 in $\mathscr{ETS}$ with the projections π_1 and π_2.*

Proof. As observed earlier due to [17], it suffices to prove that $TS_1 \times TS_2$ satisfies (A5) and (A6).

Suppose (s_1, s_2), $(s_1', s_2') \in S$ such that $(s_1, s_2) \neq (s_1', s_2')$. Then $s_1 \neq s_1'$ or $s_2 \neq s_2'$. Assume first that $s_1 \neq s_1'$. Then TS_1 being elementary, we can find a region r_1 in TS_1 such that $s_1 \in r_1$ and $s_1' \notin r_1$. From Lemma 5.9 it follows that $r_1 \times S_2$ is a region in $TS_1 \times TS_2$. Clearly $(s_1, s_2) \in r_1 \times S_2$ but $(s_1', s_2') \notin r_1 \times S_2$.

In case $s_1 = s_1'$ and $s_2 \neq s_2'$ then a similar argument can be applied using Lemma 5.10 to find a region in $TS_1 \times TS_2$ which contains (s_1, s_2) but not (s_1', s_2').

To verify (A6) consider $(s_1, s_2) \in S$ and $e \in E$ such that e is not enabled at (s_1, s_2) in $TS_1 \times TS_2$. We must show that there exists a pre-region of e in $TS_1 \times TS_2$ which does not contain (s_1, s_2). We have three cases.

Case 1. e is of the form $(e_1, *)$ where $e_1 \in E_1$. From the definition of T it is clear that if e_1 is enabled at s_1 in TS_1 then $(e_1, *)$ is enabled at (s_1, s_2) in $TS_1 \times TS_2$. Hence there exists a pre-region r_1 of e_1 in TS_1 which does not contain s_1. From Lemma 5.9 it follows that $r_1 \times S_2$ is a pre-region of $(e_1, *)$ in $TS_1 \times TS_2$ which does not contain (s_1, s_2).

Case 2. e is of the form $(*, e_2)$ for some $e_2 \in E_2$. The argument is similar to the argument for the previous case with the difference that we appeal to the elementariness of TS_2 and to Lemma 5.10.

Case 3. e is of the form (e_1, e_2) where $e_1 \in E_1$ and $e_2 \in E_2$. From the definition of T it follows that if e_1 is enabled at s_1 and e_2 is enabled at s_2 then (e_1, e_2) is enabled at (s_1, s_2). Hence e_1 is not enabled at s_1 or e_2 is not enabled at s_2. We can now use the arguments used for settling the previous two cases to settle this case, too. □

Next we wish to show that $\mathscr{ETS}$ has coproducts.

Definition 5.12. Let $TS_i = (S_i, E_i, T_i, s_{in}^i)$, $i = 1, 2$ be two elementary transition systems. Define the *coproduct* of TS_1 and TS_2 (denoted $TS_1 + TS_2$) as the structure $TS = (S, E, T, s_{in})$ where

- $S = (S_1 \times \{s_{in}^2\}) \cup (\{s_{in}^1\} \times S_2)$ with the injections $i_1 : S_1 \to S$ and $i_2 : S_2 \to S$ given by: $i_1(s) = (s, s_{in}^2)$ for every $s \in S_1$ and $i_2(s) = (s_{in}^1, s)$ for every $s \in S_2$.
- $E = E_1 \times \{*\} \cup \{*\} \times E_2$ where $*$ is not in $E_1 \cup E_2$.
- $T = \{(i_1(s), (e_1, *), i_1(s')) \mid (s, e_1, s') \in T_1\} \cup \{(i_2(s), (*, e_2), i_2(s')) \mid (s, e_2, s') \in T_2\}$.
- $s_{in} = (s_{in}^1, s_{in}^2)$.

As before, due to [17] in order to show that TS as defined above is the coproduct of TS_1 and TS_2 it suffices to prove that TS is an elementary transition system. Once again it is easy to check that TS fulfils the axioms (A1)-(A4). To verify the remaining two axioms we will use the following intermediate result.

Lemma 5.13. *Let TS_1, TS_2 and $TS_1 + TS_2$ be as in Definition* 5.12. *Let r_1 be a region of TS_1 and r_2 be a region of TS_2 such that $s_{in}^1 \in r_1$ iff $s_{in}^2 \in r_2$. Then*

(i) $r = r_1 \times \{s_{in}^2\} \cup \{s_{in}^1\} \times r_2$ *is a region of* $TS_1 + TS_2$.

(ii) *If r_1 is a pre-region (post-region) of e_1 in TS_1 then r is a pre-region (post-region) of $(e_1, *)$ in $TS_1 + TS_2$.*

(iii) *If r_2 is a pre-region (post-region) of e_2 in TS_2 then r is a pre-region (post-region) of $(*, e_2)$ in $TS_1 + TS_2$.*

Proof. Assume that $x_1 \xrightarrow{e} x_2$ and $x_3 \xrightarrow{e} x_4$ in $TS_1 + TS_2$ such that $x_1 \in r$ and $x_2 \notin r$. We must show that $x_3 \in r$ and $x_4 \notin r$. (symmetric arguments—which we omit— will apply for the case $x_1 \notin r$ and $x_2 \in r$.) Assume without loss of generality that e is of the form $(e_1, *)$. From the definition of $TS_1 + TS_2$ it follows that the x_i's must be of the form $x_i = (s_i, s_{in}^2)$ for $i = 1, 2, 3, 4$. Moreover, $s_1 \xrightarrow{e_1} s_2$ and $s_3 \xrightarrow{e_1} s_4$ in TS_1. Now using the definition of r and the hypothesis that $s_{in}^1 \in r_1$ iff $s_{in}^2 \in r_2$ it is easy to check that $s_i \in r_1$ iff $x_1 \in r$ for $i = 1, 2, 3, 4$. Consequently $s_1 \in r_1$ and $s_2 \notin r_1$ and moreover it suffices to prove that $s_3 \in r_1$ and $s_4 \notin r_1$. But this must be the case because r_1 is a region of TS_1. The remaining parts of the lemma are now easily established. □

Theorem 5.14. *Let TS_1, TS_2 and $TS_1 + TS_2$ be as in Definition 5.12. Then $TS_1 + TS_2$ is the coproduct of TS_1 and TS_2 in $\mathscr{ETS}$ with i_1 and i_2 as injections.*

Proof. As noted earlier the proof reduces to showing that TS fulfils (A5) and (A6). Assume that $s, s' \in S$ such that $s \neq s'$. Let $s = (s_1, s_2)$ and $s' = (s_1', s_2')$. Suppose that $s_1 \neq s_1'$. Then TS_1 being elementary, there exists a region r_1 in TS_1 which contains s_1 but not s_1'. Assume that $s_{in}^1 \notin r_1$. (Otherwise replace r_1 by $\bar{r}_1$ and interchange the roles of s_1 and s_1' in what follows.) Then $\emptyset$ being a region in TS_2 which does not contain s_{in}^2 we get from the previous lemma that $r_1 \times \{s_{in}^2\}$ is a region in $TS_1 + TS_2$. Clearly this region contains s but not s'. Symmetric argument applies in case $s_1 = s_1'$ and $s_2 \neq s_2'$.

To verify (A6), assume that $e \in E$ is not enabled at $s \in S$. Assume without loss of generality that e is of the form $(e_1, *)$. We have two cases.

Case 1. s is of the form (s_1, s_{in}^2). From the definition of T, it follows that e_1 is not enabled at s_1 in TS_1. Hence there exists a pre-region r_1 of e_1 in TS_1 which does not contain s_1. Suppose $s_{in}^1 \in r_1$. Then by setting $r_2 = S_2$ we can use Lemma 5.13 to get a pre-region r of $(e_1, *)$ in TS_1 which does not contain (s_1, s_{in}^2). Suppose $s_{in}^1 \notin r_1$. Then by setting $r_2 = \emptyset$ we can use Lemma 5.13 to get a pre-region r of $(e_1, *)$ in $TS_1 + TS_2$ which does not contain (s_1, s_{in}^2).

Case 2: s is of the form (s_{in}^1, s_2) with $s_2 \neq s_{in}^2$. Let r_1 be any pre-region of e_1 in TS_1. Suppose $s_{in}^1 \in r_1$. Then TS_2 being elementary, we can find a region r_2 in TS_2 which contains s_{in}^2 but not s_2. We can then use Lemma 5.13 to find pre-region r in $TS_1 + TS_2$ of $(e_1, *)$ which does not contain (s_{in}^1, s_2). On the other hand, if $s_{in}^1 \notin r_1$ then setting $r_2 = \emptyset$ we get—using Lemma 5.13—a pre-region r of $(e_1, *)$ in $TS_1 + TS_2$ which does not contain (s_{in}^1, s_2). □

Finally we note that every elementary transition system of the form $TS_0 = (\{s_0\}, \emptyset, \emptyset, s_0)$ is both an initial and final object in $\mathscr{ETS}$ because for any other object $TS = (S, E, T, s_{in})$ in $\mathscr{ETS}$ there is exactly one G-morphism f from TS_0 to $TS(f(s_0) = s_{in})$ and there is exactly one G-morphism g from TS to $TS_0(g(s) = s_0$ for every $s \in S)$. Thus $\mathscr{ETS}$ has initial and final objects which are of the form specified above.

6. Discussion

In this paper we have extended to the functorial level, the relationship between elementary transitions systems and elementary net systems established in [4]. Our extended relationship works in the presence of G-morphisms between transition systems and N-morphisms between net systems. It seems fairly natural to single out G-morphisms as the appropriate morphisms between transition systems, especially if the transition systems are meant to model the behaviour of distributed systems. As for the choice of morphisms between net systems there are a number of possibilities. We chose N-morphisms mainly because they capture exactly the relationship between the conditions of a net system and the regions of the "corresponding" transition system. A second possibility is to work with the less restrictive W-morphisms due to Winskel.

As pointed out already these morphisms are not—strictly speaking—strictly more liberal than N-morphisms because we do not require our net system to be contact-free. However, one often imposes contact-freeness anyway and hence there is a good case for choosing W-morphisms. It turns out however that in this case our main result will not go through. To see this, let TS_1 be the (infinite) transition system shown in Fig. 16(a) and $\mathbf{N}_2$ be the net system shown in Fig. 16(b).

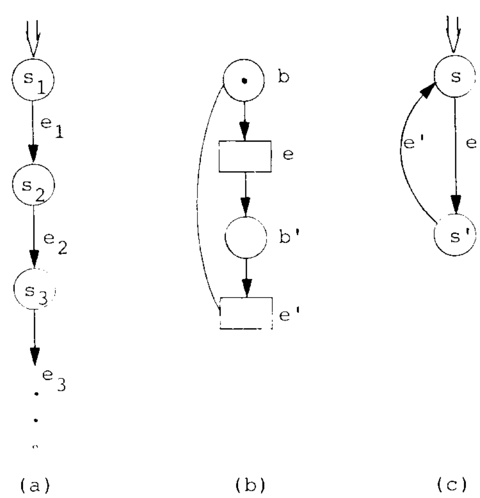

Fig. 16.

$H(\mathbf{N}_2) = TS_2$ is shown in Fig. 16(c) with $s = \{b\}$ and $s' = \{b'\}$. Consider the G-morphism $f: TS_1 \to H(\mathbf{N}_2)$ given by: $f(s_i) = s$ if i is odd and $f(s_i) = s'$ if i is even. It is difficult to pictorially represent $J(TS_1) = \mathbf{N}_1$ because R_{TS_1} has uncountably many elements. However we can identify two interesting portions of $\mathbf{N}_1$ which can be used to construct two *different* N-morphisms (β_1, η) and (β_2, η) from $\mathbf{N}_1$ to $\mathbf{N}_2$. Set $\eta(e_i) = e$ if i is odd and $\eta(e_i) = e'$ if i is even. For specifying β_1 let $r_i = \{s_1, s_2, \ldots, s_i\}$ for every positive integer i. Clearly each r_i is a non-trivial region of

TS_1. Now set

$$\beta_1 = \{(r_i, b), (r_{i+1}, b') \mid i \text{ is odd}\}.$$

Next set $\hat{r}_0 = \{s_i \mid i \text{ is odd}\}$ and $\hat{r}_1 = \{s_i \mid i \text{ is even}\}$ and define $\beta_2 = \{(\hat{r}_0, b), (\hat{r}_1, b')\}$. For the reader familiar with W-morphisms, it should be easy to check that both (β_1, η) and (β_2, η) are W-morphisms with the property $H((\beta_1, \eta)) = H((\beta_2, \eta))$. This example also illustrates a different point. Clearly, there can be no non-trivial N-morphism (β, η) (i.e. $\beta \neq \emptyset$ and $\eta \neq \emptyset$) from $U(\boldsymbol{N}_2)$, the unfolding of $\boldsymbol{N}_2$ [10], to $\boldsymbol{N}_2$. The W-morphism (β_1, η) is however in some sense the (couniversal) W-morphism from $U(\boldsymbol{N}_2)$, to $\boldsymbol{N}_2$ constructed in [4]. The existence of (β_2, η) tells us however that there is a nice N-morphism from the *saturated* version of $U(\boldsymbol{N}_2)$ (which is up to isomorphism, $\boldsymbol{N}_1$) and the *saturated* version of $\boldsymbol{N}_2$ (which is up to isomorphism, itself)! Thus there is some hope that unfoldings can be understood in term of N-morphisms as well, provided we work with saturated net systems.

Our recent work on refinement operations for net systems suggest that it might anyway be a good idea to just work with saturated net systems.

Two other proposals have been made in the literature for net morphisms. The earliest proposal is due to Petri [11]. His notion of morphisms is defined purely in terms of nest and hence no general principles can be derived concerning behaviour. Some particular results can however be stated as illustrated in [13] and [3]. Yet another proposal for net morphisms is due to Meseguer and Montanari [8]. Their net morphisms work for Petri nets in general and at present it is not clear how these morphisms—when brought down to the level of elementary net systems—relate to N-morphisms.

This brings us to possible extensions of the results reported here. There is one obvious way of lifting our results to labeled net systems. First fix Σ, an alphabet of labels. Then one extends the notion of elementary transition systems to labeled elementary transition systems. One can then demand that both G-morphisms and N-morphisms *preserve* labels. This is however a very narrow-minded method of handling labeled structures. More thought and work needs to be devoted to this important topic. Dropping various subsets of the axiom set $\{(A1), (A2), \ldots, (A6)\}$ will of course permit large classes of objects. One natural extension seems to be to just drop (A1) in order to handle 1-safe Petri nets which properly include elementary net systems. The axiom (A2) appears to be equally fundamental and dropping it seems to invite a great deal of trouble. It might be necessary to drop (A3) and (A4) once we start to define interesting operations on elementary transition systems. These issues and others are currently under study.

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