J. Math. Anal. Appl. 389 (2012) 741–753

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Well-posedness for the incompressible magneto-hydrodynamic system on modulation spaces ☆

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ARTICLE INFO

*Article history:*
Received 20 March 2010
Available online 16 December 2011
Submitted by J. Guermond

*Keywords:*
Magneto-hydrodynamic system
Well-posedness
Modulation spaces

ABSTRACT

We study the Cauchy problem for an incompressible magneto-hydrodynamics (MHD) system in the modulation space $M^s_{q,\sigma}(\mathbb{R}^n)$ $(n \geqslant 2)$ with initial data $(u_0, b_0) \in \mathbf{P}M^s_{q,\sigma}(\mathbb{R}^n) \times \mathbf{P}M^s_{q,\sigma}(\mathbb{R}^n)$. We prove that this problem is locally well-posed in such a space when $1 \leqslant q \leqslant \infty$, $1 \leqslant \sigma < \infty$ and $\frac{n(\sigma-1)}{\sigma} - 1 \leqslant s$, and globally well-posed when $1 \leqslant q \leqslant n$, $\frac{n}{n-1} \leqslant \sigma < \infty$ and $\max\{\frac{n(\sigma-1)}{\sigma} - 1, \frac{n(\sigma-2)}{\sigma}\} < s < \frac{n(\sigma-1)}{\sigma}$.



## 1. Introduction

The purpose of this paper is to study the initial value problem of the incompressible magneto-hydrodynamic (MHD) system:

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla P = 0, \\ b_t - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0, \\ \operatorname{div} u = 0, \quad \operatorname{div} b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $t \geqslant 0$ and $n \geqslant 2$ is the space dimension, $u = u(x,t) = (u_1(x,t), \dots, u_n(x,t))$, $b = b(x,t) = (b_1(x,t), \dots, b_n(x,t))$ and $P = p(x,t) + \frac{|b(x,t)|^2}{2}$ are nondimensional quantities corresponding to the velocity of the fluid, the magnetic field and total kinetic pressure, respectively, $u_0$ and $b_0$ are initial velocity and magnetic field satisfying that $\operatorname{div} u_0 = 0$ and $\operatorname{div} b_0 = 0$. Here and throughout this paper, we assume that the Reynolds number, the magnetic Reynolds number and the corresponding coefficients are all equal to 1 for simplicity.

The MHD is a mathematical model describing the motion of an electrically conducting fluid in the presence of a magnetic field, which consists essentially of the interaction between the fluid velocity and the magnetic field (see [22]). When the magnetic field $b(x,t) \equiv 0$, the MHD system becomes the incompressible Navier–Stokes equations, for which there have been a lot of work concerning well-posedness of the initial value problem in various classical function spaces. Fujita and Kato [9] proved both the global well-posedness for small initial data and the local well-posedness for large initial data in the Sobolev space $\dot{H}^s(\mathbb{R}^n)$ for $s \geqslant n/2 - 1$ in 1964. In 1984, Kato [15] established similar results in the Lebesgue spaces $L^n(\mathbb{R}^n)$. In 1985, Giga and Miyakawa [10] considered the Cauchy problem in $L^p(\Omega)$, where $\Omega$ is a bounded domain and $p \geqslant n$. Cannone [2] and Planchon [20] considered global solutions in the case of $n = 3$ for initial $u_0 \in B^{-1+3/q}_{q,\infty}(\mathbb{R}^3)$ $(3 < q \leqslant 6)$. Koch and Tataru

☆ Research supported by the National Natural Science Foundation of China (10771223).
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0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2011.12.015

[16] studied local solutions for initial data $u_0 \in vom^{-1}$ and global solutions for small initial data $u_0 \in BMO^{-1}$, Miura [19] studied the local solutions, which have time continuity in $gmo^{-1}$, for initial data $u_0 \in vmo^{-1} \cap gmo^{-1}$. Wang, Zhao and Guo [29] and Iwabuchi [13,14] studied well-posedness in the modulation space $M^s_{q,\sigma}(\mathbb{R}^n)$.

For the MHD system, the situation becomes more complicated due to the coupling effect between the velocity $u(x,t)$ and the magnetic field $b(x,t)$. Sermange and Temam [21] studied local well-posedness of the Cauchy problem of the MHD system in the $H^s(\mathbb{R}^3)$ ($s \geqslant 3$). Kozono [17] proved the existence of a classical solution for the two-dimensional MHD system in a bounded domain. In [18], Miao, Yuan and Zhang studied the MHD system in $BMO^{-1}$ and $vmo^{-1}$, and obtained global well-posedness the $BMO^{-1}$ and local well-posedness in $bmo^{-1}$ when the initial data $u_0$ is small. Wu studied the regularity of the MHD in [30].

In this paper we study well-posedness of the initial value problem of the MHD system in modulation spaces $M^s_{q,\sigma}(\mathbb{R}^n)$. The modulation space $M^s_{q,\sigma}(\mathbb{R}^n)$ is defined by imposing mixed norm estimates on the local Fourier transform on the involved functions or distributions. More precisely, let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions respectively, and let $g \in \mathcal{S}(\mathbb{R}^n) \setminus 0$ be fixed. Then the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to $g$ is the smooth function, given by

$$V_g f(t,\xi) = \mathcal{F}\big(f \cdot \overline{g(\cdot - x)}\big)(\xi).$$

Here $\mathcal{F}$ stands for the Fourier transform which takes the form

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x$$

when $f \in L^1(\mathbb{R}^n)$.

**Definition 1.1.** Let $g \in \mathcal{S}(\mathbb{R}^n) \setminus 0$. Then the modulation space $M^s_{q,\sigma}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M^s_{q,\sigma}} := \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\big|V_g f(x,\xi)\langle\xi\rangle^s\big|^p\,\mathrm{d}x\right)^{q/p}\right)^{1/q} < \infty, \tag{1.2}$$

where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$.

Modulation spaces were introduced by Feichtinger in [5] and the fundamental theory was established by Feichtinger and Gröchenig in [7,8]. For example, here they prove that $M^s_{q,\sigma}$ increases with the parameters $q,\sigma$ and decreases with the parameter $s$. Furthermore, they prove that if $g \in M^{|s|}_{1,1}$ and $f \in \mathcal{S}'$, then $f \in M^s_{q,\sigma}$, if and only if $V_g f$ satisfies the estimate in (1.2), and different choices of $g$ give rise to equivalent norms. The latter property is improve in [27], where the condition on $g$ is relaxed in $g \in M^{|s|}_{r,r}$ with $1 \leqslant r \leqslant \min\{q,q',\sigma,\sigma'\}$. Here $q'$ denotes the conjugate exponent to $q$, i.e., $\frac{1}{q}+\frac{1}{q'}=1$. We refer to [1,3,4,7,8,11,12,23–26,28,29] and their references for more facts about modulation spaces.

In our investigations it is convenient to follow the approach in [13,14,29] concerning modulation spaces. More precisely it follows from [7] that the following is true:

**Proposition 1.1.** *Let $\{\varphi_k\}_{k\in\mathbb{Z}^n} \subset C_0^\infty(\mathbb{R}^n)$ be a partition of unity satisfying the following. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying*

$$\operatorname{supp}\varphi \subset \big\{\xi \in \mathbb{R}^n \;\big|\; \|\xi\| \leqslant \sqrt{n}\big\}, \qquad \sum_{k\in\mathbb{Z}^n}\varphi(x-k) = 1 \quad \text{for any } \xi \in \mathbb{R}^n.$$

*Let $\varphi_k$ be defined by $\varphi_k := \varphi(\xi - k)$.*

*If $1 \leqslant q,\sigma \leqslant \infty$, $s \in \mathbb{R}$, then we have*

$$M^s_{q,\sigma}\big(\mathbb{R}^n\big) := \big\{f \in \mathcal{S}'\big(\mathbb{R}^n\big) \;\big|\; \|f\|_{M^s_{q,\sigma}(\mathbb{R}^n)} < \infty\big\}, \qquad \|f\|_{M^s_{q,\sigma}(\mathbb{R}^n)} := \begin{cases} (\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{s\sigma}\|\Box_k f\|^\sigma_{L^q(\mathbb{R}^n)})^{\frac{1}{\sigma}} & \text{for } 1 \leqslant \sigma < \infty, \\ \sup_{k\in\mathbb{Z}^n}\langle k\rangle^s\|\Box_k f\|_{L^q(\mathbb{R}^n)} & \text{for } \sigma = \infty, \end{cases}$$

*where $\langle k\rangle := (1+|k|^2)^{\frac{1}{2}}$ and $\Box_k f := \mathcal{F}^{-1}\varphi_k\mathcal{F}f$.*

As a standard practice, the MHD system can be reduced into the following equivalent integral form:

$$\begin{cases} u(\cdot,t) = e^{t\Delta}u_0 - \int_0^t \nabla e^{(t-\tau)\Delta}\mathbf{P}(u\otimes u)(\tau)\,\mathrm{d}\tau + \int_0^t \nabla e^{(t-\tau)\Delta}\mathbf{P}(b\otimes b)(\tau)\,\mathrm{d}\tau := \Phi_1(u,b), \\ b(\cdot,t) = e^{t\Delta}b_0 - \int_0^t \nabla e^{(t-\tau)\Delta}\mathbf{P}(u\otimes b)(\tau)\,\mathrm{d}\tau + \int_0^t \nabla e^{(t-\tau)\Delta}\mathbf{P}(b\otimes u)(\tau)\,\mathrm{d}\tau := \Phi_2(u,b) \end{cases} \tag{1.3}$$

where $\mathbf{P} := 1 + (-\Delta)^{-1}\nabla\mathrm{div}$ is the Helmholtz–Weyl projection operator and $\otimes$ denotes tensor product. Define the space $\mathbf{P}M^s_{q,\sigma}(\mathbb{R}^n)$ by

$$\mathbf{P}M^s_{q,\sigma}(\mathbb{R}^n) := \{u \in [M^s_{q,\sigma}(\mathbb{R}^n)]^n \mid \operatorname{div} u = 0 \text{ in } \mathcal{S}'(\mathbb{R}^n)\}, \qquad \|u\|_{M^s_{q,\sigma}(\mathbb{R}^n)} := \sum_{j=1}^n \|u_j\|_{M^s_{q,\sigma}(\mathbb{R}^n)}.$$

To state our main results, we also need to introduce the following function spaces $l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n)))$ which can be find in [13] and [28].

**Definition 1.2.** For $1 \leqslant \sigma, r, q \leqslant \infty$, $s \in \mathbb{R}$ and $0 < T \leqslant \infty$, we define the function spaces $l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n)))$ by

$$l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n))) := \{f \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^n) \mid \|f\|_{l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n)))} < \infty\},$$

$$\|f\|_{l^{s,\sigma}_{\square}(L^r(0,T;L^q(\mathbb{R}^n)))} := \begin{cases} (\sum_{k\in\mathbb{Z}^n}(\langle k\rangle^s \|\square_k f\|_{L^r(0,T;L^q(\mathbb{R}^n))})^\sigma)^{\frac{1}{\sigma}} & \text{if } \sigma < \infty, \\ \sup_{k\in\mathbb{Z}^n}\langle k\rangle^s \|\square_k f\|_{L^r(0,T;L^q(\mathbb{R}^n))} & \text{if } \sigma = \infty, \end{cases}$$

where $\langle k\rangle$, $\square_k f$ defined as Definition 1.1.

The main results of this paper are the following two theorems. Here it is convenient to set the space $X_T$ for $1 \leqslant q, \sigma \leqslant \infty$ and $s \geqslant -1$ as

$$X_T := \{u \in [C(0,T;M^s_{q,\sigma}(\mathbb{R}^n))]^n \mid \|u\|_{X_T} < \infty, \operatorname{div} u = 0\},$$

$$\|u\|_{X_T} := \|u\|_Y + \|u\|_Z, \qquad \|u\|_Y := \sup_{t\in(0,T)} \|u\|_{M^s_{q,\sigma}(\mathbb{R}^n)},$$

$$\|u\|_Z := \begin{cases} \|u\|_{l^{0,1}_{\square}(L^2(0,T;L^q(\mathbb{R}^n)))} & \text{if } s = -1, \\ \sup_{t\in(0,T)} t^{\frac{|s|}{2}} \|u\|_{M^0_{q,\sigma}(\mathbb{R}^n)} & \text{if } -1 < s < 0, \\ \sup_{t\in(0,T)} t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})} \|u\|_{M^s_{q,\nu}(\mathbb{R}^n)} & \text{if } 0 \leqslant s < \frac{n(\sigma-1)}{\sigma}, \\ \|u\|_{l^{s+1,\sigma}_{\square}(L^2(0,T;L^q(\mathbb{R}^n)))} & \text{if } s \geqslant \frac{n(\sigma-1)}{\sigma}, \end{cases}$$

where $\nu \in \mathbb{R}$ satisfying

$$\frac{1}{\sigma} < \frac{1}{\nu} < \frac{1}{\sigma} + \frac{n(\sigma-1)-\sigma s}{2n\sigma}.$$

**Theorem 1.1** *(Existence of local solution). Let n, q, σ, s satisfy*

$$n \geqslant 2, \qquad 1 \leqslant q \leqslant \infty, \qquad 1 \leqslant \sigma < \infty, \qquad \frac{n(\sigma-1)}{\sigma} - 1 \leqslant s.$$

*Then, for any $(u_0, b_0) \in \mathbf{P}M^s_{q,\sigma}(\mathbb{R}^n) \times \mathbf{P}M^s_{q,\sigma}(\mathbb{R}^n)$, there exists $T > 0$ such that the MHD system has a unique solution $(u, b)$ such that $u, b \in X_T$.*

**Theorem 1.2** *(Existence of global solution). Let n, q, σ, s satisfy*

$$n \geqslant 2, \qquad 1 \leqslant q \leqslant n, \qquad \frac{n}{n-1} \leqslant \sigma < \infty, \qquad \max\left\{\frac{n(\sigma-1)}{\sigma} - 1, \frac{n(\sigma-2)}{\sigma}\right\} < s < \frac{n(\sigma-1)}{\sigma}.$$

*Then there exists a positive constant $\varepsilon_0$ such that if the initial data $(u_0, b_0) \in \mathbf{P}M^s_{q,\sigma}(\mathbb{R}^n) \times \mathbf{P}M^s_{q,\sigma}(\mathbb{R}^n)$ satisfies $\|(u_0, b_0)\|_{M^s_{q,\sigma}(\mathbb{R}^n)} < \varepsilon_0$, then the MHD system has a unique global solution $(u, b)$ such that $u, b \in X$, where X defined by*

$$X := \{u \in [C(0,+\infty;M^s_{q,\sigma}(\mathbb{R}^n))]^n \mid \|u\|_X < \infty, \operatorname{div} u = 0\}.$$

$$\|u\|_X := \|u\|_{\overline{Y}} + \|u\|_{\overline{Z}}, \qquad \|u\|_{\overline{Y}} := \sup_{t\in(0,+\infty)} \|u\|_{M^s_{q,\sigma}(\mathbb{R}^n)}, \qquad \|u\|_{\overline{Z}} := \sup_{t\in(0,+\infty)} (1+t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})} \|u\|_{M^s_{r,\sigma}(\mathbb{R}^n)},$$

*where $r \in \mathbb{R}$ satisfying*

$$\max\{q, 2\} < r < 2q, \qquad \frac{1}{q} - \frac{1}{n} < \frac{1}{r} < \frac{1}{2}\left(\frac{1}{q} + \frac{1}{n}\right).$$

**Remark 1.1.** Note that when $s < -1$, $1 \leqslant \sigma < \infty$ and $b \equiv 0$, the MHD system is ill-posed in $M^s_{2,\sigma}(\mathbb{R}^n)$ (see [13]).

This paper is divided into three sections. In Section 2, we recall some properties of modulation spaces and of the propagator $e^{t\Delta}$ in the modulation spaces. In Section 3, we prove our main results.

## 2. Preliminaries

In this section, we recall some properties of modulation spaces $M^s_{q,\sigma}(\mathbb{R}^n)$ and $l^{s,\sigma}_\Box(L^r(0,T;L^q(\mathbb{R}^n)))$ which refer to [3,5,7,11,13,14,24–26,28,29], and the estimates for the propagator $e^{t\Delta}$ in these function spaces. Throughout this paper, we put $l^{s,\sigma}L^r_TL^q := l^{s,\sigma}_\Box(L^r(0,T;L^q(\mathbb{R}^n)))$ and $\|u\|_{M^s_{q,\sigma}} := \|u\|_{M^s_{q,\sigma}(\mathbb{R}^n)}$ for simplicity, and let $C$ denote the positive constant which is independent of $t$ and can change in each line.

**Proposition 2.1.** *(See [5,7,11,28,29].) If $1 \leqslant q, r, \sigma \leqslant \infty$, $s \in \mathbb{R}$ and $T > 0$ then $M^s_{q,\sigma}(\mathbb{R}^n)$ and $l^{s,\sigma}_\Box L^r_T L^q$ are complete Banach spaces.*

**Proposition 2.2.** *(See [6,13,14,24–26,28].) Let $1 \leqslant q, q_1, q_2, \sigma, \sigma_1, \sigma_2 \leqslant \infty$ and $s \in \mathbb{R}$, then we have the following continuous embeddings*:

(i) *If $\frac{1}{q} + \frac{1}{q'} = 1$, then*

$$M^0_{q,\min\{q,q'\}}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \hookrightarrow M^0_{q,\max\{q,q'\}}(\mathbb{R}^n).$$

(ii) *If $q_1 \leqslant q_2$, $\sigma_1 \leqslant \sigma_2$, $s_1 \geqslant s_2$, $1 \leqslant r \leqslant \infty$ and $T > 0$, then*

$$M^{s_1}_{q_1,\sigma_1}(\mathbb{R}^n) \hookrightarrow M^{s_2}_{q_2,\sigma_2}(\mathbb{R}^n), \qquad l^{s_1,\sigma_1}_\Box L^r_T L^{q_1} \hookrightarrow l^{s_2,\sigma_2}_\Box L^r_T L^{q_2}.$$

(iii) *If $\sigma_1 \leqslant \sigma_2$, $s_1 > s_2$, $s_1 - s_2 > n(\frac{1}{\sigma_2} - \frac{1}{\sigma_1})$, $1 \leqslant r \leqslant \infty$ and $T > 0$, then*

$$M^{s_1}_{q,\sigma_1}(\mathbb{R}^n) \hookrightarrow M^{s_2}_{q,\sigma_2}(\mathbb{R}^n), \qquad l^{s_1,\sigma_1}_\Box L^r_T L^q \hookrightarrow l^{s_2,\sigma_2}_\Box L^r_T L^q.$$

**Proposition 2.3.** *(See [3,5,13,14].) Let $1 \leqslant q, q_1, q_2, r, r_1, r_2 \leqslant \infty$, $1 < \sigma, \sigma_1, \sigma_2 < \infty$, $0 < s < \frac{n}{\sigma}$ and $T > 0$ satisfy*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \qquad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \qquad \frac{1}{\sigma} - \frac{s}{n} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} - 1, \qquad \sigma \geqslant \sigma_1, \qquad \sigma \geqslant \sigma_1.$$

*Then there exists a constant $C > 0$ such that for any $u \in M^s_{q_1,\sigma_1}(\mathbb{R}^n)$ and $v \in M^s_{q_2,\sigma_2}(\mathbb{R}^n)$, we have*

$$\|uv\|_{M^s_{q,\sigma}} \leqslant C\|u\|_{M^s_{q_1,\sigma_1}}\|v\|_{M^s_{q_2,\sigma_2}}.$$

*For any $u \in l^{s,\sigma_1}L^{r_1}_T L^{q_1}$ and $v \in l^{s,\sigma_2}L^{r_2}_T L^{q_2}$, we have*

$$\|uv\|_{l^{s,\sigma}L^r_TL^q} \leqslant C\|u\|_{l^{s,\sigma_1}L^{r_1}_TL^{q_1}}\|v\|_{l^{s,\sigma_2}L^{r_2}_TL^{q_2}}.$$

**Proposition 2.4.** *Let $1 \leqslant q, r, \sigma, \nu \leqslant \infty$, $s, \widetilde{s} \in \mathbb{R}$.*

(i) *(See [13,14,25,26].) If $q \geqslant r$, there exists a constant $C > 0$ such that*

$$\|e^{t\Delta}u\|_{M^s_{q,\sigma}} \leqslant C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|u\|_{M^s_{r,\sigma}}.$$

(ii) *(See [13,14,25,26].) If $\sigma \leqslant \nu$, there exists a constant $C > 0$ such that*

$$\|e^{t\Delta}u\|_{M^s_{q,\sigma}} \leqslant C\big(1 + t^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\big)\|u\|_{M^s_{q,\nu}}.$$

(iii) *(See [13,14].) If $s \leqslant \widetilde{s}$, there exists a constant $C > 0$ such that*

$$\|e^{t\Delta}u\|_{M^{\widetilde{s}}_{q,\sigma}} \leqslant C\big(1 + t^{-\frac{\widetilde{s}-s}{2}}\big)\|u\|_{M^s_{q,\sigma}}.$$

(iv) *There exists a constant $C > 0$ such that*

$$\|\nabla e^{t\Delta}u\|_{M^s_{q,\sigma}} \leqslant Ct^{-\frac{1}{2}}\|u\|_{M^s_{q,\sigma}}.$$

**Proof of Proposition 2.4(iv).** By Definition 1.1, we only need to prove the inequalities $\|\Box_k\nabla e^{t\Delta}u\|_{L^q(\mathbb{R}^n)} \leqslant C\frac{1}{t^{\frac{1}{2}}}\|\Box_k u\|_{L^q(\mathbb{R}^n)}$.

Using the Young's inequality and changing variables, we can get

$$\|\Box_k\nabla e^{t\Delta}u\|_{L^q(\mathbb{R}^n)} \leqslant \|\mathcal{F}^{-1}(\xi e^{-t|\xi|^2})\|_{L^1(\mathbb{R}^n)}\|\Box_k u\|_{L^q(\mathbb{R}^n)} = \frac{1}{t^{\frac{1}{2}}}\|\mathcal{F}^{-1}(\eta e^{-|\eta|^2})\|_{L^1(\mathbb{R}^n)}\|\Box_k u\|_{L^q(\mathbb{R}^n)},$$

and $\|\mathcal{F}^{-1}(\eta e^{-t|\eta|^2})\|_{L^1(\mathbb{R}^n)} < +\infty$. $\Box$

**Proposition 2.5.** *(See [13,14].) Let $1 \leqslant q \leqslant \infty$, $1 \leqslant \nu < \sigma < \infty$, $s \in \mathbb{R}$, then the following estimates hold.*

(i) $\lim_{T\to 0}\sup_{t\in(0,T)} t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\|e^{t\Delta}u\|_{M^s_{q,\nu}} = 0$ *for any* $u \in M^s_{q,\sigma}(\mathbb{R}^n)$.

(ii) $\lim_{T\to 0}\sup_{t\in(0,T)} t^{\frac{1}{2}+\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\|e^{t\Delta}u\|_{M^{s+1}_{q,\nu}} = 0$ *for any* $u \in M^s_{q,\sigma}(\mathbb{R}^n)$.

(iii) $\lim_{T\to 0}\sup_{t\in(0,T)} t^{\frac{1}{2}+\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\|\nabla e^{t\Delta}u\|_{M^s_{q,\nu}} = 0$ *for any* $u \in M^s_{q,\sigma}(\mathbb{R}^n)$.

**Proposition 2.6.** *Let $1 \leqslant q \leqslant \infty$, $1 \leqslant \sigma < \infty$, $s \in \mathbb{R}$, $0 < \alpha \leqslant 1$, then we have*

(i) $\|e^{t\Delta}u\|_{l^{s,\sigma}L_T^{\frac{2}{\alpha}}L^q} \leqslant C(1+T^{\frac{\alpha}{2}})\|u\|_{M^{s-\alpha}_{q,\sigma}}$ *for any* $u \in M^{s-\alpha}_{q,\sigma}(\mathbb{R}^n)$.

(ii) $\lim_{T\to 0}\|e^{t\Delta}u\|_{l^{s,\sigma}L^{\frac{2}{\alpha}}L^q} = 0$ *for any* $u \in M^{s-\alpha}_{q,\sigma}(\mathbb{R}^n)$.

**Proof.** Similar to the proof of the Proposition 2.12 of [13]. □

**Proposition 2.7.** *(See [13].) Let $1 \leqslant r, q, \sigma \leqslant \infty$, $s \in \mathbb{R}$ and $T > 0$.*

(i) *There exists a constant $C > 0$ such that for any $u \in l^{s,\sigma}L^1_T L^q$, we have*

$$\sup_{t\in(0,T)}\left\|\int_0^t e^{t\Delta}u(\tau),\mathrm{d}\tau\right\|_{M^s_{q,\sigma}} \leqslant C\|u\|_{l^{s,\sigma}L^1_T L^q}.$$

(ii) *There exists a constant $C > 0$ such that for any $u \in l^{s-\frac{2}{r},\sigma}L^1_T L^q$, we have*

$$\left\|\int_0^t e^{t\Delta}u(\tau),\mathrm{d}\tau\right\|_{l^{s,\sigma}L^r_T L^q} \leqslant C\big(1+T^{\frac{1}{r}}\big)\|u\|_{l^{s-\frac{1}{r},\sigma}L^1_T L^q}.$$

**Proposition 2.8.** *(See [13,14].) Let $1 \leqslant q, r, \sigma \leqslant \infty$, $s \in \mathbb{R}$ and $T > 0$, then there exists a constant $C > 0$ such that*

$$\|\nabla \mathbf{P}u\|_{M^s_{q,\sigma}} \leqslant C\|u\|_{M^{s+1}_{q,\sigma}} \quad \text{if } u \in M^{s+1}_{q,\sigma}\big(\mathbb{R}^n\big), \qquad \|\nabla \mathbf{P}u\|_{l^s_\sigma L^r_T L^q} \leqslant C\|u\|_{l^{s+1}_\sigma L^r_T L^q} \quad \text{if } u \in l^{s+1}_\sigma L^r_T L^q.$$

## 3. Proof of main results

We consider the integral form of (1.3), and rewrite it into the following form

$$\begin{cases} u(\cdot,t) = \Phi_1(u,b) = e^{t\Delta}u_0 - B(u,u) + B(b,b), \\ b(\cdot,t) = \Phi_2(u,b) = e^{t\Delta}b_0 - B(u,b) + B(b,u) \end{cases} \tag{3.1}$$

where $B(u,v) := \int_0^t \nabla e^{(t-\tau)\Delta}\mathbf{P}(u\otimes v)(\tau)\,\mathrm{d}\tau$. For $u, b \in X$, we define the norm of vector $(u,b)$ as

$$\big\|(u,b)\big\|_X = \|u\|_X + \|b\|_X.$$

For simplicity, the integral equations (3.1) can be written as

$$(u,b) = \big(\Phi_1(u,b), \Phi_2(u,b)\big) :\triangleq \mathcal{T}(u,b).$$

**Proof of Theorem 1.1.** We consider the case $s=-1$ in Step 1; the case $-1<s<0$ in Step 2; the case $0 \leqslant s < \frac{n(\sigma-1)}{\sigma}$ in Step 3 and the last case in Step 4.

Step 1. When $s=-1$, by the condition of Theorem 1.1, we have $\sigma = 1$ and $\|u\|_Y = \sup_{t\in(0,T)}\|u\|_{M^{-1}_{q,1}}$. Proposition 2.4 implies that

$$\big\|e^{t\Delta}u_0\big\|_Y \leqslant C\|u_0\|_{M^{-1}_{q,1}}, \tag{3.2}$$

and Proposition 2.6 implies

$$\big\|e^{t\Delta}u_0\big\|_Z = \big\|e^{t\Delta}u_0\big\|_{l^{0,1}L^2_T L^q} \leqslant C\big(1+T^{\frac{1}{2}}\big)\|u_0\|_{M^{-1}_{q,1}}. \tag{3.3}$$

Applying Propositions 2.8, 2.7, 2.4, 2.3 and 2.2, we have

$$\begin{aligned}\|B(u,b)\|_Y &= \sup_{t\in(0,T)}\left\|\int_0^t \nabla e^{(t-\tau)\Delta}\mathbf{P}(u\otimes b)\,d\tau\right\|_{M^{-1}_{q,1}} \leqslant C\sup_{t\in(0,T)}\left\|\int_0^t e^{(t-\tau)\Delta}(u\otimes b)\,d\tau\right\|_{M^0_{q,1}}\\ &\leqslant C\|u\otimes b\|_{l^{0,1}L^1_T L^q} \leqslant C\|u\|_{l^{0,1}L^2_T L^{2q}}\|b\|_{l^{0,1}L^2_T L^{2q}} \leqslant C\|u\|_Z\|b\|_Z, \end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\|B(u,b)\|_Z &\leqslant C\left\|\int_0^t e^{(t-\tau)\Delta}(u\otimes b)\,d\tau\right\|_{l^{1,1}L^2_T L^q} \leqslant C\big(1+T^{\frac12}\big)\|u\otimes b\|_{l^{0,1}L^1_T L^q}\\ &\leqslant C\big(1+T^{\frac12}\big)\|u\|_{l^{0,1}L^2_T L^{2q}}\|b\|_{l^{0,1}L^2_T L^{2q}} \leqslant C\big(1+T^{\frac12}\big)\|u\|_Z\|b\|_Z. \end{aligned}\tag{3.5}$$

Making use of the estimates (3.2)–(3.5) above lead to the following two estimates

$$\|\Phi_1(u,b)\|_{X_T} \leqslant C\big(1+T^{\frac12}\big)\|u_0\|_{M^{-1}_{q,1}} + C\big(1+T^{\frac12}\big)\|u\|_Z^2 + C\big(1+T^{\frac12}\big)\|b\|_Z^2,\tag{3.6}$$

$$\|\Phi_2(u,b)\|_{X_T} \leqslant C\big(1+T^{\frac12}\big)\|b_0\|_{M^{-1}_{q,1}} + 2C\big(1+T^{\frac12}\big)\|u\|_Z\|b\|_Z.\tag{3.7}$$

Now, let $0<T\leqslant 1$, $K_0 := \|(u_0,b_0)\|_{M^{-1}_{q,1}} = \|u_0\|_{M^{-1}_{q,1}} + \|b_0\|_{M^{-1}_{q,1}}$. The Proposition 2.1 allows us to define a complete metric space

$$E = \big\{(u,b)\ \big|\ u,b\in X_T,\ \|(u,b)\|_Y \leqslant 2CK_0+\varepsilon,\ \|(u,b)\|_Z\leqslant\varepsilon\big\}$$

with the metric $d(f,g)=\|f-g\|_{X_T}$. Combining (3.6), (3.7) and the fact that $0<T\leqslant 1$, it follows that for $(u,b)\in E$,

$$\|\mathcal{T}(u,b)\|_{X_T} \leqslant 2C\|(u_0,b_0)\|_{M^{-1}_{q,1}} + 2C\|(u,b)\|_Z^2 \leqslant 2CK_0 + 2C\varepsilon^2.\tag{3.8}$$

Let $T$ be small enough such that $16C\varepsilon<1$, we obtain $\|\mathcal{T}(u,b)\|_{X_T}\leqslant 2CK_0+\varepsilon$ from above inequality and $\mathcal{T}(u,b)\in E$.

On the other hand, for any $(u_1,b_1),(u_2,b_2)\in E$ we have

$$\begin{aligned}\|\Phi_1(u_1,b_1)-\Phi_1(u_2,b_2)\|_{X_T} &\leqslant C\big(\|B(u_1,u_1)-B(u_2,u_2)\|_{X_T} + \|B(b_1,b_1)-B(b_2,b_2)\|_{X_T}\big)\\ &\leqslant C\big(\|B(u_1,u_1-u_2)+B(u_1-u_2,u_2)\|_{X_T} + \|B(b_1-b_2,b_2)+B(b_1,b_1-b_2)\|_{X_T}\big)\\ &\leqslant C\big(1+T^{\frac12}\big)\big\{\big(\|u_1\|_Z+\|u_2\|_Z\big)\|u_1-u_2\|_Z + \big(\|b_1\|_Z+\|b_2\|_Z\big)\|b_1-b_2\|_Z\big\}\\ &\leqslant 8C\varepsilon\big(\|u_1-u_2\|_Z+\|b_1-b_2\|_Z\big)\\ &\leqslant \frac12\big(\|u_1-u_2\|_{X_T}+\|b_1-b_2\|_{X_T}\big), \end{aligned}\tag{3.9}$$

similarly,

$$\begin{aligned}\|\Phi_2(u_1,b_1)-\Phi_2(u_2,b_2)\|_{X_T} &\leqslant 2C\big(\|b_1\|_Z\|u_1-u_2\|_Z+\|u_2\|_Z\|b_1-b_2\|_Z\big)\\ &\leqslant 4C\varepsilon\big(\|u_1-u_2\|_Z+\|b_1-b_2\|_Z\big)\\ &\leqslant \frac12\big(\|u_1-u_2\|_{X_T}+\|b_1-b_2\|_{X_T}\big). \end{aligned}\tag{3.10}$$

Combining (3.9) and (3.10), it follows that

$$\|\mathcal{T}(u_1,b_1)-\mathcal{T}(u_2,b_2)\|_{X_T} \leqslant \frac12\|(u_1,b_1)-(u_2,b_2)\|_{X_T}.\tag{3.11}$$

From the estimates (3.8) and (3.11), we get that $\mathcal{T}$ is a contraction mapping from $E$ to $E$ when $T$ is small enough. By the Banach's fixed point theorem, we conclude that $\mathcal{T}$ has a unique fixed point $(u,b)\in E$.

Step 2. For $-1<s<0$ and $\frac{n(\sigma-1)}{\sigma}-1\leqslant s$, then $1<\sigma<2$. Similar to Step 1, we first prove the following estimates

$$\|\Phi_1(u,b)\|_{X_T} \leqslant C\big(1+T^{\frac{|s|}{2}}\big)\|u_0\|_{M^s_{q,\sigma}} + C\big(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\big)\|u\|_Z^2 + C\big(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\big)\|b\|_Z^2,\tag{3.12}$$

$$\|\Phi_2(u,b)\|_{X_T} \leqslant C\big(1+T^{\frac{|s|}{2}}\big)\|b_0\|_{M^s_{q,\sigma}} + 2C\big(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\big)\|u\|_Z\|b\|_Z,\tag{3.13}$$

where $\alpha_1,\alpha_2>0$ and $\beta_1,\beta_2\geqslant 0$.

By Proposition 2.4, we have

$$\|e^{t\Delta}u_0\|_Y \leqslant C\|u_0\|_{M^s_{q,\sigma}},$$
$$\|e^{t\Delta}u_0\|_Z = \sup_{t\in(0,T)} t^{\frac{|s|}{2}} \|e^{t\Delta}u_0\|_{M^0_{q,\sigma}} \leqslant C \sup_{t\in(0,T)} t^{\frac{|s|}{2}} \big(1+t^{\frac{s}{2}}\big)\|u_0\|_{M^s_{q,\sigma}} \leqslant C\big(1+T^{\frac{|s|}{2}}\big)\|u_0\|_{M^s_{q,\sigma}}.$$

Define a real number $\widetilde{\nu}$ with $\frac{1}{\widetilde{\nu}} = \frac{2}{\sigma} - 1$, then $\sigma < \widetilde{\nu} < \infty$ for $\sigma \geqslant 1$. Applying Propositions 2.8, 2.4, 2.3 and 2.2, we obtain

$$\begin{aligned}
\|B(u,b)\|_{M^s_{q,\sigma}} &\leqslant C\int_0^t \|e^{(t-\tau)\Delta}(u\otimes b)\|_{M^{s+1}_{q,\sigma}}\,\mathrm{d}\tau \\
&\leqslant C\int_0^t \big(1+(t-\tau)^{-\frac{1+s}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)\|u\otimes b\|_{M^0_{q,\widetilde{\nu}}}\,\mathrm{d}\tau \\
&\leqslant C\int_0^t \big(1+(t-\tau)^{-\frac{1+s}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)\|u\|_{M^0_{2q,\sigma}}\|b\|_{M^0_{2q,\sigma}}\,\mathrm{d}\tau \\
&\leqslant C\int_0^t \big(1+(t-\tau)^{-\frac{1+s}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)\|u\|_{M^0_{q,\sigma}}\|b\|_{M^0_{q,\sigma}}\,\mathrm{d}\tau \\
&\leqslant C\|u\|_Z\|b\|_Z\int_0^t \big(1+(t-\tau)^{-\frac{1+s}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)\tau^s\,\mathrm{d}\tau \\
&\leqslant C\|u\|_Z\|b\|_Z\big(T^{1+s}+T^{1-\frac{s+1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})+s}\big) \\
&\triangleq C\|u\|_Z\|b\|_Z\big(T^{\alpha_1}+T^{\beta_1}\big),
\end{aligned}$$

where $\alpha_1 > 0$ for $-1 < s < 0$ and $\frac{n(\sigma-1)}{\sigma} - 1 \leqslant s$, and by the definition of $\widetilde{\nu}$, we can get $\beta_1 \geqslant 0$, that's $\|B(u,b)\|_Y \leqslant C(T^{\alpha_1}+T^{\beta_1})\|u\|_Z\|b\|_Z$.

Applying Propositions 2.8, 2.4, 2.3 and 2.2 again, we obtain

$$\begin{aligned}
t^{\frac{|s|}{2}}\|B(u,b)\|_{M^0_{q,\sigma}} &\leqslant Ct^{\frac{|s|}{2}}\int_0^t \|e^{(t-\tau)\Delta}(u\otimes b)\|_{M^1_{q,\sigma}}\,\mathrm{d}\tau \\
&\leqslant Ct^{\frac{|s|}{2}}\int_0^t \big(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)\|u\otimes b\|_{M^0_{q,\widetilde{\nu}}}\,\mathrm{d}\tau \\
&\leqslant Ct^{\frac{|s|}{2}}\int_0^t \big(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)\|u\|_{M^0_{2q,\sigma}}\|b\|_{M^0_{2q,\sigma}}\,\mathrm{d}\tau \\
&\leqslant Ct^{\frac{|s|}{2}}\int_0^t \big(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)\|u\|_{M^0_{q,\sigma}}\|b\|_{M^0_{q,\sigma}}\,\mathrm{d}\tau \\
&\leqslant C\|u\|_Z\|b\|_Zt^{\frac{|s|}{2}}\int_0^t \big(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)\tau^s\,\mathrm{d}\tau \\
&\leqslant C\|u\|_Z\|b\|_Zt^{\frac{|s|}{2}}\big(t^{1+s}+t^{\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})+s}\big) \\
&\leqslant C\|u\|_Z\|b\|_Z\big(T^{1+s+\frac{|s|}{2}}+T^{\frac{1}{2}+\frac{|s|}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})+s}\big) \\
&\triangleq C\|u\|_Z\|b\|_Z\big(T^{\alpha_2}+T^{\beta_2}\big),
\end{aligned}$$

where $\alpha_2 > 0$ for $-1 < s < 0$ and $\frac{n(\sigma-1)}{\sigma} - 1 \leqslant s$, and by the definition of $\widetilde{\nu}$, we can get $\beta_2 \geqslant 0$, that's $\|B(u,b)\|_Z \leqslant C(T^{\alpha_2}+T^{\beta_2})\|u\|_Z\|b\|_Z$. Using the four estimates above, we obtain (3.12) and (3.13).

Combining (3.12) and (3.13), it follows that

$$\left\|\mathcal{T}(u,b)\right\|_{X_T} \leqslant C\left(1+T^{\frac{|s|}{2}}\right)\left\|(u_0,b_0)\right\|_{M^s_{q,\sigma}} + C\left(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\right)\left\|(u,b)\right\|^2_Z. \tag{3.14}$$

Now, let $0<T<1$ and $K_0 := \|(u_0,b_0)\|_{M^s_{q,\sigma}} = \|u_0\|_{M^s_{q,\sigma}} + \|b_0\|_{M^s_{q,\sigma}}$. Define a complete metric space

$$E = \left\{(u,b)\ \middle|\ u,b\in X_T,\ \left\|(u,b)\right\|_Y \leqslant 2CK_0+\varepsilon,\ \left\|(u,b)\right\|_Z \leqslant \varepsilon\right\}$$

with the metric $d(f,g)=\|f-g\|_{X_T}$. Let $T$ be small enough such that $32C\varepsilon<1$, the estimate (3.14) implies that for any $(u,b)\in E$, $\|\mathcal{T}(u,b)\|_{X_T}\leqslant 2CK_0+\varepsilon$ and $\mathcal{T}(u,b)\in E$.

On the other hand, similar to (3.9) and (3.10), for any $(u_1,b_1),(u_2,b_2)\in E$, we have

$$\begin{aligned}
&\left\|\Phi_1(u_1,b_1)-\Phi_1(u_2,b_2)\right\|_{X_T}\\
&\quad\leqslant C\left(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\right)\left\{\left(\|u_1\|_Z+\|u_2\|_Z\right)\|u_1-u_2\|_Z+\left(\|b_1\|_z+\|b_2\|_Z\right)\|b_1-b_2\|_Z\right\}\\
&\quad\leqslant 16C\varepsilon\left(\|u_1-u_2\|_{X_T}+\|b_1-b_2\|_{X_T}\right),\\
&\left\|\Phi_2(u_1,b_1)-\Phi_2(u_2,b_2)\right\|_{X_T}\\
&\quad\leqslant 2C\left(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\right)\left(\|b_1\|_Z\|u_1-u_2\|_{X_T}+\|u_2\|_{X_T}\|b_1-b_2\|_{X_T}\right)\\
&\quad\leqslant 16C\varepsilon\left(\|u_1-u_2\|_{X_T}+\|b_1-b_2\|_{X_T}\right).
\end{aligned}$$

Then

$$\left\|\mathcal{T}(u_1,b_1)-\mathcal{T}(u_2,b_2)\right\|_{X_T} \leqslant \frac{1}{2}\left\|(u_1,b_1)-(u_2,b_2)\right\|_{X_T}. \tag{3.15}$$

The estimates (3.14) and (3.15) imply that $\mathcal{T}$ is a contraction mapping from $E$ to itself if $T$ is small enough. The Banach's fixed point theorem is applied, then there exists a unique fixed point $(u,b)\in E$ of $\mathcal{T}$.

Step 3. For $0\leqslant s<\frac{n(\sigma-1)}{\sigma}$ and $\frac{n(\sigma-1)}{\sigma}-1\leqslant s$. Similar to Step 2, we first show the following estimates

$$\begin{aligned}
&\left\|\Phi_1(u,b)\right\|_{X_T}\\
&\quad\leqslant C\left(1+T^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\right)\|u_0\|_{M^s_{q,\sigma}} + C\left(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\right)\|u\|^2_Z + C\left(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\right)\|b\|^2_Z,
\end{aligned} \tag{3.16}$$

$$\left\|\Phi_2(u,b)\right\|_{X_T} \leqslant C\left(1+T^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\right)\|b_0\|_{M^s_{q,\sigma}} + 2C\left(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\right)\|u\|_Z\|b\|_Z, \tag{3.17}$$

where $\alpha_1,\alpha_2>0$ and $\beta_1,\beta_2\geqslant 0$.

By Proposition 2.4, we have

$$\begin{aligned}
&\left\|e^{t\Delta}u_0\right\|_Y \leqslant C\|u_0\|_{M^s_{q,\sigma}},\\
&\left\|e^{t\Delta}u_0\right\|_Z = \sup_{t\in(0,T)} t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\left\|e^{t\Delta}u_0\right\|_{M^s_{q,\nu}}\\
&\qquad\leqslant C\sup_{t\in(0,T)} t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\left(1+t^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\right)\|u_0\|_{M^s_{q,\sigma}} \leqslant C\left(1+T^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\right)\|u_0\|_{M^s_{q,\sigma}}.
\end{aligned}$$

Define a real number $\widetilde{\nu}$ with $\frac{1}{\widetilde{\nu}}-\frac{s}{n}=\frac{2}{\nu}-1$, then $\sigma<\widetilde{\nu}<\infty$ because of the selection of $\nu$ in Theorem 1.1 satisfying $\frac{1}{\sigma}<\frac{1}{\nu}<\frac{1}{\sigma}+\frac{n(\sigma-1)-\sigma s}{2n\sigma}$. Using Propositions 2.8, 2.4, 2.3 and 2.2, it follows that

$$\begin{aligned}
\left\|B(u,b)\right\|_{M^s_{q,\sigma}} &\leqslant C\int_0^t \left\|\nabla e^{(t-\tau)\Delta}(u\otimes b)\right\|_{M^s_{q,\sigma}}\,d\tau\\
&\leqslant C\int_0^t \left(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\right)(t-\tau)^{-\frac{1}{2}}\|u\otimes b\|_{M^s_{q,\widetilde{\nu}}}\,d\tau\\
&\leqslant C\int_0^t \left(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\right)(t-\tau)^{-\frac{1}{2}}\|u\|_{M^s_{2q,\sigma}}\|b\|_{M^s_{2q,\sigma}}\,d\tau\\
&\leqslant C\int_0^t \left(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\right)(t-\tau)^{-\frac{1}{2}}\|u\|_{M^s_{q,\sigma}}\|b\|_{M^s_{q,\sigma}}\,d\tau
\end{aligned}$$

$$\leqslant C\|u\|_Z\|b\|_Z\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\big)(t-\tau)^{-\frac{1}{2}}\tau^{-n(\frac{1}{\nu}-\frac{1}{\sigma})}\,\mathrm{d}\tau$$
$$\leqslant C\|u\|_Z\|b\|_Z\big(T^{\frac{1}{2}-n(\frac{1}{\nu}-\frac{1}{\sigma})}+T^{\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})-n(\frac{1}{\nu}-\frac{1}{\sigma})}\big)$$
$$\triangleq C\|u\|_Z\|b\|_Z\big(T^{\alpha_1}+T^{\beta_1}\big),$$

where $\alpha_1>0$ for $0\leqslant s<\frac{n(\sigma-1)}{\sigma}$, $\frac{n(\sigma-1)}{\sigma}-1\leqslant s$ and $\frac{1}{\sigma}<\frac{1}{\nu}<\frac{1}{\sigma}+\frac{n(\sigma-1)-\sigma s}{\sigma}$, and by the definition of $\widetilde{\nu}$, we can get $\beta_1\geqslant 0$, that's $\|B(u,b)\|_Y\leqslant C(T^{\alpha_1}+T^{\beta_1})\|u\|_Z\|b\|_Z$.

Using Propositions 2.8, 2.4, 2.3 and 2.2 again, we obtain

$$t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\big\|B(u,b)\big\|_{M^s_{q,\nu}}\leqslant Ct^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\int_0^t\big\|\nabla e^{(t-\tau)\Delta}(u\otimes b)\big\|_{M^s_{q,\nu}}\,\mathrm{d}\tau$$
$$\leqslant Ct^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\widetilde{\nu}})}\big)(t-\tau)^{-\frac{1}{2}}\|u\otimes b\|_{M^s_{q,\widetilde{\nu}}}\,\mathrm{d}\tau$$
$$\leqslant Ct^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\widetilde{\nu}})}\big)(t-\tau)^{-\frac{1}{2}}\|u\|_{M^s_{q,\sigma}}\|b\|_{M^s_{q,\sigma}}\,\mathrm{d}\tau$$
$$\leqslant C\|u\|_Z\|b\|_Z t^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\widetilde{\nu}})}\big)(t-\tau)^{-\frac{1}{2}}\tau^{-n(\frac{1}{\nu}-\frac{1}{\sigma})}\,\mathrm{d}\tau$$
$$\leqslant C\|u\|_Z\|b\|_Z\big(T^{\frac{1}{2}-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}+T^{\frac{1}{2}-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\widetilde{\nu}})-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\sigma})}\big)$$
$$\triangleq C\|u\|_Z\|b\|_Z\big(T^{\alpha_2}+T^{\beta_2}\big),$$

where $\alpha_2>0$ and $\beta_2\geqslant 0$ are satisfied by $0\leqslant s<\frac{n(\sigma-1)}{\sigma}$, $\frac{n(\sigma-1)}{\sigma}-1\leqslant s$, $\frac{1}{\sigma}<\frac{1}{\nu}<\frac{1}{\sigma}+\frac{n(\sigma-1)-\sigma s}{2n\sigma}$ and definition of $\widetilde{\nu}$, that's $\|B(u,b)\|_Z\leqslant C(T^{\alpha_2}+T^{\beta_2})\|u\|_Z\|b\|_Z$.

Making use of the four estimates above, we can establish (3.16) and (3.17). Similar to (3.9) and (3.10), we can get

$$\big\|\Phi_1(u_1,b_1)-\Phi_1(u_2,b_2)\big\|_{X_T}$$
$$\leqslant C\big(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\big)\big\{\big(\|u_1\|_Z+\|u_2\|_Z\big)\|u_1-u_2\|_Z+\big(\|b_1\|_Z+\|b_2\|_Z\big)\|b_1-b_2\|_Z\big\},$$
$$\big\|\Phi_2(u_1,b_1)-\Phi_2(u_2,b_2)\big\|_{X_T}\leqslant 2C\big(T^{\alpha_1}+T^{\alpha_2}+T^{\beta_1}+T^{\beta_2}\big)\big(\|b_1\|_Z\|u_1-u_2\|_Z+\|u_2\|_Z\|b_1-b_2\|_Z\big).$$

Now, we can define a complete metric space $E$ similar to Step 1 and Step 2, such that $\mathcal{T}$ is a contract and onto map on $E$. Using the Banach's fixed point theorem, $\mathcal{T}$ has a unique fixed point in $E$.

Step 4. For $s\geqslant\frac{n(\sigma-1)}{\sigma}$, Proposition 2.5 implies $\|e^{t\Delta}u_0\|_Y\leqslant C\|u_0\|_{M^s_{q,\sigma}}$ and the Proposition 2.7 implies

$$\big\|e^{t\Delta}u_0\big\|_Z=\big\|e^{t\Delta}u_0\big\|_{l^{s+1,\sigma}L^2_TL^q}\leqslant C\big(1+T^{\frac{1}{2}}\big)\|u_0\|_{M^s_{q,\sigma}}.$$

From Propositions 2.8, 2.7, 2.4, 2.3 and 2.2, we obtain

$$\big\|B(u,b)\big\|_Y=\sup\nolimits_{t\in(0,T)}\left\|\int_0^t\nabla e^{(t-\tau)\Delta}\mathbf{P}(u\otimes b)\,\mathrm{d}\tau\right\|_{M^s_{q,\sigma}}\leqslant C\sup\nolimits_{t\in(0,T)}\left\|\int_0^t e^{(t-\tau)\Delta}(u\otimes b)\,\mathrm{d}\tau\right\|_{M^{s+1}_{q,\sigma}}$$
$$\leqslant C\|u\otimes b\|_{l^{s+1,\sigma}L^1_TL^q}\leqslant C\|u\|_{l^{s+1,\sigma}L^2_TL^{2q}}\|b\|_{l^{s+1,\sigma}L^2_TL^{2q}}\leqslant C\|u\|_{l^{s+1,\sigma}L^2_TL^q}\|b\|_{l^{s+1,\sigma}L^2_TL^q}\leqslant C\|u\|_Z\|b\|_Z,$$

and

$$\big\|B(u,b)\big\|_Z\leqslant C\left\|\int_0^t e^{(t-\tau)\Delta}(u\otimes b)\,\mathrm{d}\tau\right\|_{l^{s+2,\sigma}L^2_TL^q}\leqslant C\big(1+T^{\frac{1}{2}}\big)\|u\otimes b\|_{l^{s+1,\sigma}L^1_TL^q}$$
$$\leqslant C\big(1+T^{\frac{1}{2}}\big)\|u\|_{l^{s+1,\sigma}L^2_TL^{2q}}\|b\|_{l^{s+1,\sigma}L^2_TL^{2q}}\leqslant C\big(1+T^{\frac{1}{2}}\big)\|u\|_{l^{s+1,\sigma}L^2_TL^q}\|b\|_{l^{s+1,\sigma}L^2_TL^q}\leqslant C\big(1+T^{\frac{1}{2}}\big)\|u\|_Z\|b\|_Z.$$

Thus, it follows from the four estimates above that

$$\|\Phi_1(u,b)\|_{X_T} \leqslant C\big(1+T^{\frac{1}{2}}\big)\|u_0\|_{M^s_{q,\sigma}} + C\big(1+T^{\frac{1}{2}}\big)\|u\|_Z^2 + C\big(1+T^{\frac{1}{2}}\big)\|b\|_Z^2,$$
$$\|\Phi_2(u,b)\|_{X_T} \leqslant C\big(1+T^{\frac{1}{2}}\big)\|b_0\|_{M^s_{q,\sigma}} + 2C\big(1+T^{\frac{1}{2}}\big)\|u\|_Z\|b\|_Z.$$

On the other hand, we can get

$$\|\Phi_1(u_1,b_1)-\Phi_1(u_2,b_2)\|_{X_T} \leqslant C\big(1+T^{\frac{1}{2}}\big)\big\{\big(\|u_1\|_Z+\|u_2\|_Z\big)\|u_1-u_2\|_Z + \big(\|b_1\|_Z+\|b_2\|_Z\big)\|b_1-b_2\|_Z\big\},$$
$$\|\Phi_1(u_1,b_1)-\Phi_1(u_2,b_2)\|_{X_T} \leqslant C\big(1+T^{\frac{1}{2}}\big)\big\{\|b_1\|_Z\|u_1-u_2\|_Z + \|u_2\|_Z)\|b_1-b_2\|_Z\big\}.$$

Therefore, we can use the contraction argument as Step 1 and Step 2 to obtain a fixed point, and the proof of the theorem is complete. □

In order to prove Theorem 1.2, we need the following lemma.

**Lemma 3.1.** *Let $1 \leqslant q \leqslant n$, $1 \leqslant \sigma < \infty$, $\sigma < \widetilde{\nu} < \infty$ and $\max\{q,2\} < r < 2q$ satisfy*

$$\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}} < \frac{1}{n}, \qquad \frac{1}{q}-\frac{1}{n} < \frac{1}{r} < \frac{1}{2}\left(\frac{1}{q}+\frac{1}{n}\right).$$

*Then, we have*

$$\text{(i)}\quad \sup_{t\in(0,+\infty)}\int_0^t \big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\big)\times(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,\mathrm{d}\tau < \infty. \tag{3.18}$$

$$\text{(ii)}\quad \sup_{t\in(0,+\infty)}(1+t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\int_0^t \big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\big)(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2r}}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,\mathrm{d}\tau < \infty. \tag{3.19}$$

**Proof.** We only prove the inequality (3.18), and (3.19) can be proved in a similar way.

Let $f(t)=\int_0^t(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})})(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,\mathrm{d}\tau$, and we show that $\sup_{t\in(o,+\infty)} f(t) < +\infty$ since there holds $\frac{1}{\sigma}-\frac{1}{\nu}<\frac{1}{n}$, $\frac{1}{q}-\frac{1}{n}<\frac{1}{r}<\frac{1}{2}(\frac{1}{q}+\frac{1}{n})$ and $\frac{1}{2}+\frac{n}{2}(\frac{2}{r}-\frac{1}{q})+n(\frac{1}{q}-\frac{1}{r}) \geqslant 1$ if $1\leqslant q\leqslant n$.

For $t\in(0,3)$, then

$$f(t) \leqslant C\int_0^t \big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\big)(t-\tau)^{-\frac{1}{2}}\,\mathrm{d}\tau \leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}}\,\mathrm{d}\tau + C\int_0^t (t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})-\frac{1}{2}}\,\mathrm{d}\tau$$
$$\leqslant Ct^{\frac{1}{2}} + Ct^{\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})} \leqslant C < \infty.$$

For $t\in[3,+\infty)$, we divide the interval $(0,t)$ into three parts, i.e. $(0,t) := (0,1]\cup(1,t-1]\cup(t-1,t)$, then we have

$$\int_0^1 \big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\big)(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,\mathrm{d}\tau$$
$$\leqslant C\int_0^1 \big(1+2^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\big)2^{-\frac{1}{2}}\,\mathrm{d}\tau \leqslant C < \infty,$$

and

$$\int_1^{t-1} \big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\big)(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,\mathrm{d}\tau$$
$$\leqslant C\int_1^{t-1} (t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,\mathrm{d}\tau$$
$$\leqslant C\int_1^{t-1} (t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,\mathrm{d}\tau$$

$$\leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}\tau^{-n(\frac{1}{q}-\frac{1}{r})}\,d\tau$$
$$\leqslant Ct^{\frac{1}{2}-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})-n(\frac{1}{q}-\frac{1}{r})}\int_0^1 (1-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}\tau^{-n(\frac{1}{q}-\frac{1}{r})}\,d\tau$$
$$\leqslant C<\infty,$$

the last inequality holds because of $t\geqslant 3$, $\frac{1}{2}-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})-n(\frac{1}{q}-\frac{1}{r})\leqslant 0$ and $\int_0^1(1-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}\tau^{-n(\frac{1}{q}-\frac{1}{r})}\,d\tau<\infty$ for $\frac{1}{q}-\frac{1}{n}<\frac{1}{r}<\frac{1}{2}(\frac{1}{q}+\frac{1}{n})$.

For the last part, we can obtain

$$\int_{t-1}^t \big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,d\tau$$
$$\leqslant C\int_{t-1}^t \big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)(t-\tau)^{-\frac{1}{2}}\,d\tau$$
$$\leqslant C\int_{t-1}^t (t-\tau)^{-\frac{1}{2}}\,d\tau+C\int_{t-1}^t (t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})-\frac{1}{2}}\,d\tau$$
$$\leqslant C<\infty,$$

by the inequality $\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}}<\frac{1}{n}$.

Combining the estimates above, we complete the proof of (3.18). □

**Proof of Theorem 1.2.** Similar to the proof of Theorem 1.1, we first show some useful estimates. Proposition 2.4 implies

$$\|e^{t\Delta}u_0\|_{\overline{Y}}\leqslant C\|u_0\|_{M^s_{q,\sigma}}, \tag{3.20}$$
$$\begin{aligned}\|e^{t\Delta}u_0\|_{\overline{Z}}&=\sup_{t\in(0,+\infty)}(1+t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|e^{t\Delta}u_0\|_{M^s_{r,\sigma}}\\&\leqslant C\sup_{t\in(0,+\infty)}(1+t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|u_0\|_{M^s_{q,\sigma}}\leqslant C\|u_0\|_{M^s_{q,\sigma}}.\end{aligned} \tag{3.21}$$

Define a real number $\widetilde{\nu}$ with $\frac{1}{\widetilde{\nu}}-\frac{s}{n}=\frac{2}{\sigma}-1$, then $\sigma<\widetilde{\nu}<\infty$ since there holds $\max\{\frac{n(\sigma-1)}{\sigma}-1,\frac{n(\sigma-2)}{\sigma}\}<s<\frac{n(\sigma-1)}{\sigma}$. From Propositions 2.4, 2.3 and 2.2, we have

$$\begin{aligned}\|B(u,b)\|_{\overline{Y}}&=\sup_{t\in(0,+\infty)}\|B(u,b)\|_{M^s_{q,\sigma}}\\&\leqslant C\sup_{t\in(0,+\infty)}\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}\|u\otimes b\|_{M^s_{\frac{r}{2},\widetilde{\nu}}}\,d\tau\\&\leqslant C\sup_{t\in(0,+\infty)}\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}\|u\|_{M^s_{r,\sigma}}\|b\|_{M^s_{r,\sigma}}\,d\tau\\&\leqslant C\|u\|_Z\|b\|_Z\sup_{t\in(0,+\infty)}\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\widetilde{\nu}})}\big)(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{q})}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,d\tau\\&\leqslant C\|u\|_{\overline{Z}}\|b\|_{\overline{Z}},\end{aligned} \tag{3.22}$$

the last inequality holds since the conditions of Lemma 3.1 hold. Using Propositions 2.4, 2.3, 2.2 and Lemma 3.1 again, we obtain

$$\|B(u,b)\|_{\overline{Z}}=\sup_{t\in(0,+\infty)}(1+t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|B(u,b)\|_{M^s_{r,\sigma}}$$

$$\leqslant C\sup_{t\in(0,+\infty)}(1+t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde\nu})}\big)(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{r})}\|u\otimes b\|_{M^s_{\frac{r}{2},\tilde\nu}}\,d\tau$$

$$\leqslant C\sup_{t\in(0,+\infty)}(1+t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde\nu})}\big)(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{2}{r}-\frac{1}{r})}\|u\|_{M^s_{r,\sigma}}\|b\|_{M^s_{r,\sigma}}\,d\tau$$

$$\leqslant C\|u\|_Z\|b\|_Z\sup_{t\in(0,+\infty)}(1+t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\int_0^t\big(1+(t-\tau)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde\nu})}\big)$$
$$\times(t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{n}{2r}}(1+\tau)^{-n(\frac{1}{q}-\frac{1}{r})}\,d\tau$$
$$\leqslant C\|u\|_{\overline{Z}}\|b\|_{\overline{Z}}. \tag{3.23}$$

Combining the estimates (3.20)–(3.23), we obtain the following two estimates

$$\big\|\Phi_1(u,b)\big\|_X\leqslant C\|u_0\|_{M^s_{q,\sigma}}+C\|u\|^2_{\overline{Z}}+C\|b\|^2_{\overline{Z}}, \tag{3.24}$$
$$\big\|\Phi_2(u,b)\big\|_X\leqslant C\|b_0\|_{M^s_{q,\sigma}}+2C\|u\|_{\overline{Z}}\|b\|_{\overline{Z}}. \tag{3.25}$$

On the other hand, we can get

$$\big\|\Phi_1(u_1,b_1)-\Phi_1(u_2,b_2)\big\|_X\leqslant C\big\{\big(\|u_1\|_{\overline{Z}}+\|u_2\|_{\overline{Z}}\big)\|u_1-u_2\|_{\overline{Z}}+\big(\|b_1\|_Z+\|b_2\|_{\overline{Z}}\big)\|b_1-b_2\|_{\overline{Z}}\big\}, \tag{3.26}$$
$$\big\|\Phi_2(u_1,b_1)-\Phi_2(u_2,b_2)\big\|_X\leqslant 2C\big(\|b_1\|_{\overline{Z}}\|u_1-u_2\|_{\overline{Z}}+\|u_2\|_{\overline{Z}}\|b_1-b_2\|_{\overline{Z}}\big). \tag{3.27}$$

Let $K_0:=\|(u_0,b_0)\|_{M^s_{q,\sigma}}=\|u_0\|_{M^s_{q,\sigma}}+\|b_0\|_{M^s_{q,\sigma}}$. The estimates (3.24) and (3.25) imply that $\|\mathcal{T}(u,b)\|_X<CK_0+CK_0^2$. Define a complete metric space

$$E:=\big\{(u,b)\;\big|\;u,b\in\big[C\big(0,+\infty;M^s_{q,\sigma}\big(\mathbb{R}^n\big)\big)\big]^n,\ \big\|(u,b)\big\|_X\leqslant 2CK_0\big\}$$

with the metric $d(f,g)=\|f-g\|_X$. Making use of the estimates (3.24)–(3.27), we conclude that there exists a positive constant $\varepsilon_0<\frac{1}{8C}$, when $K_0<\varepsilon_0$, $\mathcal{T}(u,b)$ becomes a contract map from $E$ into itself, and the Banach's fixed point theorem can be applied. □

## Acknowledgments

The authors would like to express their great thanks to the editor and the reviewer for valuable comments and remarks, they also thank professor Iwabuchi who kindly gave us his recent papers.

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