On the blow up phenomenon of the critical nonlinear Schrödinger equation

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Abstract

In this paper we consider the blow up phenomenon of critical nonlinear Schrödinger equations in dimension 1D and 2D. We define the minimal mass as the $L^2$ norm necessary to ignite a wave collapse and we stress its role in the blow up mechanism. Asymptotic compactness properties and $L^2$-concentration are proved. The proof relies on linear and nonlinear profile decompositions.

Keywords: Time dependent Schrödinger equation; Blow up; Mass concentration

1. Introduction

We consider the $L^2$-critical nonlinear Schrödinger equation:

$$i \partial_t u + \Delta u = \kappa |u|^{4/d} u;$$

(1.1)

with initial data

$$u(0, x) = u_0(x).$$

(1.2)
Here, $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$ stands for the Laplace operator on $\mathbb{R}^d$, $u : ]-T_*, T^*[ \times \mathbb{R}^d \to \mathbb{C}$ and $u_0 \in L^2(\mathbb{R}^d)$. The parameter $\kappa$ equal to $-1$ (respectively $1$) corresponds to the focusing (respectively defocusing) NLS, respectively. It is well known (see [4], for example) that for every $u_0 \in L^2(\mathbb{R}^d)$, there exists a unique maximal solution $u$ to (1.1), (1.2), with

$$u \in C(] - T_*, T^*[, L^2(\mathbb{R}^d)) \cap L^\gamma_{\text{loc}}(]-T_*, T^*[\times \mathbb{R}^d)), \quad \gamma := \frac{2(d+2)}{d}$$

for some $T_*, T^* > 0$. Moreover, we have the following alternative: either $T_* = T^* = +\infty$ or $\min\{T_*, T^*\} < +\infty$ and

$$\int_{-T_*}^{T^*} \int_{\mathbb{R}^d} |u|^\gamma \, dx \, dt = +\infty.$$

In addition, the conservation law

$$\int_{\mathbb{R}^d} |u(t)|^2 \, dx = \int_{\mathbb{R}^d} |u_0|^2 \, dx \quad (1.3)$$

is satisfied for all $t \in ]-T_*, T^*[$.

The local theory relies heavily on some integrability properties of the solution of the associated linear Schrödinger equation

$$\begin{cases}
i \partial_t u + \Delta u = 0, \\
u(0, x) = u_0,
\end{cases} \quad (1.4)$$

called Strichartz estimates. In fact, by using Fourier analysis, in connections with the work by Tomas [18], as in [17] or an abstract operators theory as in [7], it was proved that $e^{it\Delta} u_0$, solution of (1.4), satisfies

$$\|e^{it\Delta} u_0\|_{L^\gamma(\mathbb{R}^{d+1})} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}. \quad (1.5)$$

The local solution follows from solving the equivalent integral equation

$$u = e^{it\Delta} u_0 - i \kappa \int_0^t e^{i(t-s)\Delta} |u|^{4/d} u(s) \, ds,$$

by a standard Picard iteration method. However, this iteration method cannot by itself yield a global wellposedness. In the sub-critical case ($|u|^\alpha u$, $0 < \alpha < \frac{4}{d}$, instead of $|u|^{4/d} u$) the identity (1.3) suffices to solve (1.1), (1.2) globally. This relies on the fact that the lifespan $T$ depends only on the $L^2$ norm of the data. In the critical case the situation is more subtle and the time of existence depends on certain concentration functions of the data.
Note that the Cauchy condition (1.2) can be replaced by the following asymptotic condition:

\[
\|u(t, \cdot) - e^{it\Delta} v\|_{L^2} \longrightarrow 0. \tag{1.6}
\]

The corresponding integral equation in this case is

\[
u = e^{it\Delta} u_0 - i\kappa \int_{-\infty}^{t} e^{i(t-s)\Delta} |u|^{4/d} u(s) \, ds,
\]

and the maximal solution \( u \) belongs to \( C([-\infty, T^*[\cap L^2_{\text{loc}}(-\infty, T^[, L^\gamma(\mathbb{R}^d))]) \cap L^\gamma_{\text{loc}}([-\infty, T^*[\cap L^\gamma(\mathbb{R}^d)).
\]

In the same way one can take the asymptotic condition to be hold at \( +\infty \) instead of \( -\infty \).

The small data theory asserts that there exists \( \delta > 0 \) (related to the constant \( C \) in (1.5)) such that if

\[
\|u_0\|_{L^2(\mathbb{R}^d)} < \delta, \tag{1.7}
\]

the initial values problem (1.1), (1.2) has unique global solution \( u(t, x) \), with \( u \in (C \cap L^\infty)(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^\gamma(\mathbb{R}^d+1) \). This follows by solving the Cauchy problem (1.1), (1.2) directly in the whole space (the first step of the iteration method suffices to reach \( T^* = T^* = \infty \)). Notice that the sign of \( \kappa \) does not play any role and the global existence for small data occurs in both defocusing and focusing cases. However, for a large data and in the focusing case \( (\kappa = -1) \) blow up may occur. The blow up or “wave collapse” is an important phenomenon with many physical consequences. A lot of theoretical and numerical works are dedicated to this subject when the initial data belongs to \( H^1 \). In fact, in this space energy arguments apply and a blow up theory has been developed in the two last decades (see [4,9,16,21] and the references therein). This theory is mainly connected to the notion of ground state: the unique positive radial solution of the elliptic problem

\[
\Delta Q - Q + |Q|^{4/d} Q = 0.
\]

In [20], M.I. Weinstein exhibited the following refined Gagliardo–Nirenberg inequality:

\[
\|\psi\|_{L^{4/d+2}}^{4/d+2} \leq C_d \|\psi\|_{L^2}^{4/d} \|\nabla \psi\|_{L^2}^2, \quad \forall \psi \in H^1, \tag{1.8}
\]

with \( C_d = \frac{d+2}{d} \|Q\|_{L^2}^{-4/d} \). Combined with the conservation of energy, this implies that \( \|Q\|_{L^2} \) is the critical mass for the formation of singularities: for every \( u_0 \in H^1 \) such that

\[
\|u_0\|_{L^2} < \|Q\|_{L^2}
\]

the solution of (1.1) with initial data \( u_0 \) is global. Also, this bound is optimal. In fact, by using the conformal invariance, one constructs

\[
u(t, x) = (T^* - t)^{-d/2} e^{i(\Delta x^2/(T^*+t)) - (i|x|^2/T^*-t)} \left( \frac{x}{T^* - t} \right)
\]
a blowing up solution of (1.1) with \( \|u\|_{L^2} = \|Q\|_{L^2} \). F. Merle [10] has proved that, up the invariants of (1.1), this is the only blowing up solution with minimal mass. It is also proved (see [12,19]) that at the blow up there is a concentration phenomenon in \( L^2 \) norm: there exists a continuous functions \( x(t) \) such that

\[
\forall R > 0, \quad \liminf_{t \to T^*} \int_{|x-x(t)| \leq R} |u(t,x)|^2 \, dx \geq \int Q^2.
\]

For the case \( u_0 \in L^2 \) no results are known until 1998. The first result in this direction is due to J. Bourgain [2] in the case of dimension 2. In fact, by using a refined version of the Strichartz inequality (1.5) proved in [14] and harmonic analysis techniques, this author have proved that if \( u \) is blow up solution of (1.1), (1.2) at finite time \( T^* > 0 \), then

\[
\lim_{t \uparrow T^*} \left( \sup_{y \in \mathbb{R}^2} \int_{|x-y| < \sqrt{T^*-t}} |u(t,x)|^2 \, dx \right) > 0.
\]

Using the work by Bourgain, F. Merle and L. Vega [13] have proved, among other things, some asymptotic compactness properties in \( L^2(\mathbb{R}^2) \) up to the invariance of the equation. More precisely, there exists \( \delta_0 \) so that for a subsequence \( t_n \to T^* \), there are \( a_n, b_n, x_n, \rho_n \to 0 \) and \( H \neq 0 \) such that

\[
e^{ia_n x + ib_n |x|^2} \rho_n u(t_n, (x - x_n) \rho_n) \rightharpoonup H. \quad (1.10)
\]

Our aim in this paper is to give a better description of the blow up solutions of (1.1), (1.2) in one and two space dimensions. We define the minimal mass as the most little \( L^2 \)-norm needed the ignite the blow up phenomenon and we stress the role of the solutions with minimal mass in the collapse mechanism.\(^1\)

**Definition 1.1.** We define \( \delta_0 \) as the supremum of \( \delta \) in (1.7), such that the global existence for (1.1), (1.2) holds, with \( u \in (C \cap L^\infty)(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^\gamma(\mathbb{R}^{d+1}) \).\(^2\)

In the ball \( B_{\delta_0} := \{ u_0, \|u_0\|_{L^2(\mathbb{R}^2)} < \delta_0 \} \) evolution problem (1.1), (1.2) admits a complete scattering theory with respect to the associated linear problem.

**Definition 1.2.** A solution \( u \) of (1.1), (1.2) blows up for \( t > 0 \) if \( T^* < +\infty \) or \( T^* = +\infty \) and \( u \) does not disperse at infinity, i.e., \( \int_0^\infty \int_{\mathbb{R}^d} |u|^\gamma \, dx \, dt = +\infty \). Similarly for the backward problem.\(^3\)

\(^1\) We try to show that these solutions play a similar role to the one played by \( Q \) in \( H^1 \) context.

\(^2\) That means that the solution \( u \) is globally defined and disperses at infinity. The general consensus is that \( \delta_0(+) = +\infty \), and \( \delta_0(-) = \|Q\|_{L^2} \).

\(^3\) Note that the finite time blow up and the no dispersion phenomena are conjugated via the pseudo-conform transform.
In the first theorem we prove the existence of a blow up solutions with minimal mass (the supremum $\delta_0$ is not attained).

**Theorem 1.3.** Assume $d = 1$ or $2$. There exists an initial data $u_0 \in L^2(\mathbb{R}^d)$ with $\|u_0\|_{L^2} = \delta_0$, for which the solution $u$ of (1.1), (1.2) blows up for both $t > 0$ and $t < 0$.

**Remark 1.4.** In $H^1$ context it turns out that, up the invariants of (1.1), the only blow up solution with minimal mass which blows up for both $t > 0$ and $t < 0$ is the solitary wave $e^{i\tau} Q(x)$.

As a direct consequence of Theorem 1 and the pseudo-conform transform, we get the following:

**Corollary 1.5.** Assume $d = 1$ or $2$. There exists an initial data $u_0 \in L^2(\mathbb{R}^d)$ with $\|u_0\|_{L^2} = \delta_0$, for which the solution $u$ of (1.1), (1.2) blows up in finite time $T^* > 0$.

**Theorem 1.6.** Let $u$ be a blowing up solution of (1.1), (1.2) at finite time $T^* > 0$ such that $\|u_0\|_{L^2} < \sqrt{2} \delta_0$. Let $\{t_n\}_{n=1}^{\infty}$ be any time sequence such that, as $n \to \infty$,

$$t_n \to T^*.$$

Then there exists a subsequence of $\{t_n\}_{n=1}^{\infty}$ (still denoted by $\{t_n\}_{n=1}^{\infty}$), which satisfies the following properties. There exist

(i) a function $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2} \geq \delta_0$ such that the solution $U$ of (1.1), (1.2) with initial data $\psi$ blows up for both $t > 0$ and $t < 0$, and

(ii) a sequence $\{\rho_n, \xi_n, x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R}^d$ such that

$$\rho_n^{d/2} e^{i\xi_n \cdot x_n} u(t_n, \rho_n x + x_n) \rightharpoonup \psi.$$

Furthermore, we have

$$\lim_{n \to \infty} \frac{\rho_n}{\sqrt{T^* - t_n}} \leq \frac{1}{\sqrt{T^{**}}}.$$

where $T^{**}$ denotes the (forward) lifespan of $U$.

**Remark 1.7.** The assumption $\|u_0\|_{L^2} < \sqrt{2} \delta_0$ is merely technical. It guarantees the uniqueness of the blow up profile which is necessary to prevent the apparition of quadratic oscillations (see the proof for more details). For arbitrary large data, the asymptotic (1.10) remains the best available result.

**Remark 1.8.** Similar results of asymptotic and limiting profiles of blow up solutions in the $H^1$ context are proved by H. Nawa [15].
Remark 1.9. We do not know if $\psi$ or $\|\psi\|_{L^2}$ depends on the time sequence $\{t_n\}_{n=1}^{\infty}$. Recently, F. Merle and P. Raphael [11] have proved that, in the $H^1$ context, $Q$ is the universal blow up profile (for the strong $H^1$ convergence) for the near-critical mass solutions.

As a direct consequence, we get for the singular solutions with minimal mass the following proposition.

Corollary 1.10. Assume $d = 1$ or 2. Let $u$ be a blow up solution with minimal mass of (1.1), (1.2) at time $T^* > 0$. Let $\{t_n\}_{n=1}^{\infty}$ be any time sequence such that, as $n \to \infty$,

$$t_n \uparrow T^*.$$ 

Then there exists a subsequence of $\{t_n\}_{n=1}^{\infty}$ (still denoted by $\{t_n\}_{n=1}^{\infty}$), which satisfies the following properties. There exist

(i) a function $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = \delta_0$ such that the solution $U$ of (1.1), (1.2) with initial data $\psi$ blows up for both $t > 0$ and $t < 0$, and

(ii) a sequence $\{\rho_n, \xi_n, x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R}^d$ such that

$$\rho_n^{d/2} e^{i\xi_n \cdot x_n} u(t_n, \rho_n x + x_n) \to \psi$$

strongly in $L^2$.

Furthermore, we have

$$\lim_{n \to \infty} \frac{\rho_n}{\sqrt{T^* - t_n}} \leq \frac{1}{\sqrt{T^{**}}},$$

where $T^{**}$ denotes the (forward) lifespan of $U$.

Remark 1.11. We do not know if $\psi$ depends on the time sequence $\{t_n\}_{n=1}^{\infty}$. It is expected that, up to a multiplication by $e^{i\theta}$, $\psi = Q$. The fact that the solution $U$ associated to $\psi$ blows up for both $t > 0$ and $t < 0$ corroborates this expectation (remember Remark 1.4).

The next theorem shows that, for every data with in the ring $C = \{u_0 \in L^2, \delta_0 \leq \|u_0\|_{L^2} < \sqrt{2}\delta_0\}$, there is a concentration phenomenon in $L^2$ which occurs at the blow up time, with minimal amount $\delta_0$. More precisely, we have

Theorem 1.12. Assume $d = 1$ or 2. Let $u$ be a blowing up solution of (1.1), (1.2) at finite time $T^* > 0$ such that $\|u_0\|_{L^2} < \sqrt{2}\delta_0$. Let $\lambda(t) > 0$, such that $\frac{\sqrt{T^* - t}}{\lambda(t)} \to 0$ as $t \uparrow T^*$. There exists $x(t) \in \mathbb{R}^d$, such that

$$\liminf_{t \uparrow T^*} \int_{|x - x(t)| \leq \lambda(t)} |u(t, x)|^2 \, dx \geq \delta_0^2,$$
Remark 1.13. The problem of concentration of a minimal amount of mass at the blow up time is still open for arbitrary large.

Finally, as a classical application of the profile decomposition, there exists an a priori estimate for the $L^\gamma$ norm of every solution of (1.1), (1.2) having initial data in $B_{\delta_0}$.

Theorem 1.14. Assume $d = 1$ or 2. There exists a nondecreasing function $F : [0, \delta_0[ \to [0, \infty[$, such that for every solution $u$ of (1.1), (1.2), with $\|u_0\|_{L^2} < \delta_0$, we have

$$\|u\|_{L^\gamma(\mathbb{R}_+^{d+1})} \leq F(\|u_0\|_{L^2}).$$

(1.11)

Remark 1.15. This result, which is obvious in the case of small initial data, tells that once we have global existence in a ball of $L^2$ then there exist a priori estimates of Strichartz norm of the solutions in terms of the $L^2$ norm of the initial data. A precise description of $A$ in (1.11) remains an open problem. We know that

$$A(t) \sim t$$

for $t$ small.

The rest of the paper is structured as follows. In Section 2 we prove a result of nonlinear profile decomposition needed for the proofs of our results which are given in Section 3.

2. Profile decomposition

We start with the following definitions.

Definition 2.1. (i) For every sequence $\Gamma_n = \{\rho_n, t_n, \xi_n, x_n\}_{n=1}^\infty \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ we define the isometric operator $\Gamma_n$ on $L^\gamma(\mathbb{R}_+^{d+1})$ by

$$\Gamma_n(f)(t, x) = \rho_n^{d/2} e^{i x \cdot \xi_n} e^{-i t \xi_n^2} f(\rho_n^2 t_n + t_n, \rho_n(x - t_n \xi_n) + x_n).$$

(ii) Two sequences $\Gamma_n^j = \{\rho_n^j, t_n^j, \xi_n^j, x_n^j\}_{n=1}^\infty$ and $\Gamma_n^k = \{\rho_n^k, t_n^k, \xi_n^k, x_n^k\}_{n=1}^\infty$ are said to be orthogonal if

$$\begin{align*}
\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} &\to +\infty \quad \text{or} \quad \rho_n^j = \rho_n^k \quad \text{and} \\
\frac{|\xi_n^j - \xi_n^k|}{\rho_n^j} + |t_n^j - t_n^k| &+ \left| \frac{\xi_n^j - \xi_n^k}{\rho_n^j} t_n^j + x_n^j - x_n^k \right| &\to +\infty.
\end{align*}$$
Remark 2.2. (i) The set of solutions to (1.1) is invariant under the action of $\Gamma_n$.
(ii) If two sequences $\Gamma^j_n = \{\rho^j_n, t^j_n, \xi^j_n, x^j_n\}_{n=1}^\infty$ and $\Gamma^k_n = \{\rho^k_n, t^k_n, \xi^k_n, x^k_n\}_{n=1}^\infty$ are orthogonal then

\[
(\Gamma^j_n)^{-1} \Gamma^k_n f \rightharpoonup 0,
\]
for every $f \in L^\gamma(\mathbb{R}^{d+1})$.

The following theorem is a restatement of the parts of [13] and [3] which are relevant for us.

Theorem 2.3. Assume $d = 1$ or 2. Let $\{v_n\}_{n=1}^\infty$ be a bounded family of $L^2(\mathbb{R}^d)$. Then there exists a subsequence of $\{v_n\}_{n=1}^\infty$ (still denoted by $\{v_n\}_{n=1}^\infty$), which satisfies the following properties. There exist

(i) a family $\{V^j\}_{j=1}^\infty$ of solutions to (1.4),
(ii) a family of pairwise orthogonal sequences $\Gamma^j_n = \{\rho^j_n, t^j_n, \xi^j_n, x^j_n\}_{n=1}^\infty \subset \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ such that, for every $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, we have

\[
e^{it\Delta} v_n(x) = \sum_{j=1}^\ell \Gamma^j_n V^j(t, x) + w^\ell_n(t, x),
\]

with

\[
\lim_{n \to \infty} \|w^\ell_n\|_{L^\gamma(\mathbb{R}^{d+1})} \infty \to \infty 0.
\]

Furthermore, we have

\[
\|v_n\|_{L^2(\mathbb{R}^d)}^2 = \sum_{j=1}^\ell \|V^j\|_{L^2(\mathbb{R}^d)}^2 + \|w^\ell_n\|_{L^2(\mathbb{R}^d)}^2 + o(1), \quad n \to \infty,
\]

for every $\ell \geq 1$.

Remark 2.4. Up to change the profiles one can take $\lim_{n \to \infty} t^j_n \in \{-\infty, 0, +\infty\}$.

Before stating the nonlinear equivalent of this theorem, we need to introduce the following

Definition 2.5. Let $V$ be a solution of the linear equation (1.4) and $\Gamma_n = \{\rho_n, t_n, \xi_n, x_n\}_{n=1}^\infty$ a sequence of $\mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ such that the quantity $\{t_n\}_{n=1}^\infty$ has a limit in $[-\infty, +\infty]$
when $n$ goes to the infinity. We define the nonlinear profile $U$ associated to $\{V, \Gamma_n\}_{n=1}^{\infty}$ as the unique maximal solution\(^4\)

$$
\| (U - V)(t_n, \cdot) \|_{L^2(\mathbb{R}^d)} \to 0. \tag{2.4}
$$

The following theorem results from Theorem 2.3 and a perturbation analysis.

**Theorem 2.6.** Assume $d = 1$ or 2. Let $\{u_{0,n}\}_{n=1}^{\infty}$ be a bounded family of $L^2(\mathbb{R}^d)$ and $\{u_n\}_{n=1}^{\infty}$ the corresponding family of solutions to (1.1), (1.2). Let $\{V^i, \Gamma_{n}^{i}\}_{i=1}^{\infty}$ be the family of linear profiles associated to $\{\phi_n\}_{n=1}^{\infty}$ via Theorem 2.3 and $\{U^i\}_{i=1}^{\infty}$ the family of nonlinear profiles associated to $\{V^i, \Gamma_{n}^{i}\}_{i=1}^{\infty}$ via Definition 2.5.

Let $\{I_n\}_{n=1}^{\infty}$ to be a family of intervals containing the origin 0. Then the following statements are equivalent:

(i) For every $j \geq 1$, we have

$$
\lim_{n \to \infty} \| \Gamma_{n}^{j} U^j \|_{L^\gamma(I_n \times \mathbb{R}^d)} < \infty, \tag{2.5}
$$

(ii) $\lim_{n \to \infty} \| u_n \|_{L^\gamma(I_n \times \mathbb{R}^d)} < \infty. \tag{2.6}$

Moreover, if (i) or (ii) holds, then

$$
u_n = \sum_{j=1}^{\ell} \Gamma_{n}^{j} U^j + w_{n}^{\ell} + r_{n}^{\ell},
$$

where $w_{n}^{\ell}$ is as in (2.1) and

$$
\lim_{n \to \infty} \left( \| r_{n}^{\ell} \|_{L^\gamma(I_n \times \mathbb{R}^d)} + \sup_{t \in I_n} \| r_{n}^{\ell} \|_{L^2(\mathbb{R}^d)} \right) \to 0, \quad \ell \to \infty.
$$

**Remark 2.7.** The first implication shows that the length of the interval of existence of $u_n$ is bounded from below by the smallest of the length of the interval of existence of each profile. This is a direct effect of the pairwise orthogonality of the family $\{\rho_n^{j}, t_n^{j}, \xi_n^{j}, x_n^{j}\}$; the sum of the linear profiles is decoupled when $n$ goes to infinity and there is no interaction of the profiles inducing a smaller interval of existence than that associated to every profile. The second implication proves that there is no interaction between the profiles which generates a solution for a larger interval of existence than one of the profile separately.

\(^4\) The nonlinear profile $U$ is obtained by solving (1.1) with $U(t_0, x) = V(t_0, x)$, where $t_0 = \lim t_n$. $V$ is a Cauchy data if $t_0$ is finite and an asymptotic state (solution of (1.1), (1.6)), otherwise.
Remark 2.8. In view of (2.3) there exists $N \in \mathbb{N}$, such that, for every $j > N$, $\|V^j\|_{L^2} < \delta_0$. Thus, $U^j$ is global, for every $j > N$, and (2.5) is satisfied for those profiles. We are then concerned with a finite number of profiles $\{U^j, \ 1 \leq j \leq N\}$ only. Note also that if we put

$$I^j_n = (\rho_n^j)^2 I_n + t_n^j$$

then (2.5) means that $I^j_n \to I^j$ such that $U^j$ is well defined in $\overline{I^j}$. If $I_n \subset \mathbb{R}^+$ (respectively $I_n \subset \mathbb{R}^-$) and, for some $j$, $t_n^j \to +\infty$ (respectively $t_n^j \to -\infty$) then $I^j = \emptyset$, and (2.5) is satisfied for this index $j$.

Remark 2.9. A similar results had already been proved for wave equations By H. Bahouri and P. Gérard [1] and for $H^1$-critical Schrödinger equations by the author [8] (see also [5,6]).

2.1. Proof of Theorem 2.6

Let us first introduce some notations. In the sequel $A \preceq B$ denotes an estimate of the form $A \leq KB$ for some constant $K$. For every $I \subset \mathbb{R}$ $L^p(I \times \mathbb{R}^d)$ will be shortened to $L^p[I]$ and $\| \cdot \|$ stands for the norm:

$$\|f\|_I = \|f\|_{L^p[I]} + \sup_{t \in I}\|f(t, \cdot)\|_{L^2(\mathbb{R}^d)}.$$

Also we denote

$$F(z) = \kappa |z|^{4/d} z, \quad U^j_n = \Gamma^j_n U^j, \quad V^j_n = \Gamma^j_n V^j.$$

The rest of the proof proceeds in two steps.

**Step 1.** In this step we prove the first implication of the theorem. We set

$$r^\ell_n = u_n - \sum_{j=1}^{\ell} U^j_n - w^\ell_n,$$

where $w^\ell_n$ is as in (2.1). Function $r^\ell_n$ satisfies the following equation

$$\begin{cases}
   i \partial_t r^\ell_n + \Delta r^\ell_n = f^\ell_n, \\
   r^\ell_n(0, x) = \sum_{j=1}^{\ell} (V^j_n - U^j_n)(0, x),
\end{cases}$$

where

$$f^\ell_n := F \left( \sum_{j=1}^{\ell} U^j_n + w^\ell_n \right) - \sum_{j=1}^{\ell} F(U^j_n).$$
We shall prove
\[
\lim_{n \to \infty} \| r_n^\ell \|_{[I_n]} \to 0.
\] (2.7)

Once proved, (2.7) yields
\[
\lim_{n \to \infty} \left\| u_n \right\|_{L^\gamma[I_n]} \leq \sum_{j=1}^{\ell_0} \lim_{n \to \infty} \left\| U_n^j \right\|_{L^\gamma[I_n]} + 1
\]
for some \( \ell_0 \). According to the assumption (2.5), the right-hand side term is bounded, and (2.6) is then proved.

Let us then prove (2.7). By the inhomogeneous Strichartz estimates (see [4]), we infer
\[
\left\| r_n^\ell \right\|_{[I]} \lesssim \left\| r_n^\ell(a, \cdot) \right\|_{L^2} + \left\| f_n^\ell \right\|_{L^{\tilde{\gamma}}[I]},
\] (2.8)
for every interval \( I = [a, b] \subset I_n \). Here \( \tilde{\gamma} = \frac{2(d+2)}{d+4} \) denotes the conjugate of \( \gamma \). However, by triangle and Hölder’s inequalities, we can estimate
\[
\left\| f_n^\ell \right\|_{L^{\tilde{\gamma}}[I]} \lesssim \beta_n^\ell \sum_{\alpha=1}^{\gamma-1} \left\| \sum_{j=1}^\ell U_n^j + w_n^\ell \right\|_{L^\gamma[I]} \left\| r_n^\ell \right\|_{L^{\tilde{\gamma}}[I]},
\] (2.9)
where
\[
\beta_n^\ell := \left\| F\left( \sum_{j=1}^\ell U_n^j + w_n^\ell \right) - \sum_{j=1}^\ell F(U_n^j) \right\|_{L^{\tilde{\gamma}}[I_n]}.
\]

In this stage we need the following.

**Proposition 2.10.** Under the notations above, we have
\[
\lim_{\ell \to \infty} \left( \lim_{n \to \infty} \left\| \sum_{j=1}^\ell U_n^j + w_n^\ell \right\|_{L^\gamma[I_n]} \right) < +\infty,
\] (2.10)
and
\[
\lim_{n \to \infty} \beta_n^\ell \to 0.
\] (2.11)

**Proof.** Observe first that, in view of (2.2), if we prove
\[
\lim_{\ell \to \infty} \left( \lim_{n \to \infty} \left\| \sum_{j=1}^\ell U_n^j \right\|_{L^\gamma[I_n]} \right) < +\infty,
\] (2.12)
then (2.10) follows.
It is easy to check\(^5\) that the pairwise orthogonality of \(\{\Gamma_j^i\}_{j=1}^\infty\) yields

\[
\lim_{n \to \infty} \left\| \sum_{j=1}^\ell U_n^j \right\|_{L^\gamma[I_n]} = \lim_{n \to \infty} \sum_{j=1}^\ell \left\| U_n^j \right\|_{L^\gamma[I_n]},
\]

(2.13)

for every \(\ell \geq 1\). However, thanks to (2.3), the series \(\sum \| V_j \|_{L^2(\mathbb{R}^d)}^2\) converge. Thus, for every \(\epsilon > 0\), there exists \(\ell(\epsilon)\) such that

\[
\| V_j \|_{L^2(\mathbb{R}^d)} \leq \epsilon, \quad \forall j > \ell(\epsilon).
\]

The theory of small data asserts that, for \(\epsilon\) sufficiently small, \(U_j^i\) is global and

\[
\| U_j^i \|_{L^\gamma[\mathbb{R}]} \lesssim \| V_j \|_{L^2(\mathbb{R}^d)},
\]

which yields (since \(\gamma > 2\))

\[
\sum_{j > \ell(\epsilon)} \| U_j^i \|_{L^\gamma[\mathbb{R}]} < \infty.
\]

(2.14)

So we have to deal only with a finite number of nonlinear profiles \(\{U_j^i\}_{1 \leq j \leq \ell(\epsilon)}\). But, in view of the pairwise orthogonality of \(\{\Gamma_j^i\}_{j=1}^\infty\) and the assumption (2.5), one has

\[
\lim_{n \to \infty} \left\| \sum_{j=1}^{\ell(\epsilon)} U_n^j \right\|_{L^\gamma[I_n]} \leq \sum_{j=1}^{\ell(\epsilon)} \lim_{n \to \infty} \| U_n^j \|_{L^\gamma[I_n]} < \infty
\]

and then (2.12) follows.

The proof of (2.11) is similar to the (2.10). It uses the pairwise orthogonality of \(\{\Gamma_j^i\}_{j=1}^\infty\) and the smallness of \(\| w_n^\epsilon \|_{L^\gamma[\mathbb{R}]}\). This closes the proof of Proposition 2.10.

Coming back to the proof of (2.7) and according to (2.8)–(2.10), we can estimate

\[
\| r_n^\epsilon \|_{[I]} \lesssim \| r_n^\epsilon(\alpha, \cdot) \|_{L^2} + \beta_n^\epsilon + \| w_n^\epsilon \|_{L^\gamma[I_n]}^{\gamma-2} \| r_n^\epsilon \|_{L^\gamma[I]} + \sum_{\alpha=2}^{\gamma-1} \| r_n^\epsilon \|_{L^\gamma[I]}^\alpha
\]

\[
+ \sum_{j=1}^\ell \| U_n^j \|_{L^\gamma[I]}^{\gamma-2} \| r_n^\epsilon \|_{L^\gamma[I]} + \sum_{\alpha=2}^{\gamma-1} \| r_n^\epsilon \|_{L^\gamma[I]}^\alpha.
\]

\(^5\) See [8] for a detailed proof. The main argument is that the pairwise orthogonality leads the mixed terms to go to 0.
for every \( I = [a, b] \subset I_n \). Thanks to the smallness of \( \|w_n^{\ell}\|_{L^\gamma[I]} \) when \( n \) and \( \ell \) are large, the linear term \( \|w_n^{\ell}\|_{L^\gamma[I]}^\gamma \|r_n^{\ell}\|_{L^\gamma[I]} \) may be absorbed by the left-hand term, and we get for \( \ell \) and \( n \) large

\[
\ell \sum_{j=1}^{\ell} \|U^j_n\|_{L^\gamma[I]}^\gamma + \frac{1}{2} \sum_{\alpha=2}^{\gamma-1} \|r_n^{\ell}\|_{L^\gamma[I]}^{\alpha}.
\]

(2.15)

The next lemma shows that, under a suitable partition of \( I_n \), we can also absorb the other linear term on \( \|r_n^{\ell}\|_{L^\gamma[I]} \) in the right-hand side of (2.15).

**Lemma 2.11.** For every \( \epsilon > 0 \), there exists an integer \( p \) (which depends on \( \epsilon \) but not on \( n \) and \( \ell \)) and a partition of \( I_n \)

\[
I_n = \bigcup_{i=1}^{p} I_n^i,
\]

such that

\[
\lim_{n \to \infty} \|\sum_{j=1}^{\ell} \|U^j_n\|_{L^\gamma[I_n^j]} \| \leq \epsilon,
\]

(2.16)

for every \( 1 \leq i \leq p \) and every \( \ell \geq 1 \).

**Proof.** From (2.13) and (2.14), it follows that there exists \( \ell_1 = \ell_1(\epsilon) \) sufficiently large such that

\[
\lim_{n \to \infty} \|\sum_{j > \ell_1} \|U^j_n\|_{L^\gamma(\mathbb{R}^d+)} \| \leq \frac{\epsilon}{2}.
\]

Thus, it suffices to construct a family of partial partitions as in (2.16), for every \( 1 \leq j \leq \ell_1 \) such that

\[
\lim_{n \to \infty} \|U^j_n\|_{L^\gamma[I_n^j]} \| \leq \frac{\epsilon}{2\ell_1},
\]

(2.17)

for every \( 0 \leq i \leq p_j \). The final partition will be obtained by intersecting all the partial ones.

Let us discuss the case \( j = 1 \). We denote by \( I^1 \) the maximal interval of existence of \( U^1 \). Since,

\[
\|U^1_n\|_{L^\gamma[I_n^1]} = \|U^1\|_{L^\gamma[(\rho_n^1)^2I_n^1+I_n^1]},
\]

then the assumption (2.6) implies that there exists a closed interval \( \tilde{I}^1 \subset I^1 \) such that

\[
\|U^1\|_{L^\gamma[\tilde{I}^1]} < \infty.
\]
and \((\rho_n^1)^2 I_n + t_n^1 \subset \tilde{I}^1\), for \(n\) large. We decompose \(\tilde{I}^1\) as
\[
\tilde{I}^1 = \bigcup_{i=1}^{p_1} \tilde{I}_i^1
\]
so that
\[
\| U^1 \|_{L^\gamma[\tilde{I}_i^1]} < \frac{\epsilon}{2\ell_1}, \quad \forall \ 0 \leq i \leq p_1.
\]
This yields,
\[
\| U_n^1 \|_{L^\gamma[\tilde{I}_n^1]} < \frac{\epsilon}{2\ell_1}, \quad \forall \ 0 \leq i \leq p_1,
\]
where
\[
\tilde{I}_{n,i}^1 := \frac{\tilde{I}_i^1 - t_n^1}{(\rho_n^1)^2}.
\]
The family of intervals
\[
I_i^1 = I_n \cap \tilde{I}_{n,i}^1
\]
fulfills the condition (2.18) for \(j = 1\). In the same way we construct a partial partition, for every \(2 \leq j \leq \ell_1\). The final partition, which is obtained by intersecting all the partial ones, is finite independently of \(n\) and \(l\). This concludes the proof of Lemma 2.11.

Let us now achieve the proof of Theorem 2.6. Up to consider separately the backward and forward problem, we may write the partition (2.16) as
\[
I_n = \left[0, b_n^1\right] \cup \left[b_n^1, b_n^2\right] \cup \cdots \cup \left[b_n^{p-1}, b_n^p\right].
\]
Applying (2.15) on \(I_n^1\), it follows that
\[
\left\| r_n^\varepsilon \right\|_{L^\gamma[I_n^1]} \lesssim \left\| r_n^\varepsilon (0, \cdot) \right\|_{L^2} + \epsilon^{\gamma-2} \left\| r_n^\varepsilon \right\|_{L^\gamma[I_n^1]} + \beta_n^\varepsilon + \sum_{\alpha=2}^{\gamma-1} \left\| r_n^\varepsilon \right\|_{L^\gamma[I_n^1]}^\alpha.
\]
By choosing \(\epsilon\) sufficiently small, we obtain
\[
\left\| r_n^\varepsilon \right\|_{L^\gamma[I_n^1]} \lesssim \left\| r_n^\varepsilon (0, \cdot) \right\|_{L^2} + \beta_n^\varepsilon + \sum_{\alpha=2}^{\gamma-1} \left\| r_n^\varepsilon \right\|_{L^\gamma[I_n^1]}^\alpha.
\]
Observe that, by definition of the nonlinear profile \( U_n^j \), we have
\[
\lim_{n \to \infty} \| r_n^\ell(0, \cdot) \|_{L^2} = 0
\]
for every \( \ell \geq 1 \). This fact and a standard bootstrap argument show easily that
\[
\lim_{n \to \infty} \| r_n^\ell \|_{\{ I_n^1 \}} \to 0.
\]
This gives, in particular,
\[
\lim_{n \to \infty} \| r_n^\ell(b_n^1, \cdot) \|_{L^2} \to 0;
\]
and allows us to repeat the same argument on \( I_n^2 \). We iterate the same process for every \( 1 \leq i \leq p \). Since \( I_n = I_n^1 \cup \cdots \cup I_n^p \) and \( p \) is finite independently of \( n \) and \( l \), we get
\[
\lim_{n \to \infty} \| r_n^\ell \|_{\{ I_n \}} \to 0,
\]
which is (2.7). Remark finally that (2.2), (2.7) and the pairwise orthogonality of the family \( \{ \Gamma_n^j \}_{j=1}^\infty \) give
\[
\lim_{n \to \infty} \left\| u_n \right\|_{L^\gamma[I_n]} = \sum_{j=1}^\infty \lim_{n \to \infty} \left\| U_n^j \right\|_{L^\gamma[I_n]}.
\]
(2.20)

**Step 2.** In this part we shall prove the second implication of Theorem 2.6.

Let \( \{ I_n \}_{n=1}^\infty \) be a family of intervals containing 0 such that (2.6) holds. If (2.5) fails then, for every \( M > 0 \), there exists a family \( \tilde{I}_n \subset I_n \) containing 0 such that
\[
\lim_{n \to \infty} \left\| U_n^j \right\|_{L^\gamma[\tilde{I}_n]} < \infty,
\]
for every \( j \geq 1 \), and
\[
\sum_{j=1}^\infty \lim_{n \to \infty} \left\| U_n^j \right\|_{L^\gamma[\tilde{I}_n]} > M.
\]
The sequence of intervals \( \tilde{I}_n \) satisfies the statement (i) of Theorem 2.6. This gives, in particular (remember (2.20)),
\[
\lim_{n \to \infty} \left\| u_n \right\|_{L^\gamma[\tilde{I}_n]} = \sum_{j=1}^\infty \lim_{n \to \infty} \left\| U_n^j \right\|_{L^\gamma[\tilde{I}_n]} > M.
\]
This leads to
\[
\lim_{n \to \infty} \left\| u_n \right\|_{L^\gamma[I_n]} \geq \lim_{n \to \infty} \left\| u_n \right\|_{L^\gamma[\tilde{I}_n]} > M.
\]
for every \( M > 0 \), which means that
\[
\lim_{n \to \infty} \| u_n \|_{L^\gamma[I_n]} = +\infty.
\]
This contradicts (2.6). Thus (2.5) holds and the proof of Theorem 2.6 is complete.

3. Proof of the main results

3.1. Proof of Theorem 1.3

In this paragraph we shall give a partial proof of Theorem 1.3. More precisely, for the moment we prove only that there exists an initial data \( u_0 \in L^2(\mathbb{R}^d) \) with \( \| u_0 \|_{L^2} = \delta_0 \) such that the corresponding solution of (1.1), (1.2) blows up for \( t > 0 \) or \( t < 0 \). In the proof of Theorem 1.6, we shall show that there exists an initial data \( u_0 \in L^2(\mathbb{R}^d) \) with \( \| u_0 \|_{L^2} = \delta_0 \), for which the solution \( u \) of (1.1), (1.2) blows up for both \( t > 0 \) and \( t < 0 \).

From the definition of \( \delta_0 \) it follows that there exists a family of initial data \( \{ u_{0,n} \}_{n=1}^\infty \) in \( L^2(\mathbb{R}^d) \) such that
\[
\| u_{0,n} \|_{L^2(\mathbb{R}^d)} \to_{n \to \infty} \delta_0
\]
and the family sequence of \( \{ u_n \}_{n=1}^\infty \) of corresponding solutions to (1.1), (1.2) are not global. By time translation and scaling, we may assume that \( \{ u_{0,n} \}_{n=1}^\infty \) is well defined on \( [0, 1] \), and
\[
\| u_n \|_{L^\gamma([0,1] \times \mathbb{R}^d)} \to_{n \to \infty} +\infty.
\]

Let \( \{ U^j, V^j, \rho^j_n, s^j_n, \xi^j_n, x^j_n \} \) be the family of linear and nonlinear profiles associated to \( \{ u_n \}_{n=1}^\infty \) via Theorems 2.3 and 2.6. We claim that at least one of \( U^{j_0} \) is a blowing up solution. Otherwise, the equivalence in Theorem 2.6 implies that
\[
\lim_{n \to \infty} \| u_n \|_{L^\gamma([0,1] \times \mathbb{R}^d)} < \infty,
\]
which contradicts (3.1). On the one hand, by definition of \( B_{\delta_0} \), it ensures that
\[
\| V^{j_0} \|_{L^2(\mathbb{R}^d)} \geq \delta_0.
\]
On the other hand, (2.3) implies that
\[
\sum_{j \geq 0} \| V^{j_0} \|_{L^2(\mathbb{R}^d)}^2 \leq \lim_{n \to \infty} \| u_{0,n} \|_{L^2(\mathbb{R}^d)}^2 = \delta_0^2.
\]
This yields
\[
\| U^{j_0} \|_{L^2(\mathbb{R}^d)} = \| V^{j_0} \|_{L^2(\mathbb{R}^d)} \leq \delta_0.
\]
Thus, we infer
\[ \| U^{j_0} \|_{L^2(\mathbb{R}^d)} = \delta_0. \]

Recall that \( U^{j_0} \) is solution of (1.1) satisfying \( U(s^{j_0}, x) = V(s^{j_0}, x) \), where \( s^{j_0} = \lim s_{n}^{j_0} \). If \( s^{j_0} = 0 \) then \( U^{j_0} \) is a blow up solution of (1.1), (1.2) with minimal mass. If \( s^{j_0} = \infty \) then we take \( U^{j_0}(t_0 + \cdot, \cdot) \) where \( t_0 \) is a finite time in which \( U^{j_0} \) is defined.

Finally, if \( U^{j_0} \) blows up at infinity we use the pseudo-conformal transformation:
\[ \tilde{U}^{j_0}(t, x) = (T^* - t)^{-d/2} e^{-i|x|^2/(T^* - t)} \tilde{U}^{j_0} \left( \frac{1}{T^* - t}, \frac{x}{T^* - t} \right), \]
to get a solution with minimal mass which blows up at finite time \( T^* \).

### 3.2. Proof of Theorem 1.6

Take \( u \) to be a solution of (1.1), (1.2) which blows up at finite time \( T^* > 0 \) and \( \{t_n\}_{n=1}^{\infty} \) to be a time sequence going to \( T^* \) as \( n \to \infty \). We set
\[ u_n(t, x) = u(t_n + t, x). \]
\( \{u_n\}_{n=1}^{\infty} \) is a family solutions of (1.1), (1.2) on \( I_n = [-t_n, T^* - t_n] \) satisfying
\[ \int_{\mathbb{R}^d} |u_n|^2 \, dx = \int_{\mathbb{R}^d} |u_0|^2 \, dx, \quad \forall n \in \mathbb{N}. \]

In the last line we have used the conservation of \( L^2 \)-norm for Eq. (1.1). Also, the blow up of \( u \) at time \( T^* \) implies that
\[ \lim_{n \to \infty} \| u_n \|_{L^\gamma(I_n \times \mathbb{R}^d)} = +\infty. \]

By applying Theorem 2.3 to the sequence \( \{u_0(0, \cdot)\} \), we obtain, for some subsequence of \( \{u_n\}_{n=1}^{\infty} \), a family of linear profiles \( \{V^j, \Gamma^j\}_{j=1}^{\infty} \) such that (2.1)–(2.3) hold. Also, applied to the sequence \( I_n = [0, T^* - t_n] \), the second implication in Theorem 2.6 implies that there exists some \( j_0 \) such that the nonlinear profile \( \{U^{j_0}, \rho_n^{j_0}, s_n^{j_0}, \xi_n^{j_0}, x_n^{j_0}\} \) satisfies
\[ \lim_{n \to \infty} \| U^{j_0} \|_{L^\gamma(I_n^{j_0} \times \mathbb{R}^d)} = +\infty, \quad (3.2) \]
where
\[ I_n^{j_0} := [s_n^{j_0}, (\rho_n^{j_0})^2(T^* - t_n) + s_n^{j_0}]. \]

We set
\[ s^{j_0} = \lim_{n \to \infty} s_n^{j_0}. \]
It is easy to see that (3.2) implies that \( s^{j_0} \neq +\infty \) (otherwise \( I_{j_0} \to \emptyset \) and (3.2) is impossible). Thus, one of the two cases holds: either \( s^{j_0} = -\infty \) either \( s^{j_0} = 0 \) (remember Remark 2.4). In the latter case \( U^{j_0} \) is solution of (1.1), (1.2) with initial data \( V^{j_0}(0, \cdot) \); and (3.2) means that \( U^{j_0} \) blows up at time \( T^{j_0} \in ]0, +\infty[ \) and

\[
\lim_{n \to \infty} (\rho_{j_0} n)^2 (T^* - t_n) \geq T^{j_0} \tag{3.3}
\]

If we assume also that \( \|u_0\|_{L^2} < \sqrt{2}\delta_0 \) then, thanks to (2.3), there is at most one linear profile with \( L^2 \)-norm greater than \( \delta_0 \). That means that the profile \( U^{j_0} \) founded above is the only blowing up nonlinear profile (since all the other profiles have \( L^2 \)-norm lesser than \( \delta_0 \) and then they are global). By repeating the same argument in \( I_{j_0} = [-t_0, 0] \), we get

\[
\lim_{n \to \infty} \|U^{j_0}\|_{L^\gamma(I_{j_0} \times \mathbb{R}^d)} = +\infty, \quad \tilde{I}_{j_0} = \left[ -(\rho_{j_0} n)^2 (t_0) + s^{j_0} n, s^{j_0} n \right]. \tag{3.4}
\]

This implies, in particular, that \( \lim s^{j_0}_n := s^{j_0} \neq -\infty \). Since, as proved earlier,\(^6\) \( s^{j_0} \neq +\infty \), then \( s^{j_0} = 0 \) and the solution \( U^j \) of (1.1) with initial data \( V^{j_0}(0, \cdot) \) blows up also for \( t < 0 \). Thus, the nonlinear profile \( U^{j_0} \) is solution of (1.1) which blows up for both \( t < 0 \) and \( t > 0 \).\(^7\)

The linear decomposition yields

\[
(\Gamma_n^{j_0})^{-1}(e^{it\Delta}u(t_0, \cdot)) = V^{j_0} + \sum_{1 \leq j \leq l, j \neq j_0} (\Gamma_n^{j_0})^{-1}(\Gamma_n^j V^j) + +(\Gamma_n^{j_0})^{-1}w_n^j.
\]

The pairwise orthogonality of the family \( \{\Gamma_n^j\}_{j=1}^\infty \) implies

\[
(\Gamma_n^{j_0})^{-1}\Gamma_n^j V^j \rightharpoonup 0 \quad \text{weakly}
\]

for every \( j \neq j_0 \). Then

\[
(\Gamma_n^{j_0})^{-1}(e^{it\Delta}u(t_0, \cdot)) \rightharpoonup V^{j_0} + \tilde{w}^j,
\]

where \( \tilde{w}^j \) denote the weak limit of \( \{(\Gamma_n^{j_0})^{-1}w_n^j\}_{n \geq 0} \). However, we have

\[
\|\tilde{w}^j\|_{L^\gamma(\mathbb{R}^{d+1})} \leq \lim_{n \to \infty} \|\tilde{w}_n^j\|_{L^\gamma(\mathbb{R}^{d+1})} \to 0.
\]

\(^6\) Note that without the technical assumption \( \|u_0\|_{L^2} < \sqrt{2}\delta_0 \) the profiles satisfying (3.2) and (3.4) are not necessarily identical. The assumption \( \|u_0\|_{L^2} < \sqrt{2}\delta_0 \) is to guarantee, via the orthogonality relation (2.3), the uniqueness of singular profile. Without this we are not able the prove that \( \lim s^{j_0}_n \) is finite, which creates quadratic oscillations and the result cannot be better than (1.10).

\(^7\) This completes, in particular, the remainder part of the proof of Theorem 1.3. In fact, if \( \|u_0\|_{L^2} = \delta_0 \) then \( \|U^{j_0}\|_{L^2} = \delta_0 \) which means that the nonlinear profile \( U^{j_0} \) is solution of (1.1) with minimal mass which blows up for both \( t < 0 \) and \( t > 0 \).
Thereby, by uniqueness of weak limit, we get
\[ \tilde{w}^l = 0 \]
for every \( l \geq j_0 \). Hence, we obtain
\[ (\mathbf{F}_n^{j_0})^{-1}(e^{i\Delta t u(t_n, \cdot)}) \rightharpoonup V^{j_0}. \]
We need the following lemma (the proof is easy, see [13, Lemma 3.23] for two spatial dimensions).

**Lemma 3.1.** Let \( \{\varphi_n\}_{n \geq 1} \) and \( \varphi \) be in \( L^2(\mathbb{R}^d) \). The following statements are equivalent.

(i) \( \varphi_n \rightharpoonup \varphi \) weakly in \( L^2(\mathbb{R}^d) \).
(ii) \( e^{it\Delta} \varphi_n \rightharpoonup e^{it\Delta} \varphi \) weakly in \( L^r(\mathbb{R}^{d+1}) \).

Applied to \( (\mathbf{F}_n^{j_0})^{-1}(e^{i\Delta t u(t_n, \cdot)}) \), Lemma 3.1 yields
\[ e^{-is_n\Delta}(\rho_n^{d/2} e^{ix_n \xi_n} e^{i\theta_n} u(t_n, \rho_n x + x_n)) \rightharpoonup V^{j_0}(0, \cdot), \]
with
\[ s_n = s_n^{j_0}, \quad \rho_n = \frac{1}{\rho_n^{j_0}}, \quad \theta_n = \frac{x_n^{j_0} \xi_n^{j_0}}{\rho_n^{j_0}}, \quad x_n = \frac{-x_n^{j_0}}{\rho_n^{j_0}}, \quad \xi_n = \frac{-\xi_n^{j_0}}{\rho_n^{j_0}}. \]
Up to a subsequence, we can assume that \( e^{i\theta_n} \rightharpoonup e^{i\theta} \); and since \( s_n \to 0 \), we get
\[ \rho_n^{d/2} e^{ix_n \xi_n} u(t_n, \rho_n x + x_n) \rightharpoonup V, \quad \text{(3.5)} \]
where \( V = e^{-i\theta} V^{j_0}(0, \cdot) \). The associated solution is \( e^{-i\theta} U^{j_0} \). Let us finally note that estimate (3.3) gives
\[ \lim_{n \to \infty} \frac{\rho_n}{\sqrt{T^* - t_n}} \leq \frac{1}{\sqrt{T^*_{j_0}}}, \]
as claimed. This ends the proof of Theorem 1.6.

### 3.3. Proof of Corollary 1.10

If in the context of the proof of Theorem 1.3 we assume also that
\[ \|u_n\|_{L^2} = \|u(0, \cdot)\|_{L^2} = \delta_0 \]
then the identities (2.3) yields

\[ \| V^0 \|_{L^2} \leq \delta_0. \]  \hspace{1cm} (3.6)

Thus, it follows that

\[ \| V^0 \|_{L^2} = \delta_0. \]

This implies that there exists a unique profile \( V^1 \) and the weak limit in (3.5) is, in fact, strong.

### 3.4. Proof of Theorem 1.12

Let \( u \) be a solution of (1.1), (1.2), with \( \| u_0 \|_{L^2} < \sqrt{2} \delta_0 \), which blows up at finite time \( T^* > 0 \). Let \( \{t_n\}_{n=1}^\infty \) be any time sequence such that, as \( n \to \infty \),

\[ t_n \uparrow T^*. \]

According to Theorem 1.10, there exist \( V \in L^2(\mathbb{R}^d) \) with \( \| V \|_{L^2} \geq \delta_0 \) and a sequence \( \{\rho_n, \xi_n, x_n\} \subset \mathbb{R}^*_+ \times \mathbb{R}^d \times \mathbb{R}^d \) such that, up to a subsequence,

\[ (\rho_n)^{d/2} e^{i x \cdot \xi_n} u(t_n, \rho_n x + x_n) \rightharpoonup V \] \hspace{1cm} (3.7)

and

\[ \lim_{n \to \infty} \frac{\rho_n}{\sqrt{T^* - t_n}} \leq A \] \hspace{1cm} (3.8)

for some \( A \geq 0 \). From (3.7), it follows that

\[ \lim_{n \to \infty} \frac{1}{(\rho_n)^d} \int_{|x| \leq R} |u(t_n, \rho_n x + x_n)|^2 \, dx \geq \int_{|x| \leq R} |V|^2 \, dx, \]

for every \( R > 0 \). Thus,

\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq R \rho_n} |u(t_n, x)|^2 \, dx \geq \int_{|x| \leq R} |V|^2 \, dx. \] \hspace{1cm} (3.9)

Since \( \frac{\sqrt{T^* - t}}{\lambda(t)} \to 0 \) as \( t \uparrow T^* \), it follows from (3.8) that \( \frac{\rho_n}{\sqrt{\lambda(t_n)}} \to 0 \) and then

\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq \lambda(t_n)} |u(t_n, x)|^2 \, dx \geq \int_{|x| \leq R} |V|^2 \, dx \]
for every $R > 0$, which means that
\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{\{ |x-y| \leq \lambda(t_n) \}} |u(t_n, x)|^2 \, dx \geq \int_{\mathbb{R}^d} |V|^2 \, dx \geq \delta_0^2.
\]
This yields finally
\[
\liminf_{t \to T} \sup_{y \in \mathbb{R}^d} \int_{\{ |x-y| \leq \lambda(t) \}} |u(t, x)|^2 \, dx \geq \delta_0^2.
\]
Since, for every $t$, the function $y \mapsto \int_{\{ |x-y| \leq \lambda(t) \}} |u(t, x)|^2 \, dx$ is continuous and goes to 0 at infinity, then there exists a family $x(t)$ such that
\[
\sup_{y \in \mathbb{R}^d} \int_{\{ |x-y| \leq \lambda(t) \}} |u(t, x)|^2 \, dx = \int_{\{ |x-x(t)| \leq \lambda(t) \}} |u(t, x)|^2 \, dx,
\]
which concludes the proof of Theorem 1.12.

3.5. Proof of Theorem 1.14

Assume that the a priori estimate (1.11) fails, then there exists some sequence $\{u_{0,n}\}_{n=1}^\infty$ of $L^2$ with
\[
\sup_n \|u_{0,n}\|_{L^2} < \delta_0,
\]
such that the corresponding sequences $\{u_n\}_{n=1}^\infty$ of solutions of (1.1), (1.2) satisfies
\[
\lim_{n \to \infty} \|u_n\|_{L^\gamma(\mathbb{R}^{d+1})} \to \infty.
\]
From (2.3), it follows that
\[
\|V_j\|_{L^2} < \delta_0, \quad \forall j \geq 1.
\]
Thus, the nonlinear profiles $U_j$ are global and the statement (i) $\Rightarrow$ (ii) of Theorem 2.6 yields
\[
\lim_{n \to \infty} \|u_n\|_{L^\gamma} < \infty,
\]
which contradicts (3.10) and proves the existence of some function $F$ satisfying (1.11). This concludes the proof of Theorem 1.14.
References