Character Tables of Certain Association Schemes
Coming from Finite Unitary and Symplectic Groups

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In this paper we study the character tables of the association schemes obtained
from the action of (i) finite unitary groups acting on the set of nonisotropic
projective points, and (ii) finite symplectic groups acting on the set of nonisotropic
projective lines. It is shown that these character tables are controlled by the
character tables of corresponding smaller association schemes which are obtained
from the action of the group $PGL(2, q)$ on the cosets by cyclic subgroups $Z_{q+1}$ and
$Z_{q-1}$, respectively. © 1991 Academic Press, Inc.

INTRODUCTION

A commutative association scheme is a pair $\mathfrak{A} = (X, \{R_i\})$ of a
finite set $X$ and a collection of non-empty relations $R_i$ on $X$ which satisfies
the following conditions:

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(1) \( R_0 = \{(x, x) | x \in X\} \).

(2) \( R_0, R_1, \ldots, R_d \) gives a partition of \( X \times X \).

(3) For each \( i \in \{0, 1, \ldots, d\} \), we get \( R_i := \{(y, x) | (x, y) \in R_j\} = R_j \) for some \( j \in \{0, 1, \ldots, d\} \).

(4) For each fixed ordered triplet of \( i, j, h \in \{0, 1, \ldots, d\} \), the cardinality of the set

\[ \{z \in X | (x, z) \in R_i, (z, y) \in R_j\} \]

is constant (\( = p_i^h \)) whenever \( (x, y) \in R_h \).

(5) \( p_i^h = p_j^h \) for any \( i, j, h \in \{0, 1, \ldots, d\} \).

Let \( A_j (i = 0, 1, \ldots, d) \) be the adjacency matrix with respect to the relation \( R_i \), and let \( \mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle \) be the Bose–Mesner algebra of the association scheme \( \mathcal{X} \). The algebra \( \mathcal{A} \) has the unique set of primitive idempotents \( E_0, E_1, \ldots, E_d \) (which form a basis of \( \mathcal{A} \)). Let \( P = (p_{ij}) \) be the \((d + 1) \times (d + 1)\) matrix of base change defined by \( p_{0j} = p_j(i) \), where

\[ A_j = \sum_{i=0}^{d} p_j(i) E_i. \]

The matrix \( P \) is called the first eigenmatrix of the association scheme \( \mathcal{X} \) and is also called the character table of the association scheme \( \mathcal{X} \). The reader is referred to \([1, 4]\) for more details as well as an explanation of why the matrix \( P \) is considered to be a generalization (analogue) of the character table of a finite group.

Let a finite group \( G \) act on a finite set \( X \) transitively and let the permutation representation be multiplicity-free, i.e., the permutation representation is decomposed into a sum of distinct irreducible representations. Then we get a commutative association scheme. (Note that if the rank of the permutation representation, i.e., the number of orbits on \( X \) of the stabilizer of one point, is \( l + 1 \) then the class number \( d \) of the association scheme is equal to \( l \).) The Bose–Mesner algebra \( \mathcal{A} \) of this association scheme is the centralizer (or Hecke) algebra of the permutation representation. The purpose of this paper is to study the following two kinds of association schemes obtained from the action of classical groups on certain sets of nonisotropic subspaces: (i) the finite unitary groups acting on the set of nonisotropic projective points, and (ii) the finite symplectic groups acting on the set of nonisotropic projective lines.

The studies for unitary groups and symplectic groups are treated in Sections I and II, respectively, and they will be read independently. The contents of each section is described at the beginning of each section. A very interesting fact about these character tables is that they are controlled
by the character tables of smaller association schemes (playing a role somewhat similar to that of Weyl groups in Chevalley groups) which are obtained from the action of the group $\text{PGL}(2, q)$ acting on the cosets by cyclic subgroups $Z_{q+1}$ and $Z_{q-1}$, respectively.

The study given in this paper is a continuation of our previous papers (Bannai and Song [4] and Bannai, Hao, and Song [2]). In [2] we studied the character tables of association schemes obtained from various finite orthogonal groups acting on nonisotropic projective points, and we observed the following phenomena: The character tables of the full orthogonal group $\text{GO}_m^+(q)$ acting on the set of nonisotropic projective points are controlled by the character table of the group $\text{PGL}(2, q)$. (The character tables of the simple group $\text{O}_m^+(q) \equiv \text{PGL}_m^+(q)$, with $q$ odd, are controlled by the character table of the group $\text{PSL}(2, q)$.) The character tables of the group $\text{GO}_{m+1}^+(q)$ (or the simple group $\text{O}_{m+1}^+(q)$) acting on the set of positive-type and negative-type nonisotropic projective points are controlled by the character tables of the association schemes obtained from the actions of the group $\text{PGL}(2, q)$ on the cosets by dihedral subgroups $D_{2(q-1)}$ and $D_{2(q+1)}$, respectively. Thus, combining all these results, we obtain the following table.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Spaces</th>
<th>Controlling association schemes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{GO}_m^+(q)$</td>
<td>Nonisotropic points</td>
<td>$\text{PGL}(2, q) \times \text{PGL}(2, q)/\text{PGL}(2, q)$</td>
</tr>
<tr>
<td>($\text{O}_m^+(q), q$ odd)</td>
<td>(Half of nonisotropic points)</td>
<td>$\text{PSL}(2, q) \times \text{PSL}(2, q)/\text{PSL}(2, q)$</td>
</tr>
<tr>
<td>$\text{O}_{m+1}^+(q)$</td>
<td>Positive-type nonisotropic points</td>
<td>$\text{PGL}(2, q)/D_{2(q-1)}$</td>
</tr>
<tr>
<td>$\text{O}_{m+1}^-(q)$</td>
<td>Negative-type nonisotropic points</td>
<td>$\text{PGL}(2, q)/D_{2(q+1)}$</td>
</tr>
<tr>
<td>$U_m^+(q)$</td>
<td>Nonisotropic points</td>
<td>$\text{PGL}(2, q)/Z_{q+1}$</td>
</tr>
<tr>
<td>$\text{SP}_{2m}(q)$</td>
<td>Nonisotropic lines</td>
<td>$\text{PGL}(2, q)/Z_{q-1}$</td>
</tr>
</tbody>
</table>

Throughout the paper, we say that (the character tables of) a class of association schemes are controlled by (the character table of) a particular association scheme if every character table in the class is systematically determined by that of the particular association scheme either by means of simple substitution on a certain parameter, or by taking a constant scalar multiple of the entries, perhaps with a little adjustment by applying the orthogonality relations of the characters. (See Theorems 1.4, 2.4.2, and 2.6.1; also theorems in [2, 4]).

I. The Character Table of the Association Scheme $\mathcal{A}(\text{GU}(n, q^2), \Omega)$

In this section we study the character table of the association scheme $\mathcal{A}(\text{GU}(n, q^2), \Omega)$ obtained from the action of the full unitary group $\text{GU}(n, q^2)$ on the set $\Omega$ of nonisotropic projective points. (Here we take
Let $V$ be an $n$-dimensional vector space over $\mathbb{F}_{q^2}$ and let $H(x)$ be the canonical nonsingular Hermitian form

$$H(x) = x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n,$$

where $x = (x_1, x_2, \ldots, x_n) \in V$. We denote the set of all 1-dimensional nonisotropic subspaces of $V$ with respect to $H$ by $\Omega$ and use the same symbol $x$ to denote the 1-dimensional subspace in $\Omega$ which is spanned by $x \in V$. It is known that the general unitary group

$$GU(n, q^2) = \{ T \in GL(n, q^2) \mid TT^* = I \}$$

acts on $\Omega$ transitively, and this action yields a symmetric association scheme of class $q$ if $n \geq 3$ and of class $q - 1$ if $n = 2$. We denote this association scheme by $\mathcal{X}(GU(n, q^2), \Omega)$ and define its association relations $R_0, R_1, \ldots, R_q$ by

$$R_0 = \{(x, x) \mid x \in \Omega\}$$

$$R_1 = \left\{(x, y) \in \Omega \times \Omega \mid \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$R_i = \left\{(x, y) \in \Omega \times \Omega \mid \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^* = \begin{pmatrix} 1 & 1 \end{pmatrix} \right\}$$

for $i = 2, 3, \ldots, q$, where $g$ is a primitive element of $\mathbb{F}_q$. 

**Section I consists of the following subsections.**

1.1. The association schemes $\mathcal{X}(GU(n, q^2), \Omega)$.
1.2. The parameters $p^n_i$ of the association scheme $\mathcal{X}(GU(n, q^2), \Omega)$ for $n \geq 3$.
1.3. The parameters $d^n_i$ of the association scheme $\mathcal{X}(GU(2, q^2), \Omega)$ and the relationships between the $p^n_i$ and $d^n_i$.
1.4. The character tables of $\mathcal{X}(GU(n, q^2), \Omega)$ $(n \geq 3)$ are controlled by the character table of $\mathcal{X}(GU(2, q^2), \Omega)$.
That is, \( \mathcal{A}(GU(n, q^2), \Omega) = (\Omega, \{ R_i \}_{0 \leq i \leq q}) \) if \( n \geq 3 \), and \( \mathcal{A}(GU(2, q^2), \Omega) = (\Omega, \{ R_i \}_{0 \leq i \leq q-1}) \) with

\[
\begin{align*}
|\Omega| &= q^{n-1}\{q^n - (-1)^n\}/(q + 1) \\
k_i &= q^{n-2}\{q^{n-1} - (-1)^{n-1}\}/(q + 1) \\
k_i &= q^{n-2}\{q^{n-1} - (-1)^{n-1}\} \quad \text{for} \quad i = 2, 3, \ldots, q - 1 \\
k_q &= q^{2n-3} + (-1)^{n-1}(q - 1)q^{n-2} - 1.
\end{align*}
\]

For more information on this association scheme and its parameters, we refer to [9].

1.2. The Parameters \( p_{ij}^h \) of the Association Scheme \( \mathcal{A}(GU(n, q^2), \Omega) \) for \( n \geq 3 \)

The intersection numbers \( p_{ij}^h \) of \( \mathcal{A}(GU(n, q^2), \Omega) \) were completely computed by Wei in [9]. Thus the proof is omitted here.

**Lemma 1.2** [9]. Denote the set \( \{2, 3, \ldots, q - 1\} \) by \( [q - 1] \).

\[
p_{11} = q^{n-3}\{q^{n-2} - (-1)^{n-2}\}/(q + 1) \\
p_{1j} = q^{n-3}\{q^{n-2} - (-1)^{n-2}\}, \quad \text{for} \quad j \in [q - 1] \\
p_{ij} = \begin{cases} 
q^{n-3}\{q^{n-1} + q^{n-2} + (-1)^nq^2 - (-1)^n\} & \text{if } g^{i+j-2} = g^{i-1} + g^{j-1} \\
q^{n-3}\{q^{n+1} + q^{n-2} - (-1)^nq - (-1)^n\} & \text{if } g^{i+j-2} \neq g^{i-1} + g^{j-1},
\end{cases}
\]

for all \((i, j) \in [q - 1]^2\).

For \((h, i, j) \in [q - 1]^3\),

\[
p_{ij}^h = \begin{cases} 
q^{n-3}\{q^{n-1} + q^{n-2} - (-1)^n\} & \text{if } (h, i, j) \in T_0 \\
q^{n-3}\{q^{n-1} + q^{n-2} - (-1)^nq - (-1)^n\} & \text{if } (h, i, j) \in T_1 \\
q^{n-3}\{q^{n-1} + q^{n-2} + (-1)^nq - (-1)^n\} & \text{if } (h, i, j) \in T_2,
\end{cases}
\]

where \( T_0, T_1, \) and \( T_2 \) are defined as follows:

(i) If \( q \) is an odd prime power, then

\[
T_0 = \{(h, i, j) \in [q - 1]^3 | \Delta = 0\} \\
T_1 = \{(h, i, j) \in [q - 1]^3 | \Delta \neq 0, \Delta \text{ is a square element in } \mathbb{F}_q^*\} \\
T_2 = \{(h, i, j) \in [q - 1]^3 | \Delta \text{ is a nonsquare element in } \mathbb{F}_q^*, \}
\]

where \( \Delta = g^{-\gamma(j-1)} \{ g^{h+i+j-3} + g^{j+h-3} + g^{j+i-3} - g^{h+i+j-3} \} - 4g^{h+i+j-3} \).
(ii) If \( q = 2^r \), then
\[
T_0 = \{(h, i, j) \in [q - 1]^3 | A = 0\}
\]
\[
T_1 = \{(h, i, j) \in [q - 1]^3 | A \neq 0, D(A^{-2}g^{h+i-i-1}) = 0\}
\]
\[
T_2 = \{(h, i, j) \in [q - 1]^3 | A \neq 0, D(A^{-2}g^{h+i-i-1}) = 1\},
\]
where \( A = g^{h+i-2} + g^{i-1} + g^{h-1} + g^{h+i-j-1} \), and \( D(i) = t + t^2 + t^4 + \ldots + t^{2^r-1} \).

**Proof:** See [9, pp. 301-305].

The other parameters are computed by the basic equalities

(i) \( p_{ij}^h = p_{ji}^h \)

(ii) \( \sum_{j=1}^{q} p_{ij}^h = \begin{cases} \lfloor k_i - 1 \rfloor & \text{if} \ h = i \\ k_i & \text{if} \ h \neq i \end{cases} \)

(iii) \( k_i p_{hj}^h = k_h p_{ij}^h = k_j p_{hi}^h. \)

1.3. The Parameters \( a_{ij}^h \) of the Association Scheme \( \mathcal{A}(GU(2, q^2), \Omega) \) and the Relationships between the \( p_{ij}^h \) and \( a_{ij}^h \)

The parameters of the association scheme \( \mathcal{A}(GU(2, q^2), \Omega) = (\Omega, \{R_i\}_{0 \leq i \leq q-1}) \) are given as follows.

\[
k_0 = k_1 = 1
\]
\[
k_2 = k_3 = \ldots = k_{q-1} = q + 1
\]
\[
a_{i1} = 0
\]
\[
a_{ij} = 0 \quad \text{for} \quad j \in [q - 1]
\]
\[
a_{ij} = \begin{cases} q + 1 & \text{if} \ g^{i+j-2} = g^{i-1} + g^{j-1} \\ 0 & \text{if} \ g^{i+j-2} \neq g^{i-1} + g^{j-1} \text{for} \ i, j \in [q - 1] \end{cases}
\]
\[
a_{ij} = \begin{cases} 1 & \text{if} \ (h, i, j) \in T_0 \\ 0 & \text{if} \ (h, i, j) \in T_1 \\ 2 & \text{if} \ (h, i, j) \in T_2, \end{cases}
\]

where \([q - 1], T_0, T_1, \text{and} T_2\) are as in Lemma 1.2.

Therefore, we have the following relations between the two sets of intersection numbers of the association schemes \( \mathcal{A}(GU(n, q^2), \Omega) \), with \( n \geq 3 \), and \( \mathcal{A}(GU(2, q^2), \Omega) \).

**Lemma 1.3.** Let \( p_{ij}^h \) and \( a_{ij}^h \) be the intersection numbers for \( \mathcal{A}(GU(n, q^2), \Omega) \), \( n \geq 3 \), and \( \mathcal{A}(GU(2, q^2), \Omega) \), respectively. Then
\[
p_{ij}^h = (-1)^{n-2}q^{n-2-a_{ij}^h} + q^n - 3q^{n-1} + q^{n-2} - (-1)^n(q - (-1)^n)\]
for \((h, i, j) \in \{1, 2, ..., q-1\}^3\) except for the cases

\[ p_{11}^h = q^{n-3}(q^{n-2} - (-1)^n)/(q + 1) \quad \text{for } h = 1, 2, 3, ..., q - 1, \]

and

\[ p_{ij}^h = p_{ji}^h = (q^{n-3} - (-1)^{n+3})(q^{n-2} - (-1)^n) \]

for \(h \in \{1, 2, ..., q-1\}, j \in \{2, 3, ..., q-1\} \).

**Proof.** By Lemma 1.2 and the above paragraph, it is immediate.

In order to complete the intersection matrices we give the remaining parameters in \(p_{ij}^h\) for which \(a_{ij}^h\) are not defined.

\[ p_{1a}^1 = p_{a1}^1 = \{q^{n-3} - (-1)^n\}^3 \{q^{n-2} - (-1)^n\} \]

\[ p_{i1}^h = q^{n-3}\{q^{n-2} - (-1)^n\}/(q + 1) \quad \text{for } h = 2, 3, ..., q - 1. \]

\[ p_{ij}^1 = q^{n-3}\{q^{n-2} + (-1)^n\}/(q + 1) \]

\[ p_{ij}^j = p_{ji}^j = q^{2n-5} \quad \text{for } j = 2, 3, ..., q \]

\[ p_{ij}^h = p_{ji}^h = \begin{cases} q^{n-3}(q^{n-1} + q^{n-2}) & \text{if } i \neq j, i, j \in \{2, 3, ..., q\} \\ q^{n-3}\{q^{n-1} + q^{n-2} + (-1)^n\} & \text{if } i = j = 2, 3, ..., q - 1 \\ q^{n-3}\{q^{n-1} + q^{n-2} - (-1)^n\} - 2 & \text{if } i = j = q \end{cases} \]

\[ p_{ij}^h = p_{ji}^h = \begin{cases} q^{n-3}\{q^{n-1} + q^{n-2} - (-1)^n\} - 1 & \text{if } h = j = 2, 3, ..., q - 1 \\ q^{n-3}\{q^{n-1} + q^{n-2} - (-1)^n\} - (-1)^n \} & \text{if } h \neq j, h \in \{1, 2, ..., q - 1\}, j \in \{2, 3, ..., q\}. \]

1.4. The Character Tables of \(X(GU(n, q^2), \Omega)\) \((n \geq 3)\) Are Controlled by the Character Table of \(X(GU(2, q^2), \Omega)\)

We now describe how we get the character table of the association scheme \(X(GU(n, q^2), \Omega)\) for arbitrary \(n \geq 3\) from that of \(X(GU(2, q^2), \Omega)\).

**Theorem 1.4.** Let \(\bar{P} = (\bar{p}_j(i))\) and \(P = (p_j(i))\) be the character tables of \(X(GU(n, q^2), \Omega)\) and \(X(GU(2, q^2), \Omega)\), respectively. Then

\[ \bar{p}_j(i) = (-1)^n \cdot q^{n-2} \cdot p_j(i) \]

(for all \(i, j \in \{1, 2, ..., q - 1\}\)).
Remark. For the $q$th-row and $q$th-column entries for $\bar{P}$, which the matrix $P$ does not have, we have the following.

\[
\begin{align*}
\bar{p}_i(q) &= (-1)^q q^{n-3} \\
\bar{p}_j(q) &= (-1)^q(q + 1) q^{n-3} \\
\bar{p}_k(q) &= (-1)^{q-1}(q - 2) q^{n-2} + (-1)^{q-1}(q - 1) q^{n-3} - 1.
\end{align*}
\]

Proof of Theorem 1.4. Let $\bar{B}_i$ and $B_i$ denote the $i$th-intersection matrices of $\mathcal{X}(GU(n, q^2), \Omega)$ and $\mathcal{X}(GU(2, q^2), \Omega)$, and $\bar{P}_i$ and $P_i$ denote the diagonal matrices with the diagonal entries $\bar{p}_i(0), \bar{p}_i(1), ..., \bar{p}_i(q)$ and $p_i(0), p_i(1), ..., p_i(q - 1)$, respectively. It is enough to show that for each $i = 1, 2, ..., q$ the equality $\bar{B}_i \cdot \bar{P} = \bar{P} \cdot P_i$ holds if $B_i \cdot P = P \cdot P_i$ holds for each $i = 1, 2, ..., q - 1$.

The following computation reveals that the $(f, g)$-entry of $\bar{B}_i \cdot \bar{P}$ is in fact the same as the $(f, g)$-entry of $\bar{P} \cdot P_i$, for $i, f, g \in \{2, 3, ..., q - 1\}$. (That is, $\sum_{h=0}^{q} p_f^h \cdot \bar{P}_h(g) = \bar{P}_f(i) \cdot \bar{P}_i(g)$.)

\[
\begin{align*}
\sum_{h=0}^{q} & \ p_f^h \cdot \bar{P}_h(g) \\
&= p_0^f \cdot \bar{P}_0(g) + \sum_{h=1}^{q-1} p_f^h \cdot \bar{P}_h(g) + p_f^q \cdot \bar{P}_q(g) \\
&= k_i \cdot \delta_{gf} \\
&\quad + \sum_{h=1}^{q-1} \left[ (-1)^{n-2} q^{n-2} \cdot a_f^h + q^{n-3} \{ q^{n-1} + q^{n-2} - (-1)^n q - (-1)^n \} \right] \\
&\quad \cdot (-1)^n \ q^{n-2} \cdot p_h(g) \\
&\quad + \{ q^{n-3} (q^{n-1} + q^{n-2}) + (-1)^n \ q^{n-2} \cdot \delta_{gf} \} \{ (-1)^n \ q^{n-2} - 1 \} \\
&= k_i \cdot \delta_{gf} + q^{2n-4} \sum_{h=1}^{q-1} a_f^h p_h(g) \\
&\quad + (-1)^n \ q^{2n-5} \{ q^{n-1} + q^{n-2} - (-1)^n q - (-1)^n \} \sum_{h=1}^{q-1} p_h(g) \\
&\quad + (-1)^n q^{2n-5} (q^{n-1} + q^{n-2}) - q^{n-3} (q^{n-1} + q^{n-2}) \\
&\quad + (-1)^n (q^{n-7} - 1) \cdot \delta_{gf}.
\end{align*}
\]
\[ q^{2n-4} \sum_{h=0}^{q-1} a_{j}^{h} \cdot p_{h}(g) = q^{2n-4} p_{f}(i) \cdot p_{i}(g) = \tilde{\rho}_{f}(i) \cdot \tilde{\rho}_{i}(g), \]

where \( \delta_{ij} = 0 \) or 1 depending on whether \( i \neq f \) or \( i = f \). We used the relations between \( p_{h}^{a} \) and \( a_{h}^{a} \) in Lemma 1.3 and between \( \tilde{\rho}_{f}(i) \) and \( p_{f}(i) \) as above and also used the basic equalities \( k_{i} = p_{i}^{0} \) and \( \sum_{h=1}^{q} p_{h}(g) = -p_{0}(g) = -1 \). The equalities for the rest of the entries, as well as the cases for \( i = 1 \) and \( i = q \), are checked in the same manner, so we omit the proof.

For our visibility we may depict the character tables as follows.

\[
\tilde{\rho} = \begin{bmatrix}
1 & k_{1} & k_{2} & k_{3} & \cdots & k_{q-1} & k_{q} \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(-1)^{q} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(-1)^{q+1} & (-1)^{q+1} & (-1)^{q+1} & (-1)^{q+1} & (-1)^{q+1} & (-1)^{q+1} & (-1)^{q+1} \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]

\[
p = \begin{bmatrix}
1 & 1 & q+1 & q+1 & \cdots & q+1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}.
\]

II. THE CHARACTER TABLE OF THE ASSOCIATION SCHEME \( \mathcal{X}(Sp_{2n}(q), \Omega) \)

In this section we study the character table of the association scheme \( \mathcal{X}(Sp_{2n}(q), \Omega) \) obtained from the action of the symplectic group \( Sp_{2n}(q) \) acting on the set \( \Omega \) of nonisotropic lines. These association schemes are of class \( q + 2 \) if \( n \geq 3 \) and of class \( q + 1 \) if \( n = 2 \). We first study the relations between the character tables for \( n \geq 3 \) and the character table for \( n = 2 \). Namely, we prove that the character tables for \( n \geq 3 \) are controlled by the character table for \( n = 2 \). Then, in the last subsection, we prove that for odd prime power \( q \) the character table of the association scheme for \( n = 2 \) is again controlled by the character table of the smaller association scheme.
The action of the group $SO_3(q)$ on an orbit $\Theta$ of length $q(q+1)$ in its natural action on the 3-dimensional vector space. Since the action of $SO_3(q)$ on $\Theta$ is isomorphic to the action of the group $PGL(2, q)$ on the cosets by a cyclic subgroup $Z_{q-1}$ of order $q-1$, we get the claim mentioned in the Introduction for an odd prime power $q$.

For an even prime power $q$ it is also proved that the character table of the association scheme obtained from the action of $PGL(2, q)$ on the cosets by a cyclic subgroup $Z_{q-1}$ controls the character table of $\mathcal{A}(Sp_4(q), \Omega)$, and thus that of $\mathcal{A}(Sp_{2n}(q), \Omega)$, but we omit the details.

Section II consists of the following subsections.

2.1. The Association Schemes $\mathcal{A}(Sp_{2n}(q), \Omega)$

Let $V$ be a $2n$-dimensional ($n \geq 2$) vector space over $F=GF(q)$ of a prime power, endowed with a non-singular skew-symmetric bilinear form. Let $G$ be the symplectic groups $Sp_{2n}(q)$ on $V$. That is, $G = \{ A \in GL_{2n}(q) \mid A \cdot K \cdot A' = K \}$, where $K = ( \begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix} )$ is the representing matrix of the bilinear form with respect to a suitable basis. Let $\Omega$ be the set of all nonisotropic projective lines ($2$-dimensional subspaces) on $V$. For each $Y \in \Omega$, we use the same symbol $Y$ to denote a $2 \times 2n$ matrix which represents the $2$-dimensional subspace $Y$, so that

$$\Omega = \{ Y \in \mathcal{M}_{2 \times 2n}(F) \mid \text{rank}(YKY') = 2 \}.$$
where

\[
C_i = \begin{cases}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & \\
\end{cases} \quad \text{if } i = 1
\]

\[
\begin{cases}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
-1 & 0 & 0 & g^{i-1} \\
0 & -1 & -g^{i-1} & 0 \\
\end{cases} \quad \text{for } i = 2, 3, \ldots, q
\]

\[
\begin{cases}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & \\
\end{cases} \quad \text{if } i = q + 1
\]

\[
R_{q+2} = \{(Y, Z) \in \Omega \times \Omega | \dim(Y \cap Z) = 1\}.
\]

2.2. The Intersection Numbers \(p_{ij}^h\) of the Association Scheme \(\mathcal{A}(Sp_{2n}(q), \Omega)\) for \(n \geq 3\)

All the structure constants of \(\mathcal{A}(Sp_{2n}(q), \Omega)\) for \(n \geq 3\), which will be discussed in this subsection, are also found in [7], in a work of Hao and Wei. However, the paper [7] is written in Chinese and is not easily accessible, so we restate some of the results in [7] together with a sketch of their proofs in the following two lemmas.

The number of nonisotropic projective lines on the \(2n\)-dimensional symplectic space is easily computed (cf. [8, Chap. 2]). That is,

\[
|\Omega| = \frac{(q^{2n} - 1)q^{2(n-1)}}{q^2 - 1}.
\]

**Lemma 2.2.1.** The valencies of the association scheme \(\mathcal{A}(Sp_{2n}(q), \Omega)\) are given as

\[
k_1 = \frac{q^{2n-4}(q^{2n-2} - 1)}{q^2 - 1}
\]

\[
k_2 = k_3 = \cdots = k_{q-1} = q^{2n-3}(q^{2n-2} - 1)
\]

\[
k_q = q(q^{2n-4} - 1)(q^{2n-2} - 1)
\]

\[
k_{q+1} = (q + 1)q^{2n-4}(q^{2n-2} - 1)
\]

\[
k_{q+2} = (q + 1)(q^{2n-2} - 1).
\]
Proof. \( k_1 = |\{X \in \Omega | (Y, X) \in R_1\}| \) is the number of lines orthogonal to a fixed line \( Y \) in \( \Omega \), so \( k_1 \) equals the number of nonisotropic projective lines in the \((2n-2)\)-dimensional symplectic space. For \( i = 2, 3, \ldots, q \), let

\[
Y = \begin{pmatrix} e_1 \\ e_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}
\]

and put

\[
X = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_1 \cdots u_{2n} \\ v_1 \cdots v_{2n} \end{pmatrix}.
\]

From the condition \((Y, X) \in R_i\), we must have \( u_i = 0 \) and \( u_{n-i} = 1 \). Thus the number of selections of \( u \) is \( q^{2n-2} - 1 \). For a fixed \( u \), the number of choices in \( v \) is independent of the choice of \( u \). Assume \( u = e_{n+1} + e_{n+2} = (0 \cdots 0110 \cdots 0) \); we now count the \( v \) such that \((Y, X) \in R_i\) with \( X = (\cdot) \). Such \( v \) must satisfy the condition that \( v_1 = -1 \), \( v_{n+1} = 0 \), and \(-v_1 + v_2 = g^{i-1}\). Therefore, the number of choices in \( v \) is \( q^{2n-3} \) for \( i = 2, 3, \ldots, q-1 \), and \((q^{2n-4} - 1)q \) for \( i = q \). Together with the choices of \( u \), we have

\[
k_i = \begin{cases} q^{2n-3}q^{2n-2} - 1 & \text{for } i = 2, 3, \ldots, q-1 \\ q(q^{2n-4} - 1)(q^{2n-2} - 1) & \text{for } i = q. \end{cases}
\]

For \( k_{q+2} \), we determine the number of vectors \( v = (v_1, v_2, \ldots, v_{2n}) \) satisfying

\[
\begin{pmatrix} e_1 \\ e_{n+1} \\ v \end{pmatrix} K \begin{pmatrix} e_1 \\ e_{n+1} \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\]

Because \( v_1 = -1 \) and \( v_{n+1} = 0 \), the number is \( q^{2n-2} - 1 \). Therefore, \( k_{q+2} = (q+1)(q^{2n-2} - 1) \) since each projective line contains \( q+1 \) points.

In the sequel, \( w(n) \) will denote the number of nonisotropic projective lines in \( 2n \)-dimensional symplectic space so that

\[
w(n) = \frac{(q^{2n} - 1)q^{2(n-1)}}{q^2 - 1}.
\]

Also, \( \lambda_b(n) \) will denote the number of solutions \((x_1, x_2, \ldots, x_{2n})\) of the equation \( x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n} = b \) in \( V = F^{2n}, F = GF(q) \). In fact,

\[
\lambda_b(n) = \begin{cases} q^n + q^{n-1}(q^n - 1) & \text{if } b = 0 \\ q^{n-1}(q^n - 1) & \text{if } b \in F^*. \end{cases}
\]
We now compute the intersection numbers of the association scheme \( \mathcal{A}(Sp_{2n}(q), \Omega) \) in the following lemma.

**Lemma 2.2.2.** Let \( \{ p_{ij}^h \} \) be the set of intersection numbers of the association scheme \( \mathcal{A}(Sp_{2n}(q), \Omega) \). Then

\[
(1) \quad p_{ij}^1 = \begin{cases} 
q^{2n-6}(q^{2n-4} - 1)/(q^2 - 1) & \text{for } j = 1, \\
q^{2n-5}(q^{2n-4} - 1) & \text{for } j = 2, 3, \ldots, q - 1, \\
q(q^{2n-6} - 1)(q^{2n-4} - 1) & \text{for } j = q, \\
q^{2n-6}(q + 1)(q^{2n-4} - 1) & \text{for } j = q + 1, \\
(q + 1)(q^{2n-4} - 1) & \text{for } j = q + 2.
\end{cases}
\]

For \( i = 2, 3, \ldots, q \) and \( j = 2, 3, \ldots, q \),

\[
p_{ij}^1 = \begin{cases} 
q^{2n-4}(q^2 - 1)(q^{2n-4} + q - 1) & \text{if } g^{i+j-2} = g^{i-1} + g^{j-1}, \\
q^{2n-4}(q^2 - 1)(q^{2n-4} - 1) & \text{if } g^{i+j-2} \neq g^{i-1} + g^{j-1}.
\end{cases}
\]

\( p_{i,q+2}^1 = 0 \) \quad for \quad \text{if } i = 2, 3, \ldots, q \) and \( i = q + 2 \).

(2)\quad i = 2, 3, \ldots, q \) and \( j = 2, 3, \ldots, q \),

\[
p_{ij}^{q+1} = \begin{cases} 
q^{2n-4}(q^{2n-2} - q^{2n-4} - q + 1) & \text{if } i \neq q, j \neq q, g^{i+j-2} = g^{i-1} + g^{j-1}, \\
q^{2n-4}(q^{2n-2} - q^{2n-4} + 1) & \text{if } i \neq q, j \neq q, g^{i+j-2} \neq g^{i-1} + g^{j-1}, \\
q^{2n-4}(q^{2n-2} - q^{2n-4} - q^2 + 1) & \text{if } i = q, or j = q, i \neq j, \\
q^{2n-4} - 1)(q^{2n-2} - q^{2n-4} - q^2) & \text{if } i = j = q.
\end{cases}
\]

(3)\quad For \( i = 2, 3, \ldots, q \) and \( j = 2, 3, \ldots, q \),

\[
p_{ij}^{q+2} = \begin{cases} 
q^{4n-6} & \text{if } i \neq q, j \neq q, i \neq j, \\
q^{2n-3}(q^{2n-3} - 1) & \text{if } i = j \neq q, \\
q^{2n-2}(q^{2n-4} - 1) & \text{if } i = q, or j = q, but i \neq j, \\
(q^{2n-4} - 1)(q^{2n-2} - q^2 - q) & \text{if } i = j = q.
\end{cases}
\]

(4)\quad Let \( N = \{ 2, 3, 4, \ldots, q - 1 \} \).
(i) If \( q \) is odd prime power, then let
\[
T_0 = \{(h, i, j) \in \mathbb{N}^3 \mid \Delta = 0\}
\]
\[
T_1 = \{(h, i, j) \in \mathbb{N}^3 \mid \Delta \neq 0, \Delta \text{ is a square element in } F^*\}
\]
\[
T_2 = \{(h, i, j) \in \mathbb{N}^3 \mid \Delta \text{ is a non-square element in } F^*\},
\]
where \( \Delta = g^{-(i-1)}\left(\left(g^{h+i-2} + g^{i+j-2} + g^{j-h-2} - g^{h+i+j-3}\right)^2 - 4g^{h+i+j-3}\right). \)

(ii) If \( q \) is even prime power, then let
\[
T_0 = \{(h, i, j) \in \mathbb{N}^3 \mid \Delta = 0\}
\]
\[
T_1 = \{(h, i, j) \in \mathbb{N}^3 \mid \Delta \neq 0, (1 + g^{h-1})(1 + g^{i-1}) \Delta^{-2} \in \{x^2 + x \mid x \in F\}\}
\]
\[
T_2 = \{(h, i, j) \in \mathbb{N}^3 \mid \Delta \neq 0, (1 + g^{h-1})(1 + g^{i-1}) \Delta^{-2} \notin \{x^2 + x \mid x \in F\}\},
\]
where \( \Delta = g^{h-1} + g^{i-1} + g^{h+i-1} + g^{h+i-2} - 2. \)

Then, for \( h, i, j \in \mathbb{N} \),
\[
p_n^{\delta} = \begin{cases} q^{2n-4}(q^{2n-2} - q^{2n-4} + 1) & \text{for } (h, i, j) \in T_0. \\ q^{2n-4}(q^{2n-2} - q^{2n-4} + q + 1) & \text{for } (h, i, j) \in T_1. \\ q^{2n-4}(q^{2n-2} - q^{2n-4} - q + 1) & \text{for } (h, i, j) \in T_2. \end{cases}
\]

**Proof.** (1) Let \( Y = (\epsilon_1, \epsilon_2), Z = (\epsilon_1, \epsilon_2) \). Then the number of \( X \in \Omega \) such that \((Y, X) \in R_1\) and \((X, Z) \in R_1\) is \( w(n - 2) = (q^{2n-4} - 1) q^{2n-6}/(q^2 - 1) \). Hence we have \( p_{11}^{\delta} \).

For \( j = 2, 3, ..., q \), set
\[
X = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_{2n} \\ y_2 & y_2 & \cdots & y_{2n} \end{pmatrix},
\]
then
\[
YKX' = \begin{pmatrix} x_{n+1} & y_{n+1} \\ -x_1 & -y_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
and
\[
ZKX' = \begin{pmatrix} x_{n+2} & u_{n+2} \\ -x_2 & -y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

It follows that
\[
uKu' = x_3 y_{n+3} + x_4 y_{n+4} + \cdots + x_n y_{2n} - (-1 + x_{n+3} y_3 + x_{n+4} y_4 + \cdots + x_{2n} y_n)
\]
\[
= g^{i-1}.
\]
So, for $j = 2, 3, ..., q - 1$,
\[
p_{ij} = \lambda_i(2n - 4) - q^{2n-5}(q^{2n-4} - 1)
\]

and
\[
p_{ij} = \lambda_0(2n - 4) - q(q^{2n-4} - 1) = q(q^{2n-6} - 1)(q^{2n-4} - 1).
\]

For $i = 2, 3, ..., q$ and $j = 2, 3, ..., q$, set
\[
X = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_1 & \cdots & x_n & x_{n+1} & \cdots & x_{2n} \\ y_1 & \cdots & y_n & y_{n+1} & \cdots & y_{2n} \end{pmatrix}.
\]

Then we may assume that
\[
YKX' = \begin{pmatrix} x_{n+1} & y_{n+1} \\ -x_1 & -y_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

From the condition $(Y, X) \in R_i$, we must have
\[
uKv' = x_2 y_{n+2} + \cdots + x_n y_{2n} - (-1 + x_{n+2} y_2 + \cdots + x_{2n} y_n) = g^{i-1}.
\]

Simultaneously, from $(Z, X) \in R_j$, we have
\[
\begin{pmatrix} x_{n+2} & -x_2 \\ y_{n+2} & -y_2 \end{pmatrix} \begin{pmatrix} 0 & g^{j-1} \\ -g^{j-1} & 0 \end{pmatrix} \begin{pmatrix} x_{n+2} & y_{n+2} \\ -x_2 & -y_2 \end{pmatrix} = \begin{pmatrix} 0 & g^{j-1} \\ -g^{j-1} & 0 \end{pmatrix},
\]
or
\[
x_2 y_{n+2} - x_{n+2} y_2 = g^{j-1}.
\]

That is, $x_3 y_{n+3} + \cdots + x_n y_{2n} - (x_{n+3} y_3 + \cdots + x_{2n} y_n) = g^{i-1} - g^{i-j} - 1$. Hence

\[
p_{ij} = \begin{cases} q(q^2 - 1) \lambda_i(2n - 4) & \text{if } g^{i-1} - g^{i-j} - 1 = 0 \\ q(q^2 - 1) \lambda_1(2n - 4) & \text{if } g^{i-1} - g^{i-j} - 1 \neq 0 \\ q^{2n-4}(q^2 - 1)(q^{2n-4} + q - 1) & \text{if } g^{i+j-2} = g^{i-1} + g^{j-1} \\ q^{2n-4}(q^2 - 1)(q^{2n-4} - 1) & \text{if } g^{i+j-2} \neq g^{i-1} + g^{j-1}. \end{cases}
\]

For $i = 1, 2, ..., q$ and $j = q + 2$, setting
\[
X = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & x_2 & 0 & \cdots & 0 & x_{n+2} & 0 & \cdots & 0 \\ y_1 & y_2 & \cdots & \cdots & \cdots & y_1 & y_2 & \cdots & y_{2n} \end{pmatrix},
\]

we have $YKX' = \left( \begin{smallmatrix} 0 \\ -y_1 \end{smallmatrix} \right)$, i.e., rank of $YKX' \leq 1$ so $p_{i(q+2)} = 0$ for $i = 2, 3, ..., q$. For $i = 1$, from $YKX' = \left( \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right)$ and $uKv' = x_2 y_{n+2} - x_{n+2} y_2 \neq 0$, we have $p_{1(q+2)} = (q + 1)(q^{2n-4} - 1)$. 

Parts (2) and (3) can be obtained in a similar way.

(4) For given \((Y, Z) \in R_h\), to count all \(X \in \Omega\) such that \((Y, X) \in R_j\), \((X, Z) \in R_i\) for \(h, i, j \in \{2, 3, \ldots, q-1\}\), we set

\[
Y = \begin{pmatrix}
e_1 & 0 & \cdots & 0 & \cdots & 0 \\
e_{n+1} & 1 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

\[
Z = \begin{pmatrix}
e_2 + e_{n+1} \\
-e_1 + (g^h - 1) e_{n+2}
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\
-1 & 0 & \cdots & \cdots & 0 & 0 & g^{h-1} - 1 & 0 & \cdots & 0
\end{pmatrix}
\]

and

\[
X = \begin{pmatrix}
u_1 & \cdots & x_n & x_{n+1} & \cdots & x_{2n} \\
v_1 & \cdots & y_n & y_{n+1} & \cdots & y_{2n}
\end{pmatrix}
\]

Then we may assume that

\[
YKX' = \begin{pmatrix}
x_{n+1} & y_{n+1} \\
-x_1 & -y_1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad uKv' = g^{j-1}.
\]

Also denoting \(c = \begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix}
\)

\(ZKX'\), we must have

\[
c' \begin{pmatrix}
0 \\
g^{-1}
\end{pmatrix} c = \begin{pmatrix}
0 & g^{j-1} \\
g^{-1} & 0
\end{pmatrix},
\]

That is,

(a) \(x_{n+2} y_2 = \frac{g^{h+i-j-1} - 1 - y_{n+2}}{1 - g^{h-1}} + x_2 + x_2 y_{n+2}\).

Together with \(uKv' = g^{j-1}\), we have, therefore,

(b) \(x_3 y_{n+3} + \cdots + x_n y_{2n} - (x_{n+3} y_3 + \cdots + x_{2n} y_n)\)

\[
= g^{j-1} - 1 + x_2 + \frac{g^{h+i-j-1} - 1 - y_{n+2}}{1 - g^{h-1}}.
\]

It is easy to see that there are \(q\) pairs \((x_2, y_{n+2})\) satisfying

(c) \(g^{j-1} - 1 + x_2 + \frac{g^{h+i-j-1} - 1 - y_{n+2}}{1 - g^{h-1}} = 0\).

For each of the pair \((x_2, y_{n+2})\) satisfying (c), we have \(\lambda_0(2n-4)\) choices in
In the $q$ solutions of Eq. (c), the number of $(x_2, y_{n+2})$ satisfying $x_2 + y_2 = 0$ depends on the discriminant $\Delta$ of the quadratic equation (d). This number is 1, 2, or 0 according to whether $\Delta = 0$, a nonzero square, or a nonsquare, respectively, if the characteristic of the field $F$ is not even. Since the number of solutions of

$$g^{h+i-j-1} - 1 - y_{n+2} + x_2 + x_2 y_{n+2} = 0$$

is $q - 1$. Therefore, if $\Delta = 0$, then

$$p^h = \{1 \cdot (2q - 1) + (q - 1)(q - 1)\} \lambda_0(2n - 4)$$

$$+ \{(q - 2)(2q - 1) + (q - 1)(q^2 - 2q + 2)\} \lambda_1(2n - 4)$$

$$= q^{2n - 4}(q^{2n - 2} - q^{2n - 4} + 1).$$

If $\Delta$ is a nonzero square, then

$$p^h = \{2 \cdot (2q - 1) + (q - 2)(q - 1)\} \lambda_0(2n - 4)$$

$$+ \{(q - 3)(2q - 1) + (q - 1)(q^2 - 2q + 3)\} \lambda_1(2n - 4)$$

$$= q^{2n - 4}(q^{2n - 2} - q^{2n - 4} + q + 1).$$

If $\Delta$ is a nonsquare, then

$$p^h = q(q - 1) \cdot \lambda_0(2n - 4) + \{(q - 1)(2q - 1)$$

$$+ (q - 1)(q^2 - 2q + 1)\} \lambda_1(2n - 4)$$

$$= q^{2n - 4}(q^{2n - 2} - q^{2n - 4} - q + 1).$$

If the characteristic of $F$ is 2, then Eq. (d) becomes

$$(d') x_{n+2} y_2 = (1 + g^{h-1}) x_2^2 + (g^{k-1} + g^{i-1} + g^{h-i-j-1} + g^{h+i-2}) x_2$$

$$+ (g^{i-1} + 1).$$
Hence if $\Delta = g_{h-1} + g_{h-1} + g_{h+i} g_{i-1} + g_{h+i} g_{i-1} = 0$, then in the $q$ solutions of (c) only one solution satisfies $x_n + 2y_n = 0$. If $\Delta \neq 0$, let

$$x_2 = \frac{(g_{h-1} + g_{i-1} + g_{h+i} g_{i-1} + g_{h+i} g_{i-2})(1 + g_{h-1})}{1 + g_{h-1}};$$

then, from (d)', we have

$$x_n + 2y_2 = \frac{\Delta^2}{1 + g_{h-1}} \{t^2 + t + (1 + g_{h-1})(1 + g_{i-1}) \cdot \Delta^{-2}\}.$$

Hence, when $(1 + g_{h-1})(1 + g_{i-1}) \Delta^{-2} \in \{x^2 + x | x \in F\}$ or $(1 + g_{h-1})(1 + g_{i-1}) \Delta^{-2} \notin \{x^2 + x | x \in F\}$, the number of pairs $(x_2, y_{n+2})$ satisfying $x_n + 2y_2 = 0$ is 2 or 0, respectively. So we complete the proof of (4). 

The rest of the parameters are computed from the above parameters by applying the basic identities

$$|\Omega| = 1 + \sum_{i=1}^{q+2} k_i,$$

$$p_{ij}^h = p_{ji}^h,$$

$$\sum_{j=0}^{q+2} p_{ij}^h = k_i,$$

$$k_1 p_{ij}^h = k_h p_{ij}^h.$$

2.3. The Intersection Numbers $a_{ij}^h$ of the Association Scheme $\mathcal{A}(Sp_4(q), \Omega)$ and Their Relationship to the $p_{ij}^h$

The action of $Sp_4(q)$ on the set $\Omega$ of nonisotropic projective lines carries the symmetric association scheme $\mathcal{A}(Sp_4(q), \Omega)$ of class $q + 1$ (which is one less than the class number for $\mathcal{A}(Sp_{2n}(q), \Omega), n \geq 3$). Since the association relations are defined in exactly the same way as those for $\mathcal{A}(Sp_{2n}(q), \Omega)$ with the only exception that $R_q$ vanishing in $\mathcal{A}(Sp_4(q), \Omega)$, we label them $R_0, R_1, \ldots, R_{q-1}, R_{q+1}, R_{q+2}$. (Note that $R_q$ is missing here.) Then, using the same notation as before, we have the following parameters for $\mathcal{A}(Sp_4(q), \Omega)$.

$$|\Omega| = q^2(q^2 + 1),$$

$$k_1 = 1,$$

$$k_2 = \cdots = k_{q-1} = q(q^2 - 1),$$

$$k_{q+1} = k_{q+2} = (q + 1)(q^2 - 1).$$

The intersection numbers $a_{ij}^h$ are given as follows:

Let $i, j \in \{2, 3, \ldots, q-1\}$. 


\( a_{ij}^p = \begin{cases} q(q^2 - 1) & \text{if } g^{i+j-2} = g^{i-1} + g^{j-1} \\ 0 & \text{otherwise} \end{cases} \)

\( a_{ij}^q = \begin{cases} q^2 & \text{if } (h, i, j) \in T_0 \\ q^2 + q & \text{if } (h, i, j) \in T_1 \\ q^2 - q & \text{if } (h, i, j) \in T_2 \end{cases} \)

\( a_{ij}^{q+1} = \begin{cases} q^2 - q & \text{if } g^{i+j-2} = g^{i-1} + g^{j-1} \\ q^2 & \text{otherwise} \end{cases} \)

\( a_{ij}^{q+2} = \begin{cases} q^2 & \text{if } i \neq j \\ q^2 - q & \text{if } i = j \end{cases} \)

\( a_{q+2 q+2}^{q+1} = q^2. \)

Hence we have the following obvious relation between the two sets of parameters \( \{a^p_{ij}\} \) of \( \mathcal{X}(Sp_4(q), \Omega) \) and \( \{p^h_{ij}\} \) of \( \mathcal{X}(Sp_{2n}(q), \Omega) \).

For \( h = 1, 2, \ldots, q - 1, q + 1 \),

\[
p^h_{ij} = \begin{cases} q^{2n-4}a^h_{11} + q^{2n-6}(q^2-4 - 1)/(q^2 - 1), & \text{for } j = 1, \\ q^{2n-4}a^h_{1j} + q^{2n-5}(q^2-4 - 1), & \text{for } 2 \leq j \leq q - 1, \\ q^{2n-4}a^h_{1q+1} + q^{2n-5}(q+1)(q^2-4 - 1), & \text{for } j = q + 1, \end{cases}
\]

\( p^h_{1q+2} = (q + 1)(q^2-4 - 1), \)

\( p^h_{q+2} = q^{2n-4}a^h_{1q+2}, \) for \( 2 \leq h \leq q - 1 \), and \( h = q + 1 \).

For \( h = 1, 2, \ldots, q - 1, q + 1 \), and \( i = 2, 3, \ldots, q - 1 \),

\[
p^h_{ij} = \begin{cases} q^{2n-4}a^h_{ii} + q^{2n-6}(q^2-4 - 1)(q^2-4 - 1) & \text{for } 2 \leq j \leq q - 1, \\ q^{2n-4}a^h_{iq+1} + q^{2n-5}(q+1)(q^2-4 - 1)(q^2-4 - 1) & \text{for } j = q + 1, \\ q^{2n-4}a^h_{iq+2} & \text{for } j = q + 2; i \neq h, \\ q^{2n-4}(a^h_{iq+2} + q + 1) - (q + 1) & \text{for } j = q + 2; i = h, \end{cases}
\]

\( p^q_{ij}^{q+2} = \begin{cases} q^{2n-4}(q^2-4 - 1)/(q^2 - 1) & \text{for } i = j = 1, \\ q^{4n-8}a^q_{ij}^{q+2} & \text{for } i = 1; j = 2, 3, \ldots, q - 1, q + 1, q + 2, \\ q^{4n-8}a^q_{ij}^{q+2} & \text{for } i = 2, 3, \ldots, q - 1; j = 2, 3, \ldots, q - 1, q + 1; i \neq j, \\ q^{4n-8}(a^q_{ij}^{q+2} + q) - q^{2n-3} & \text{for } 2 \leq i = j \leq q - 1, \\ q^{2n-4}a^q_{ij}^{q+2} & \text{for } j = q + 2; i = 2, 3, \ldots, q - 1, \end{cases}\)
2.4. The Character Tables of $X(\text{Sp}_{2n}(q), \Omega)$ ($n \geq 3$) Are Controlled by the Character Table of $X(\text{Sp}_4(q), \Omega)$

We now show that the character tables of $X(\text{Sp}_{2n}(q), \Omega)$ are controlled by the character table of $X(\text{Sp}_4(q), \Omega)$ for each $n \geq 3$.

Theorem 2.4.1. Let $\tilde{P} = (\tilde{p}_j(i))$ and $P = (p_j(i))$ be the character tables of the association schemes $X(\text{Sp}_{2n}(q), \Omega)$ and $X(\text{Sp}_4(q), \Omega)$, respectively. Then by a suitable arrangement of rows and columns, we have the following relations.

(i) For $i = 1, 2, \ldots, q-1$ and $j = 1, 2, \ldots, q-1, q+1$,
$$\tilde{p}_j(i) = q^{2n-4}p_j(i).$$

(ii) For $i = q+1, q+2$,
$$\tilde{p}_j(i) = q^{3n-6}p_j(i).$$

(iii) $\tilde{p}_{q+2}(i) = p_{q+2}(i) = -(q+1)$, for $i = 1, 2, \ldots, q-1$
$$\tilde{p}_{q+2}(q+1) = q^{2n-4}(p_{q+2}(q+1) + 2q+1) - q^{n-1} - q - 1.$$ 

(iv) $\tilde{p}_j(i) = \begin{cases} -q^{2n-3} + q, & \text{for } i = 1, 2, \ldots, q-1 \\ -q^{3n-5} - q^{2n-3} + q^{n-1} + q, & \text{for } i = q+1 \\ q^{3n-5} - q^{2n-3} - q^{n-1} + q, & \text{for } i = q+2. \end{cases}$

(v) $\tilde{p}_j(q) = \begin{cases} q^{2n-5}, & \text{if } j = 1 \\ -q^{2n-3} - q^{2n-4}, & \text{if } j = 2, 3, \ldots, q-1 \\ -q^{2n-4} - q^{2n-5}, & \text{if } j = q+1 \\ -(q+1), & \text{if } j = q+2 \\ q^{2n-2} - q^{2n-3} - q^{2n-4} + q, & \text{if } j = q. \end{cases}$

Proof. Let $\bar{B}_i$ and $B_i$ be the $i$th intersection matrices, whose $(j, h)$-entries are $p_{jh}^i$ and $a_{jh}^i$, of $X(\text{Sp}_{2n}(q), \Omega)$ and $X(\text{Sp}_4(q), \Omega)$, respectively.
It is sufficient to show that for each \( j = 1, 2, \ldots, q + 2 \), \( \bar{B}_j \cdot \bar{P} = 'P \cdot \bar{P}_j \), where \( \bar{P}_j \) is the diagonal matrix with the diagonal entries \( \bar{p}_j(0), \bar{p}_j(1), \ldots, \bar{p}_j(q + 2) \). However, the equalities in each corresponding entry of \( \bar{B}_j \cdot \bar{P} \) and \( 'P \cdot \bar{P}_j \) are checked one by one through tedious but straightforward computations from the equality \( \bar{B}_j \cdot 'P = 'P \cdot \bar{P}_j \) and the relationship between \( p_{ij}^k \) and \( a_{ij}^k \), and \( \bar{p}_j(i) \) and \( p_j(i) \). So we omit the detail.

The explicit tables as well as the relations are depicted in Section 2.6.

2.5. The Association Scheme \( \mathcal{A}(SO_3(q), \Theta) \) and Their Intersection Numbers \( b^h_i \) for an Odd Prime Power \( q \)

Let \( \Theta \) be the set \( \{ x \in V_3(q) \mid f(x) = 1 \} \), where \( f(x) \) is the associated quadratic form \( f(x) = 2x_1x_2 + x_3^2 \) of the orthogonal geometry on \( V_3(q) \). Then \( SO_3(q) \) acts transitively on \( \Theta \) and this action is isomorphic to the action of \( PGL(2, q) \) on the cosets by the cyclic subgroup of the order \( q - 1 \), where \( q \) is an odd prime power.

From this action we get a symmetric association scheme \( \mathcal{A}(SO_3(q), \Theta) \) of class \( q + 1 \) whose association classes are defined by the \( GO_3(q) = \langle SO_3(q) \times Z_2 \rangle \)-orbits on \( \Theta \times \Theta \), which are described as follows.

Let \( g \) be a primitive element of \( GF(q) \), and \( i_* \) be the smallest positive integer \( i \) such that \( g^i = (1 + 1)^2 \) in \( GF(q) \). Let

\[
\begin{align*}
R_0 &= \{(x, x) \mid x \in \Theta \} \\
R_1 &= \{(x, -x) \mid x \in \Theta \} \\
R_i &= \{(x, y) \mid y \neq \pm x, f(x - y) = g^{i-1} \} \quad \text{for} \quad i = 2, 3, \ldots, i_* - 1 \\
R_i &= \{(x, y) \mid y \neq \pm x, f(x - y) = g^i \} \quad \text{for} \quad i = i_* + 1, i_* + 2, \ldots, q - 1 \\
R_{q+1} &= \{(x, y) \mid y \neq \pm x, f(x - y) = g^{i_*} \} \\
R_{q+2} &= \{(x, y) \mid y \neq \pm x, f(x - y) = 0 \}.
\end{align*}
\]

Then \( \mathcal{A}(SO_3(q), \Theta) = (\Theta, \{ R_0, R_1, R_2, \ldots, R_{q-1}, R_{q+1}, R_{q+2} \}) \). (Note that \( SO_3(q) \) is transitive on each \( R_i \) for \( 0 \leq i \leq q - 1 \), but not transitive on \( R_{q+1} \) and \( R_{q+2} \).)

The valencies of \( \mathcal{A}(SO_3(q), \Theta) \) are given by

\[
\begin{align*}
k_0 &= k_1 = 1 \\
&= k_2 = k_3 = \cdots = k_{q-1} = q - 1 \\
k_{q+1} &= k_{q+2} = 2(q - 1).
\end{align*}
\]

The intersection numbers \( b^h_i \) of \( \mathcal{A}(SO_3(q), \Theta) \) are given as follows.
For $h, i, j \in \{2, 3, 4, \ldots, q - 1\}$,

$$b_{ij}^h = \begin{cases} q - 1 & \text{if } g^{i+j-2} = g^{i-1} + g^{j-1} \\ 0 & \text{otherwise} \end{cases}$$

$$b_{ij}^k = \begin{cases} 1 & \text{if } (h, i, j) \in T_0 \\ 2 & \text{if } (h, i, j) \in T_1 \\ 0 & \text{if } (h, i, j) \in T_2 \end{cases}$$

$$b_{ij}^{q+1} = \begin{cases} 0 & \text{if } g^{i+j-2} = g^{i-1} + g^{j-1} \\ 1 & \text{otherwise} \end{cases}$$

$$b_{ij}^{q+2} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

$$b_{q+2,q+2}^{q+2} = 1,$$

where $T_0, T_2$, and $T_2$ are defined as in part (4) of Lemma 2.2.2 and the rest of the intersection numbers are computed from the above and by the basic equalities.

Hence, as before, we have the following relations between the two sets of parameters $\{a_{ij}^h\}$ and $\{b_{ij}^h\}$ of $\mathcal{X}(Sp_4(q), \Omega)$ and $\mathcal{X}(SO_3(q), \Theta)$, respectively,

$$a_{ij}^h = q \cdot b_{ij}^h + q^2 - q$$

for all $h, i, j \in \{2, 3, 4, \ldots, q - 1, q + 1, q + 2\}$ except for the cases

$$a_{q+1+q+2}^{q+1} = (q(b_{q+1+q+2}^{q+1} + 2) + (q^2 - q) - 2$$

$$a_{i+2,i}^{q+1} = q(b_{i+2,i}^{q+1} + 1) + (q^2 - q) - 1 \quad \text{for } i \geq 2$$

$$a_{q+1,j}^{q+1} = q(b_{q+1,j}^{q+1} + 1) + (q^2 - q) - 1 \quad \text{for } h, j \in \{2, 3, \ldots, q - 1\}$$

and $g^{h+j-2} = g^{h-1} + g^{j-1}$.

### 2.6. The Character Table of $\mathcal{X}(Sp_4(q), \Omega)$ Is Controlled by the Character Table of $\mathcal{X}(SO_3(q), \Theta)$ when $g$ Is an Odd Prime Power

We have the following theorem which shows that the character table of $\mathcal{X}(Sp_4(q), \Omega)$ is controlled by the character table of $\mathcal{X}(SO_3(q), \Theta)$, and thus, together with Theorem 2.4.1, so is the character table of $\mathcal{X}(Sp_{2n}(q), \Omega)$ for each $n \geq 3$ and for each odd prime power $q$. 
Theorem 2.6.1. Let $P = (p_j(i))$ and $T = (t_j(i))$ be the character tables of $\mathcal{A}(Sp_4(q), \Omega)$ and $\mathcal{A}(SO_3(q), \Theta)$, respectively. Then for $i \in \{1, 2, ..., q-1, q+1, q+2\}$ and $j \in \{2, 3, ..., q-1\}$,

$$p_j(i) = q \cdot t_j(i).$$

and $p_{q+1}(i) = \begin{cases} q(t_{q+1}(i) + 1) - 1 & \text{if } i = 1, 2, ..., (q-1)/2, q+2, \\
q(t_{q+1}(i) - 1) + 1 & \text{if } i = (q+1)/2, (q+3)/2, ..., q+1,
\end{cases}$

$p_{q+2}(i) = q(t_{q+2}(i) - 1) + 1$.

Proof. Omitted.

Finally, the character tables $T$, $P$, and $\bar{P}$ depicted below by suitable permutations in rows and columns for our visibility.

Remark. The entries $t_j(i)$ of the character table $T$ of $\mathcal{A}(SO_3(q), \Theta)$ have been calculated explicitly by W. M. Kwok and appear in his Ph.D. thesis (The Ohio State University, 1989).

| T | | | | | |
|---|---|---|---|---|
| 1 & 1 & q-1 & q-1 & ... & ... & q-1 & 2(q-1) & 2(q-1) |
| 1 & 1 & ... & ... & ... & ... & ... & ... & ... |
| ... & ... & ... & ... & ... & ... & ... & ... & ... |
| 1 & 1 & t_j(i) & ... & ... & ... & ... & ... & ... |
| ... & ... & ... & ... & ... & ... & ... & ... & ... |
| 1 & -1 & ... & ... & ... & ... & ... & ... & ... |
| 1 & 1 & -2 & -2 & ... & ... & -2 & q-3 & q-3 |
| 1 & -1 & 0 & 0 & ... & ... & 0 & -(q-1) & q-1 |

| P | | | | | |
|---|---|---|---|---|
| 1 & 1 & q(q^2-1) & q(q^2-1) & ... & ... & q(q^2-1) & (q+1)(q^2-1) & (q+1)(q^2-1) |
| 1 & 1 & ... & ... & ... & ... & ... & -(q+1) & -(q+1) |
| ... & ... & ... & ... & ... & ... & ... & ... & ... |
| ... & ... & ... & ... & ... & ... & ... & ... & ... |
| ... & ... & p_j(i) = q \cdot t_j(i) & ... & ... & ... & ... & -(q+1) & ... |
| ... & ... & ... & ... & ... & ... & ... & ... & ... |
| 1 & -1 & ... & ... & ... & ... & ... & q+1 & ... |
| 1 & 1 & -2q & -2q & ... & ... & -2q & (q-2)(q-1) & (q-2)(q-1) |
| 1 & -1 & 0 & 0 & ... & ... & 0 & (q^2-1) & q^2-1 |
\[
\bar{\rho} = \begin{bmatrix}
1 & k_1 & k_2 & k_3 & \cdots & \cdots & k_{q-1} & k_{q+1} & k_{q+2} & k_q \\
1 & q^{2n-4} & \cdots & \cdots & \cdots & \cdots & \cdots & -q^{2n-4}(q+1) & -(q+1) & -q^{2n-3} + q \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & q^{2n-4} & \cdots & \cdots & \cdots & \cdots & \cdots & -q^{2n-4}(q+1) & -(q+1) & -q^{2n-3} + q \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & q^{2n-4} & \cdots & \cdots & \cdots & \cdots & \cdots & q^{2n-4}(q+1) & -(q+1) & -q^{2n-3} + q \\
1 & -q^{2n-4} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & q^{2n-6} & -q^{2n-5} & q^{2n-3} & \cdots & \cdots & \cdots & -q^{2n-5} - q^{2n-4} & q^{2n-6}(q^2 - q - 1) - q^{2n-3} & q^{2n-2} - q^{n-1} - q - 1 & -q^{2n-5} - q^{2n-3} + q^{n+1} + q \\
1 & -q^{2n-6} & q^{2n-5} & -q^{2n-3} & \cdots & \cdots & \cdots & q^{2n-5} - q^{2n-3} & -q^{2n-6}(q^2 - q - 1) - q^{2n-3} & q^{2n-2} + q^{n-1} - q - 1 & q^{2n-5} - q^{2n-3} - q^{n+1} + q \\
1 & q^{2n-5} & -q^{2n-3} & -q^{2n-4} & \cdots & \cdots & \cdots & -q^{2n-3} - q^{2n-4} & -q^{2n-4} - q^{2n-5} & -(q+1) & q^{2n-2} - q^{2n-3} - q^{2n-4} + q 
\end{bmatrix}
\]
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