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DIFFERENTIABLE LINKS

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1. DEFINITIONS

S^q DENOTES the unit sphere in Euclidean space R^{q+1} and D^{q+1} the disk bounded by S^q.

Let $(p) = (p_1, p_2, \dots, p_r)$ be a sequence of r positive integers p_i and suppose that each $p_i < m$, where m is a positive integer.

A *(p)-link in* S^{m+1} is an oriented submanifold $K^{(p)} = K^{p_1} \cup \dots \cup K^{p_r}$ in S^{m+1}, each K^{p_i} being diffeomorphic to S^{p_i}. The K^{p_i} are the components of the link.

A *(p)-link map in* S^{m+1} is a differentiable imbedding in S^{m+1} of the disjoint union of the spheres S^{p_i}.

Two (p)-links $K_1^{(p)}$ and $K_2^{(p)}$ in S^{m+1} are *isotopic* if there exists an orientation preserving diffeomorphism of S^{m+1} onto itself which maps each $K_1^{p_i}$ on $K_2^{p_i}$ with preservation of the orientations. A (p)-link in S^{m+1} is *trivial* if it is the boundary of a disjoint union of disks imbedded in S^{m+1}. Two trivial (p)-links are isotopic.

2. OPERATIONS ON LINKS

(*a*) *Addition*. Given two (p)-links $K_1^{(p)}$ and $K_2^{(p)}$ in S^{m+1}, we define a new (p)-link in S^{m+1}, called their sum and whose isotopy class depends only on the isotopy classes of $K_1^{(p)}$ and $K_2^{(p)}$. After an isotopy, we can suppose that $K_1^{(p)}$ and $K_2^{(p)}$ are contained respectively in the nothern and the southern hemisphere of S^{m+1}. Then we join $K_1^{p_i}$ to $K_2^{p_i}$ by a thin tube to get a new imbedded p_i-sphere as follows: we construct imbeddings $T_i : D^{p_i} \times [-1, +1] \to S^{m+1}$ whose restrictions to $D^{p_i} \times \{-1\}$ and $D^{p_i} \times \{+1\}$ are imbeddings in $K_1^{p_i}$ and $K_2^{p_i}$ preserving and reversing orientation resp.; moreover the image of T_i does not intersect elsewhere the union of $K_1^{(p)}$, $K_2^{(p)}$ and the image of T_j for $j \neq i$. From the union of the two (p)-links and the images of the T_i, we remove the tubes $T_i(D_0^{p_i} \times [-1, +1])$ and we round the corners along the edges $T_i(\partial D^{p_i} \times \{-1 \cup +1\})$; $D_0^{p_i}$ denotes the interior of the disk D^{p_i}.

This addition, defined for isotopy classes of (p)-links in S^{m+1}, is commutative and associative. The class of a trivial (p)-link is a unit element.

(*b*) *Contraction*. Let $(p) = (p_1, \dots, p_r)$ and $(q) = (q_1, \dots, q_s)$ be two sequences of positive integers. A contraction $\gamma : (p) \to (q)$ is a surjective map such that $\gamma(p_i)$ is an integer equal to p_i. Given a (p)-link $K^{(p)}$ in S^{m+1} and a contraction $\gamma : (p) \to (q)$, we define a q-link

$K^{(q)} = K^{q_1} \cup \ldots \cup K^{q_s}$ in S^{m+1}, where K^{q_j} is obtained by joining as above the spheres K^{p_i} with tubes, where p_i runs over the integers mapped onto q_j by γ. We also call this operation the contraction defined by γ.

Note that the contraction commutes with the addition.

(*c*) *Projection and inclusion.* Let $(q) \subset (p)$ be a subsequence of (p). Given a (p)-link in S^{m+1}, we get a (q)-link in S^{m+1} by dropping all components whose index does not belong to (q). This operation will be called a *projection.* Conversely, given a (q)-link in S^{m+1}, we get a (p)-link in S^{m+1} by completing it with the boundary of the disjoint union of disks imbedded in S^{m+1} and not intersecting the given (q)-link. This operation will be called an *injection.*

These two operations commute with addition.

3. GROUP OF *h*-COBORDISM CLASSES OF LINKS

Recall that a homotopy sphere is h-cobordant to zero if it is diffeomorphic to the boundary of a homotopy disk. A homotopy disk is a differentiable compact manifold, contractible, and whose boundary is simply connected. We extend the previous definition of a (p)-link $K^{(p)}$ by assuming that its components are homotopy spheres h-cobordant to zero.

Two (p)-links $K_0^{(p)}$ and $K_1^{(p)}$ in S^{m+1} are h-cobordant if there exists in the product $S^{m+1} \times [0, 1]$ a submanifold W cutting transversally the boundary of $S^{m+1} \times [0, 1]$ along two closed submanifolds W_0 and W_1 such that:

(1) $W_0 \subset S^{m+1} \times \{0\} = S^{m+1}$ is isotopic to $K_0^{(p)}$

$W_1 \subset S^{m+1} \times \{1\} = S^{m+1}$ is isotopic to $K_1^{(p)}$

(2) W_0 and W_1 are deformation retracts of W.

A (p)-link in S^{m+1} is h-cobordant to a trivial link (or h-cobordant to zero) if it is the boundary of the disjoint union of homotopy disks imbedded in the disk D^{m+1}.

It is clear that two isotopic (p)-links are h-cobordant. The converse is also true, according to Smale [7], if each $p_i > 4$ (recall that we assumed $p_i < m$).

All the operations defined above for isotopy classes are also defined for h-cobordism classes. The following is elementary.

PROPOSITION. *The h-cobordism classes of (p)-links in* S^{m+1} *form an abelian group* $\Sigma_{(p)}^{m+1}$ *with respect to the addition.*

A contraction $\gamma : (p) \to (q)$ *defines a contraction homomorphism* $\Sigma_{(p)}^{m+1} \to \Sigma_{(q)}^{m+1}$.

An inclusion $(q) \subset (p)$ *induces a projection* $\Sigma_{(p)}^{m+1} \to \Sigma_{(q)}^{m+1}$ *and an inclusion* $\Sigma_{(q)}^{m+1} \to \Sigma_{(p)}^{m+1}$ *so that* $\Sigma_{(q)}^{m+1}$ *is identified to a direct summand of* $\Sigma_{(p)}^{m+1}$.

4. LINKING ELEMENTS

Let $(p) = (p_1, \ldots, p_r)$. Denote by $\vee_i S^{m-p_i}$ the wedge of spheres $S^{m-p_1} \vee \ldots \vee S^{m-p_r}$ (i.e. the one point compactification of the disjoint union of the open disks $D_0^{m-p_i}$). The sphere S^{m-p_i} is identified with a subspace of this wedge. Given a (p)-link $K^{(p)}$ in S^{m+1}, there exists a map g of $\vee_i S^{m-p_i}$ in the complement X of $K^{(p)}$ whose homotopy class is well defined by the following condition: $g(S^{m-p_i})$ is homotopic in X to a $(m-p_i)$-sphere which is the fiber of a tubular neighborhood of K^{p_i} and with linking number $+1$ with K^{p_i}. *This map induces a homotopy equivalence up to dimension* $m-1$. (We assumed that $m - p_i > 1$). The homotopy of X will be identified with the homotopy of $\vee_i S^{m-p_i}$ (up to dimension $m-1$) by this equivalence.

An inclusion $(q) \subset (p)$ induces an inclusion $\vee_j S^{m-q_j} \to \vee_i S^{m-p_i}$ and a retraction $\vee_i S^{m-p_i} \to \vee_j S^{m-q_j}$. A contraction $\gamma : (p) \to (q)$ induces the map $\vee_i S^{m-p_i} \to \vee_j S^{m-q_j}$, where each sphere S^{m-p_i} is mapped onto $S^{m-\gamma(p_i)}$ with degree 1.

Via the above equivalence, the component K^{p_i} imbedded in the complement of $\cup_{j\neq i} K^{p_j}$ defines an *element* λ^i of $\pi_{p_i}(\vee_{j\neq i} S^{m-p_j})$, *called the* i^{th}*-linking element of* $K^{(p)}$. This element depends only on the h-cobordism class of $K^{(p)}$ and provides a homorphism of the group $\Sigma^{m+1}_{(p)}$ in the group $\pi_{p_i}(\vee_{j\neq i} S^{m-p_j})$. This homomorphism behaves naturally with respect to the homomorphisms induced by inclusion, retraction and contraction.

According to Hilton [4], one has the following direct sum decomposition

$$\pi_{p_i}(\vee_{j\neq i} S^{m-p_j}) = \sum_{j\neq i} \pi_{p_i}(S^{m-p_j}) + \sum_{j<k\neq i} \pi_{p_i}(S^{2m-p_j-p_k-1}) + \ldots$$

where the summand $\pi_{p_i}(S^{m-p_j})$ is imbedded by composition with the inclusion $\alpha_j : S^{m-p_j} \to \vee_{k\neq i} S^{m-p_k}$ and $\pi_{p_i}(S^{2m-p_j-p_k-1})$ by composition with the Whitehead product $[\alpha_j, \alpha_k]$, etc.

According to this decomposition, one can write

$$\lambda^i = \sum_{j\neq i} \lambda^i_j + \sum_{j<k\neq i} \lambda^i_{jk} + \ldots .$$

The linking element λ^i_j of $K^{(p)}$ is nothing else than the element of $\pi_{p_i}(S^{m-p_j})$ defined by K^{p_i} imbedded in the complement of K^{p_j} which has the homotopy type of S^{m-p_j}.

Natural questions are:

(1) How far is a link determined by its linking elements?

(2) Which elements can appear as linking elements of a link?

In general, the linking elements are not independent. For instance it is known (see Shapiro and Kervaire [5]) that the elements λ^i_j and λ^j_i are equal up to sign by stable suspension. A symmetry relation for the λ^i_{jk} (which generalizes the relation (2) in theorem 6) will be proved in a joint paper with B. Steer.

5. THE STABLE RANGE

By definition it is the range where

$$m > 3p_i/2 + 1/2 \qquad \text{for all } i.$$

This implies that all the linking elements λ^i_j belong to *stable* homotopy groups of spheres $\pi_{p_i}(S^{m-p_j})$ and that all the other elements $\lambda^i_{jk}, \ldots$ vanish, because they belong to trivial homotopy groups of spheres.

The following theorem follows immediately from theorem 1 of [1] (or also from theorem 3 of [1]).

THEOREM. *If $m > 3p_i/2 + 1/2$, the map associating to a (p)-link map in S^{m+1} the $r(r-1)/2$ linking elements λ^i_j, $i < j$, induces a bijective map between isotopy classes of (p)-links maps in S^{m+1} and the group* $\sum_{i<j} \pi_{p_i}(S^{m-p_j})$.

6. THE LIMIT CASE

This is the case where $m = 3p_i/2 + 1/2$, so that we can write

$$p_i = 2d - 1 \qquad \text{and} \qquad m + 1 = 3d.$$

The group Σ^{3d}_{2d-1} of h-cobordism classes of $(2d-1)$-spheres imbedded in S^{3d} (link with one component) is isomorphic to Z for d even > 2 (cf. [3]) and to a quotient of Z_2 for d odd > 1.

For a link in S^{3d} whose components are $(2d-1)$-spheres, the possibly non vanishing linking elements are the $\lambda^i_j \in \pi_{2d-1}(S^{2d})$ and the $\lambda^i_{jk} \in \pi_{2d-1}(S^{2d-1})$.

The following theorem is proved using the methods of [3] and [6].

THEOREM. *The h-cobordism class of a link in S^{3d} whose components are $(2d-1)$-spheres, for $d > 3$ and $\neq 7$, is characterized by the class of each component and the linking elements λ^i_j, λ^i_{jk} submitted to the symmetry relations:*

(1) $S\lambda^i_j = (-1)^{d+1} S\lambda^j_i$, $i \neq j$, *where S is the suspension homomorphism,*

(2) $\lambda^i_{jk} = (-1)^j \lambda^j_{ik} = \lambda^k_{ij}$ $i < j < k$.

We mean that, given two such links, if the knot type of their corresponding components are the same and the corresponding linking elements are equal, then these two links are equivalent. Conversely, given a knot type for each component and elements λ^i_j and λ^i_{jk} satisfying the symmetry relations (1) and (2), there exists a link corresponding to these elements. The correspondence is of course an isomorphism of groups.

This theorem is not true for $d = 3$. If we take the link with 3 components which is described in [3], p. 463, and if we join the components S_2 and S_3 with a tube, we get a link with two components which are unknotted and such that $\lambda^2_1 = 0$ and λ^1_2 is a generator of the kernel of the suspension $S : \pi_{2d-1}(S^d) \to \pi_{2d}(S^{d+1})$. For $d = 3$, this kernel is zero; nevertheless this link is not trivial, because it can be proved that $\Sigma^9_5 = Z_2$.

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