Reachability indices of periodic positive systems via positive shift-similarity

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Abstract

The problem of the paper is the formulation of the concept of reachability indices of periodic linear positive systems. A positive system is one in which the positive orthant is an invariant subset with respect to positive input signals. The concept of shift-similarity for periodic positive systems is defined and the associated cyclically augmented system is considered. The results are the relation of the positive similarity of both representations, the formulation of the reachability indices, and a detailed example.
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1. Introduction

In this paper, we focus our attention on N-periodic linear discrete-time positive systems. Our main purpose is to obtain a canonical form of a realization of such a system. These forms are worth studying as they characterize among other things when a system is positively reachable. The choice of the reachability indices is basic to accomplish canonical forms of positively reachable systems and therefore, it is thus of interest to formulate a concept of reachability indices of periodic positive systems and to show how they are computed.

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The construction of the reachability indices of a discrete-time system follows from the Brunovsky indices (see [8]). Necessary and sufficient conditions to assign invariant factors of the system under state feedback are given in [22]. Other papers related to this topic are [21,27]. In [10], these indices are defined for descriptor systems from a state-space approach and using its transfer matrix.

The study of periodic systems is interesting by their application in some engineering problems. For example, the implementation of a multirate digital control can be used to the discretization of a continuous system (see [1]) and in this case, a periodic discrete-time model is considered. The reachability indices play a crucial role in the design theory of multirate sample-data control systems taking account of the computation time of the control law and/or tracking ability (see [17]). In [16] the reachability indices of a system are related to those of the augmented system for it. For periodic systems, the structural properties have been analyzed, for instance, in [15]. Besides that, using special kinds of coprime decompositions, the relation between the reachability and observability indices of a periodic descriptor systems at consecutive times is analyzed in [11].

When positive constraints are considered the reachability problem requires a thorough investigation. In control theory, the structural properties of positive reachability, controllability and observability have been studied in depth (see [3,9,14,18,19,23]) for invariant cases. Under such limitations, Coxson and Shapiro show that the maximum reachability index is always bounded by n (see [12,13]). In [5], the positive reachability indices are introduced, in addition, such indices constitute a sequence of invariants under positive similarity transformations. Some studies have been lately carried out in order to get upper bounds on the maximum reachability index of positive 2-D systems (for instance, see [2]).

In the periodic case, it is worth mentioning [6] where it is shown that reachability (complete controllability) of an N-periodic positive system is equivalent to those reachability (complete controllability) of N associated invariant-time positive systems. Furthermore, this same property of a periodic positive system is characterized through cally augmented system associated (see [4]).

The paper has been structured as follows: In Section 2, the basic definitions are given as well as the main results used in the paper. A new concept of positive shift-similarity for periodic positive systems is introduced in Section 3. Finally, the reachability indices of periodic positive systems are detailed in Section 4. Some significant examples are indicated to clarify this wording.

2. Preliminaries and definitions

Throughout this paper *N-periodic linear discrete-time positive control systems* are considered by means of this equation:

$$x(k + 1) = A(k)x(k) + B(k)u(k), \quad k \in \mathbb{Z}, \tag{1}$$

where $A(k)$ and $B(k)$ are periodic matrices of period $N \in \mathbb{N}$ with nonnegative entries, i.e., $A(k) = A(k + N) \in \mathbb{R}_+^{n \times n}$, $B(k) = B(k + N) \in \mathbb{R}_+^{n \times m}$, $x(k) \in \mathbb{R}_+^n$ is the nonnegative *state* vector and $u(k) \in \mathbb{R}_+^m$ is the nonnegative *control* or *input* vector. We denote this system by $(A(\cdot), B(\cdot))_N$.

An N-periodic positive system (1) is related to an *invariant-time positive cyclically augmented system* (abbreviated by CAS) through a system decomposition (see [20,24]). The construction of this system is carried out taking into consideration the evolution of the system during a period of time, in particular, this stems from stacking the equations of a periodic system at times $\{k, k + 1, \ldots, k + N - 1\}$, considering both the state vector and the input vector of the new system defined as concatenated vectors of the inputs and the states of systems (1), that is, $\hat{x}(k) = \text{col}[x(k), x(k +$

1), ..., $x(k + N - 1)$] and $\hat{u}(k) = \text{col}[u(k), u(k+1), \ldots, u(k+N-1)]$ as well as taking the following relations:

$$z(k) = M(n, N, 1)^{k-1}\hat{x}(k), \quad u_e(k) = M(m, N, 1)^k\hat{u}(k),$$

where $M(j, N, 1)$ is a weakly cyclic matrix of index N (see [25]), namely

$$M(j, N, 1) = \begin{bmatrix} O & I_j \\ I_{(N-1)j} & O \end{bmatrix}$$

being j the size of blocks, N the number of them, the shift of one block in this case 1 and I_q the identity matrix of order q. Moreover, O represents throughout the paper a zero matrix of a suitable size. We observe that $M(j, N, 1)$ is the generator of a cyclic group of order N as well as $M(j, m, 1)^{-1} = M(j, m, m - 1)$.

Therefore, the *invariant-time positive cyclically augmented system* associated with a periodic system is given by

$$z(k + 1) = A_e z(k) + B_e u_e(k), \tag{2}$$

where $A_e \in \mathbb{R}_+^{nN \times nN}$ is a weakly cyclic matrix of index N (see [25]), that is,

$$A_e = \begin{bmatrix} O & A(0) \\ A & O \end{bmatrix},$$

with $A = \text{diag}[A(1), \ldots, A(N-1)]$ and $B_e = \text{diag}[B(0), B(1), \ldots, B(N-1)] \in \mathbb{R}_+^{nN \times mN}$. We denote the invariant-time positive system (CAS) given in (2) by (A_e, B_e).

In contributions [15,26], the standard definitions of both a reachable state and a reachable system can be found for periodic general systems. These are expressed jointly for the systems considered in this work as follows:

Definition 1. A periodic positive system $(A(\cdot), B(\cdot))_N$ is said to be *positively reachable at time s* (*from* 0) if, for any nonnegative state $x_f \in \mathbb{R}_+^n$, there exists a nonnegative input sequence transferring the state of the system from the origin to x_f at time s, $x(s) = x_f$, in finite time. It is *positively reachable* if it is reachable at time s, for all $s \in \mathbb{Z}$.

Let us remark that if $N = 1$, such a definition is equivalent to positive reachability for the invariant case (see [7]).

Also, in [15], Grasselli introduces an example illustrating that the property of reachability of a periodic linear discrete-time system depends on k, that is, reachability at time k does not imply reachability at any time. Specifically, the periodic system of period 3 given by $m = n$, $A(\bar{k} - 2) = A(\bar{k}) = B(\bar{k} - 2) = B(\bar{k}) = O \in \mathbb{R}^{n \times n}$, $B(\bar{k} - 1) = I \in \mathbb{R}^{n \times n}$ and by any matrix $A(\bar{k} - 1) \in \mathbb{R}^{n \times n}$ is reachable at fixed time $\bar{k} - 1$ while the subspaces of reachable states at time $\bar{k}$ and $\bar{k} - 2$ are equal to zero, then this system is not reachable at time $\bar{k}$ and $\bar{k} - 2$. Hence, the periodic system is not reachable at all.

The relation between the reachability property of an N-periodic positive system and its associated CAS is well-known.

Theorem 1 (see [4]). *Consider a periodic positive system $(A(\cdot), B(\cdot))_N$ and its respective CAS (A_e, B_e). Then, $(A(\cdot), B(\cdot))_N$ is positively reachable if and only if (A_e, B_e) is positively reachable.*

In addition, the set of reachable states at time s in k steps of $(A(\cdot), B(\cdot))_N$ corresponds to the cone $\langle \mathcal{R}_k(A(\cdot), B(\cdot), s) \rangle$ generated by the columns of the reachability matrix given by (see [4,6]),

$$\mathscr{R}_k(A(\cdot), B(\cdot), s)$$
$$= [B(s-1)|A(s-1)B(s-2)| \cdots |\phi_A(s, s-k+1)B(s-k)]$$
$$s \in \mathbb{Z} \tag{3}$$

where the transition matrix of the states $\phi_A(k, k_0)$ of system (1) is

$$\phi_A(k, k_0) = A(k-1)A(k-2) \cdots A(k_0) \quad \text{if } k \neq k_0$$

and $\phi_A(k_0, k_0) = I_n$. By means of periodicity of $(A(\cdot), B(\cdot))_N$, it suffices to study the set of reachable states belonging to the cones $\langle \mathscr{R}_k(A(\cdot), B(\cdot), s) \rangle$ for every $s = 0, 1, \ldots, N-1$.

We bear in mind that for $N = 1$, the expression (3) gives the reachability matrix in k steps of an invariant system, which, in the case of (A_e, B_e), is

$$\mathscr{R}_k(A_e, B_e) = [B_e \ A_e B_e \cdots A_e^{k-1} B_e].$$

The following result yields the relationship between the k-reachability cones of the systems (1) and (2).

Proposition 1 (see [4]). For each $k \in \mathbb{Z}$,
$$\langle \mathscr{R}_k(A_e, B_e) \rangle = \mathbb{R}_+^{nN} \text{ if and only if } \langle \mathscr{R}_k(A(\cdot), B(\cdot), s) \rangle = \mathbb{R}_+^n \quad \forall s \in \mathbb{Z}.$$

It is largely spread (see [7]) that (A_e, B_e) is positively reachable if and only if the reachability matrix $\mathscr{R}_{nN}(A_e, B_e)$ contains a monomial submatrix of order nN. We recall that a monomial vector is a positive multiple of a unit basis vector and a monomial matrix is a matrix whose columns are different monomial vectors.

We can now state the following characterization for positive reachability of a periodic system in terms of its corresponding reachability matrix at time s in nN steps, $\mathscr{R}_{nN}(A(\cdot), B(\cdot), s)$, $\forall s \in \mathbb{Z}$.

Proposition 2. A periodic positive system $(A(\cdot), B(\cdot))_N$ is positively reachable $\forall s \in \mathbb{Z}$ if and only if the reachability matrix at time s in nN steps, $\mathscr{R}_{nN}(A(\cdot), B(\cdot), s)$, contains a monomial submatrix of order n, $\forall s \in \mathbb{Z}$.

We pay heed to the fact that this result is a consequence of combining, in the order pointed out, Theorem 1, the comment presented in the paragraph just after Proposition 1 and the same proposition.

3. Shift-similarity of periodic positive systems

When considering systems without nonnegative constraints, the reachability property is invariant with respect to similarity transformations of a linear system. However, a positive positively reachable system is by a similarity transformation not necessarily transferred to a positively reachable system although such a transformation preserves the positiveness of the final system. We notice that considering these last systems as general systems, without constrains, both of them are reachable but not in a positive sense. For the sake of brevity, we refer the reader to [5] for a more detailed example.

As a consequence, to guarantee that the property of positive reachability is invariant regarding suitable transformations, we define a new concept of shift-similarity for periodic positive systems,

which extends the positive similarity of the invariant case given in [5]. We recall that this aforegoing notion of the invariant case entails the use of a suitable monomial matrix. For this reason together with the fact that the inverse of a monomial matrix is also positive, we make use again of monomial matrices to introduce this new formulation:

Definition 2. Let $(\widehat{A}(\cdot), \widehat{B}(\cdot))_N$ and $(A(\cdot), B(\cdot))_N$ be two periodic positive systems. Such systems are said to be *positively shift-similar* if there exists an N-periodic collection of monomial matrices $M(j) = M(j + N), j = 1, 2, \ldots, N$, satisfying

$$
\begin{aligned}
M^{-1}(j)A(j)M(j - 1) &= \widehat{A}(j), \\
M^{-1}(j)B(j) &= \widehat{B}(j), \quad j = 1, 2, \ldots, N.
\end{aligned}
\tag{4}
$$

The relation of positive shift-similarity between two periodic positive systems and their associated CAS can be connected with the following result:

Proposition 3. Let $(\widehat{A}(\cdot), \widehat{B}(\cdot))_N$ and $(A(\cdot), B(\cdot))_N$ be two periodic positively shift-similar systems and $(\widehat{A}_e, \widehat{B}_e)$ and (A_e, B_e) their cyclically augmented systems, respectively. Then, $(\widehat{A}_e, \widehat{B}_e)$ and (A_e, B_e) are positively similar.

Proof. By definition of the positive shift-similarity of periodic systems, there exists an N-periodic collection of monomial matrices $M(j) = M(j + N), j \in \mathbb{Z}$, satisfying (4).

Taking $M_e = \text{diag}[M(0), M(1), \ldots, M(N - 1)]$ then this matrix is monomial and moreover

$$
\begin{aligned}
M_e^{-1} A_e M_e &= \widehat{A}_e, \\
M_e^{-1} B_e &= \widehat{B}_e.
\end{aligned}
$$

Hence, using Definition 2 when $N = 1$ then $(\widehat{A}_e, \widehat{B}_e)$ and (A_e, B_e) are by the same token positively similar.

In the example below, we illustrate that two CAS can be positively similar and however the associated N-periodic systems are not so.

Example 1. Let $(A(\cdot), B(\cdot))_3$ and $(\widehat{A}(\cdot), \widehat{B}(\cdot))_3$ be the 3-periodic positive system given by the following matrices:

$$
A(0) = \widehat{A}(1) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A(1) = \widehat{A}(2) = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}
$$

and

$$
A(2) = \widehat{A}(0) = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}
$$

and $B(s), \widehat{B}(s) \in \mathbb{R}_+^{2 \times 1}, \forall s \in \mathbb{Z}$, being $B(0) = B(1) = B(2) = \widehat{B}(0) = \widehat{B}(1) = \widehat{B}(2) = O$ where O is the zero matrix of a suitable size.

Then these systems are associated with the CAS given by (A_e, B_e) and $(\widehat{A}_e, \widehat{B}_e)$, respectively, where

$$A_e = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \end{bmatrix}$$

and

$$\widehat{A}_e = \begin{bmatrix} 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \end{bmatrix}$$

and $B_e = \widehat{B}_e = O$.

Taking

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

it is verifiable that (A_e, B_e) and $(\widehat{A}_e, \widehat{B}_e)$ are positively similar, however there are not two 2-square nonnegative monomial matrices $M(0)$ and $M(1)$ such that $M(1)^{-1}A(1)M(0) = \widehat{A}(1)$, what implies that $(A(\cdot), B(\cdot))_3$ and $(\widehat{A}(\cdot), \widehat{B}(\cdot))_3$ are not positively similar.

In [5] it was proven for invariant systems that positive reachability is invariant via positive similarity, that is, if we consider that a positively reachable system (A_e, B_e) is positively similar to $(\widehat{A}_e, \widehat{B}_e)$, then, $(\widehat{A}_e, \widehat{B}_e)$ is also positively reachable. This fact allows us to establish the same property formerly stated for the periodic case.

Theorem 2. *Let $(A(\cdot), B(\cdot))_N$ be a positively reachable system. If this system is positively similar to the system $(\widehat{A}(\cdot), \widehat{B}(\cdot))_N$ then $(\widehat{A}(\cdot), \widehat{B}(\cdot))_N$ is positively reachable too.*

Proof. By Theorem 1, $(A(\cdot), B(\cdot))_N$ is positively reachable if and only if (A_e, B_e) is positively reachable. Moreover, by Proposition 3, $(\widehat{A}_e, \widehat{B}_e)$ is positively similar to (A_e, B_e). Then, $(\widehat{A}_e, \widehat{B}_e)$ is also positively reachable, since if two invariant systems are similar and one of them is positively reachable then the other one possesses the equal feature (see [5]). Now, using Theorem 1 again the periodic system $(\widehat{A}(\cdot), \widehat{B}(\cdot))_N$ is positively reachable.

4. Reachability indices of periodic positive systems

Before introducing the concept of rechability indices of an N-periodic positive system, for the convenience of the reader, we repeat the relevant material from [5] without proofs, thus giving our exposition self-contained.

To start, we recall that for a system (A, B) without positive constraints the reachability indices can be deduced analyzing the linearly independent columns (in the usual sense of independence) of the reachability matrix with respect to the vectors appeared earlier in this same matrix $\mathcal{R}_n(A, B)$. Namely, if $B = [b_1|b_2|b_3|\cdots|b_m]$, these indices can be obtained from the following table (constructing it in the same order as the vectors appearing in the reachability matrix):

	b_1	b_2	b_3	...	b_m
B	×	×	⊗	...	×
AB	×	×		...	⊗
A^2B	×	⊗		...	⊗
⋮	⋮	⋮		⋮	⋮
$A^{n-1}B$	⊗			...	⊗

where the symbol × denotes a linearly independent vector with respect to the previously considered vectors (both on the same line and in the previous rows) and the symbol ⊗ stands for the linearly dependent vectors.

At the time we come across a linearly dependent vector of a jth column, starting from this position all the remaining vectors in the same column are linearly dependent too. Thus, we take the sequence

$$\{b_1, Ab_1, \ldots, A^{k'_1-1}b_1, b_2, Ab_2, \ldots, A^{k'_2-1}b_2, \ldots, b_m, Ab_m, \ldots, A^{k'_m-1}b_m\} \tag{5}$$

consisting of m chains of length k'_i of linearly independent vectors, extracted from the columns b_i, for all $i = 1, 2, \ldots, m$, in the reachability matrix. Then, the reachability indices $\{k_1, k_2, \ldots, k_m\}$ are the ordered sequence obtained from the sequence $\{k'_1, k'_2, \ldots, k'_m\}$ applied in equation (5).

We tackle this problem now for an invariant-time positive system (A, B). As we mentioned in the preliminaries the characterization of positive reachability is given in terms of the monomial vectors of the $R_n(A, B)$. Then, the attention must be drawn to detect the linearly independent monomial columns in this matrix, it being understood that a nonnegative vector is said to be linearly independent as regards to a system S of nonnegative vectors if it does not belong to the cone spanned by them (i.e., it cannot be written as linear combinations of the elements in S with nonnegative coefficients). In this case, the reachability indices can be obtained using the same table

	b_1	b_2	b_3	...	b_m
B	×	×	⊗	...	×
AB	×	×	×	...	⊗
A^2B	⊗	×		...	⊗
A^3B	×	⊗		...	×
⋮	⋮		⋮		
$A^{n-1}B$		⊗		...	

but with × denoting linearly independent monomial vectors with respect to the preceding considered vectors (both on the same line and in the previous rows) and ⊗ denoting the remaining vectors.

Moreover, we observe that, as is displayed on the table, there are examples where a linearly independent monomial vector can appear after a nonmonomial vector, i.e., in a nonconsecutive manner.

We note that it can be found on the table above a maximum of m different chains of length p_i of linearly independent monomial vectors obtained nonnecessarily in a consecutive way in the reachability matrix from each column b_i, for all $i = 1, 2, \ldots, m$ of matrix B. Thus, each chain can be formed by the union of consecutive subsequences of lengths p_{ki}, for all $k = 1, 2, \ldots, l_i$, of linearly independent monomial vectors. Hence, the indices

$$\{p_{11}, p_{21}, \ldots, p_{l_1 1}; p_{12}, p_{22}, \ldots, p_{l_2 2}; \ldots; p_{1m}, p_{2m}, \ldots, p_{l_m m}\}$$

are called the p-numbers of the positive system (A, B) and

$$p_i = p_{1i} + p_{2i} + \cdots + p_{l_i i}, \quad i = 1, \ldots, m, \tag{6}$$

coincides with the number of linearly independent monomial vectors obtained from the ith column b_i.

After setting right the prior numbers p_i (in a nonincreasing way), specifically, $p_{i_1} \geqslant p_{i_2} \geqslant \cdots \geqslant p_{i_m}$, the set of p-numbers must be rewritten placing firstly the subsequence ordered likewise of p-numbers whose elements totalize p_{i_1}, secondly the subsequence ordered (in equal manner) of p-numbers whose elements totalize p_{i_2} and so on up to complete all indices, i.e.,

$$\{p_{j_1 i_1}, p_{j_2 i_1}, \ldots, p_{j_{l_1} i_1}; p_{j_1 i_2}, p_{j_2 i_2}, \ldots, p_{j_{l_2} i_2}; \ldots; p_{j_1 i_m}, p_{j_2 i_m}, \ldots, p_{j_{l_m} i_m}\}.$$

These new indices are named sequence of reachability indices of (A, B).

Summing up, the *reachability indices* of (A, B) are taken as *the properly ordered sequence of p numbers where such p-numbers provide the number of linearly independent monomial vectors achieved consecutively for each subsequence obtained from every column of matrix B.*

For a periodic positive system $(A(\cdot), B(\cdot))_N$ and each $s \in \mathbb{Z}$, by Proposition 2, the property of positive reachability is characterized from the monomial vectors within the reachability matrix at time s in nN steps

$$\mathscr{R}_{nN}(A(\cdot), B(\cdot), s) = [B(s-1)|A(s-1)B(s-2)| \cdots |\phi_A(s, s-nN+1)B(s-nN)]$$

which can be rewritten as

$$\begin{aligned}
\mathscr{R}_{nN}(A(\cdot), B(\cdot), s) = [&\Phi_A(s, s)B(s-1)|\Phi_A(s, s-1)B(s-2)| \cdots \\
&|\Phi_A(s, s-N+1)B(s-N)| \\
&|\Phi_A(s+N, s)B(s-1)|\Phi_A(s+N, s-1)B(s-2)| \cdots \\
&|\Phi_A(s+N, s-N+1)B(s-N)| \cdots \\
&|\Phi_A(s+(n-1)N, s)B(s-1)| \cdots \\
&|\Phi_A(s+(n-1)N, s-N+1)B(s-N)]
\end{aligned}$$

By periodicity of the system, it is enough to study such property for times $s = 0, 1, 2, \ldots, N-1$ (or equivalently for $s = 1, \ldots, N$). Without a loss of generality, from now on we regard $B(s-j)$ as

$$B(s-j) = [b_1^{s-j}|b_2^{s-j}|b_3^{s-j}| \ldots |b_m^{s-j}], j = 1, \ldots, N.$$

Aiming at detecting linearly independent monomial vectors and taking as a basis the procedure entered into invariant positive case, we display below the column vectors of $\mathscr{R}_{nN}(A(\cdot), B(\cdot), s)$ obtained from each column of $B(s-j)$, $j = 1, \ldots, N$. That is, for each $s = 1, \ldots, N$, we construct the following table:

	b_1^{s-1}	b_2^{s-1}	...	b_m^{s-1}	b_1^{s-2}	...	b_m^{s-2}	...	b_1^{s-N}	...	b_m^{s-N}
		$j=1$				$j=2$		...		$j=N$	
$\Phi_A(s, s-j+1)B(s-j)$	×	×	...	×	⊗	...	×		×	...	⊗
$\Phi_A(s+N, s-j+1)B(s-j)$	×	×	...	⊗	⊗	...	⊗		×	...	×
$\Phi_A(s+2N, s-j+1)B(s-j)$	⊗	×	...	⊗		...			×	...	⊗
⋮	⋮		⋮		⋮				⋮		
$\Phi_A(s+(n-1)N, s-j+1)B(s-j)$	⊗										

where the notations × and ⊗ symbolize the same as in the prior case.

Thus, for each time $s = 1, \ldots, N$, we can derive from each table the corresponding p-numbers or just p_s-numbers as well as the p_{si}-indices according to equation (6).

Putting in order the above indices in the same line of reasoning as before, we can introduce the reachability indices in the periodic case as follows:

Definition 3. The positive reachability indices at time s, $s = 0, 1, \ldots, N-1$, of $(A(\cdot), B(\cdot))_N$, are chosen as the properly ordered sequence of p_s numbers where such p_s-numbers provide the number of linearly independent monomial vectors achieved consecutively for each subsequence obtained from each column of $B(s-j)$, $j = 1, 2, \ldots, N$, on the preceding tables.

We illustrate all the above-mentioned steps with the following example:

Example 2. Let us consider a 3-periodic linear discrete-time positive system with a state-space of dimension $n = 6$, given by

$$[A(0)|B(0)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$[A(1)|B(1)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and

$$[A(2)|B(2)] = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and $A(k+3) = A(k)$ and $B(k+3) = B(k)$, $\forall k \in \mathbb{Z}_+$.

Let us analyse whether such a system is positively reachable or not and at the same time, let us deduce the positive reachability indices at each time $s = 1, 2, 3$. The property of positive reachability holds whenever each reachability matrix at time s in $nN = 18$ steps $\mathscr{R}_{18}(A(\cdot), B(\cdot), s)$ for $s = 1, 2, 3$ contains a monomial submatrix of order $n = 6$. We observe that these matrices are the following ones:

$$\mathscr{R}_{18}(A(\cdot), B(\cdot), 1)$$
$$= [\Phi_A(1, 1)B(0)|\Phi_A(1, 0)B(-1)|\Phi_A(1, -1)B(-2)|$$
$$\quad |\Phi_A(4, 1)B(0)|\Phi_A(4, 0)B(-1)|\Phi_A(4, -1)B(-2)|\cdots|\Phi_A(16, -1)B(-2)]$$
$$\mathscr{R}_{18}(A(\cdot), B(\cdot), 2)$$
$$= [\Phi_A(2, 2)B(1)|\Phi_A(2, 1)B(0)|\Phi_A(2, 0)B(-1)|$$
$$\quad |\Phi_A(5, 2)B(1)|\Phi_A(5, 1)B(0)|\Phi_A(5, 0)B(-1)|\cdots|\Phi_A(17, 0)B(-1)]$$
$$\mathscr{R}_{18}(A(\cdot), B(\cdot), 3)$$
$$= [\Phi_A(3, 3)B(2)|\Phi_A(3, 2)B(1)|\Phi_A(3, 1)B(0)|$$
$$\quad |\Phi_A(6, 3)B(2)|\Phi_A(6, 2)B(1)|\Phi_A(6, 1)B(0)|\cdots|\Phi_A(18, 1)B(0)]$$

Now, we construct the tables with the different monomial vectors from the corresponding reachability matrices:

For $s = 1$:

	b_1^0	b_2^0	b_3^0	b_1^2	b_2^2	b_3^2	b_1^1	b_2^1	b_3^1
		$j = 1$			$j = 2$			$j = 3$	
$\Phi_A(1, 2-j)B(1-j)$		$B(0)$			$A(0)B(2)$			$A(0)A(2)B(1)$	
$\Phi_A(4, 2-j)B(1-j)$	$A(0)A(2)A(1)B(0)$			$A(0)A(2)A(1)A(0)B(2)$			$A(0)A(2)A(1)A(0)A(2)B(1)$		
	$\vdots$	$\vdots$		$\vdots$			$\vdots$		
$\Phi_A(16, 2-j)B(1-j)$	$\Phi_A(16, 1)B(0)$			$\Phi_A(16, 0)B(2)$			$\Phi_A(16, -1)B(1)$		

That is,

	b_1^0	b_2^0	b_3^0	b_1^2	b_2^2	b_3^2	b_1^1	b_2^1	b_3^1
$\Phi_A(1, 2-j)B(1-j)$	e_6	e_1	$e_3 + e_4$	0	0	e_5	0	$2e_5 + e_2 + e_4 + e_3$	0
$\Phi_A(4, 2-j)B(1-j)$	e_3	e_2	e_4	0	0	e_5	0	$e_5 + 2e_2 + e_4$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Therefore, the number of linearly independent monomial vectors obtained from each column of $B(j)$ at time $s = 1$ is

$$B(0) \to \{2, 2, 1\}, \quad B(2) \to \{0, 0, 1\}, \quad B(1) \to \{0, 0, 0\}$$

Therefore the p_1-numbers are $\{2; 2; 1|0; 0; 1|0; 0; 0\}$ and the p_{1i}-indices are $p_{11} = 5$, $p_{12} = 1$ and $p_{13} = 0$. Hence, the positive reachability indices at time $s = 1$ are the numbers $\{2; 2; 1|1; 0; 0|0; 0; 0\}$. Since the sum of these numbers is equal to $n = 6$ then, by Proposition 2, the periodic system is positively reachable at time $s = 1$.

For the sake of brevity, for $s = 2, 3$, we give only the details of the pertaining tables and indices. Thus, for $s = 2$:

	b_1^1	b_2^1	b_3^1	b_1^0	b_2^0	b_3^0	b_1^2	b_2^2	b_3^2
$\Phi_A(2,3-j)B(2-j)$	0	$e_1+e_2+e_3+$ $e_4+e_5+e_6$	0	e_6	e_5	e_4+e_2	0	0	e_1
$\Phi_A(5,3-j)B(2-j)$	0	$2e_3+e_1+e_2+e_4$	0	e_4	e_3	e_2	0	0	e_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Then, the number of linearly independent monomial vectors derived from the columns of $B(j)$ at time $s = 2$ is

$$B(1) \to \{0; 0; 0\}, \quad B(0) \to \{2; 2; 1\}, \quad B(2) \to \{0; 0; 1\}$$

Thus, the positive reachability indices at time $s = 2$ are the numbers $\{2; 2; 1|1; 0; 0|0; 0; 0\}$. Since the addition of these p_2-numbers is $n = 6$ then, by Proposition 2, the periodic system is positively reachable at time $s = 2$.

For $s = 3$ ($s = 0$):

	b_1^2	b_2^2	b_3^2	b_1^1	b_2^1	b_3^1	b_1^0	b_2^0	b_3^0
$\Phi_A(3,4-j)B(3-j)$	e_3	e_4	e_2	0	$2e_1+e_2+e_3+$ $2e_4+e_5+e_6$	0	e_6	e_1+e_4	e_3+e_4 $+e_5$
$\Phi_A(6,4-j)B(3-j)$	0	0	e_2	0	$2e_1+e_2+e_3+$ e_4+e_5	0	e_3+e_4	e_1	e_5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Finally, the number of linearly independent monomial vectors taken out from the columns of $B(j)$ at time $s = 3$ is

$$B(2) \to \{1; 1; 1\}, \quad B(1) \to \{0; 0; 0\}, \quad B(0) \to \{1; 1; 1\}$$

Moreover, the positive reachability indices at time $s = 3$ are the numbers $\{1; 1; 1|1; 1; 1|0; 0; 0\}$ whose sum is $n = 6$ then, by Proposition 2, the periodic system is positively reachable at time $s = 3$.

Therefore, from the monomial vectors considered in the construction of the indices, we can extract the following monomial matrices:

$$M(1) = [e_6 e_3 | e_1 e_2 | e_4 | e_5],$$

$$M(2) = [e_6 e_4 | e_5 e_3 | e_2 | e_1],$$

$$M(3) = M(0) = [e_3 | e_4 | e_2 | e_6 | e_1 | e_5].$$

Then, we have found a positively similar system given by

$$[\widehat{A}(0)|\widehat{B}(0)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\widehat{A}(1)|\widehat{B}(1)] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$[\widehat{A}(2)|\widehat{B}(2)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and $\widehat{A}(k+3) = \widehat{A}(k)$ and $\widehat{B}(k+3) = \widehat{B}(k)$, $\forall k \in \mathbb{Z}_+$.

Focusing on the previous example, we can check that this last system has the same positive reachability indices as the initial periodic system. Therefore, the reachability indices seems to be invariant under positive shift-similarity for this example.

On the other hand, the positive reachability indices of the respective CAS (A_e, B_e) are related to the reachability indices of the periodic systems. In fact, with the goal of obtaining them, we construct the table below, to take up less space, just for the nonzero column of the matrix B_e:

	b_{e1}	b_{e2}	b_{e3}	b_{e5}	b_{e7}	b_{e8}	b_{e9}
B_e	e_6	e_1	$e_3 + e_4$	$e_7 + e_8 + e_9 + e_{10}$ $+e_{11} + e_{12}$	e_{15}	e_{16}	e_{14}
$A_e B_e$	e_{12}	e_{11}	$e_{10} + e_8$	$e_{14} + e_{15} + e_{17}$ $+e_{18} + 2e_{13} + 2e_{16}$			e_5
$A_e^2 B_e$	e_{18}	$e_{13} + e_{16}$	$e_{15} + e_{16} + e_{17}$	$2e_2 + e_3 + e_4 + e_5$			e_7
$A_e^3 B_e$	e_3	e_2	e_4	$e_7 + e_8 + 2e_9 + e_{10}$			e_{14}
$A_e^4 B_e$	e_{10}	e_9	e_8	$2e_{13} + e_{14} + e_{15}$ $+e_{16} + e_{17}$			e_5
$A_e^5 B_e$	$e_{15} + e_{16}$	e_{13}	e_{17}	$2e_2 + e_4 + e_5$			e_7

Hence, p-numbers of (A_e, B_e) are the indices $\{5; 2, 3; 3; 0; 0; 0; 1; 1; 3\}$. It is worth mentioning that we have included two zeros in the set of p-numbers associated with four and six columns taking apart from the previous table.

Thus, we can verify that the positive reachability indices of (A_e, B_e) are

$$\{p_{11}; p_{12}, p_{22}; p_{13}; p_{14}; p_{15}; p_{16}; p_{17}; p_{18}; p_{19}\} = \{5; 3, 2; 3; 3; 1; 1; 0; 0; 0\}$$

which via positive similarity are equal to those of $(\widehat{A}_e, \widehat{B}_e)$ being this last one the corresponding CAS in accordance with the system $(\widehat{A}(\cdot), \widehat{B}(\cdot))_N$ above.

If now we pay our attention to the number of monomial vectors derived from the columns in $B(0)$, $B(1)$ and $B(2)$ at time s, for $s = 0, 1, 2$, this implies that the addition of the p_s-numbers is

$$s = 0 \rightarrow \{5, 5, 3\}, \quad s = 1 \rightarrow \{0, 0, 0\}, \quad s = 2 \rightarrow \{1, 1, 3\}.$$

Note that the sequence of p_{s_i}-numbers $\{5, 5, 3\} \cup \{0, 0, 0\} \cup \{1, 1, 3\}$ once ordered, coincides with that of the sum of p-numbers for associated CAS.

This fact together with proposition 1 allows us to establish the relation between the positively reachable states at time s of the periodic system and the positively reachable states of the associated CAS which we describe in the following conclusion:

Remark 1. The sequences of monomial vectors appropriately considered in the periodic case according to the preceding tables are linked to a proper selection of the monomial vectors appearing on the table associated with the invariant cyclically augmented system. This selection is made taking into account that each step on one of the periodic tables is related to N-steps on the table in the invariant system.

The relation between the reachability indices of a positive periodic system and the reachability indices of the CAS associated deserves to be studied so as to deduce a canonical form and find a complete sequence of invariants. These open-problems can be handle in the short future.

Acknowledgments

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