The Spectral Decomposition of Covariance Matrices for the Variance Components Models

Shi Jian-Hong\textsuperscript{a,},\textsuperscript{*} Wang Song-Gui\textsuperscript{b}

\textsuperscript{a}College of Mathematics and Computer Science, Shanxi Normal University, Linfen, China
\textsuperscript{b}College of Applied Science, Beijing University of Technology, Beijing, China

Received 26 January 2006
Available online 21 August 2006

Abstract

The aim of this paper is to propose a simple method to determine the number of distinct eigenvalues and the spectral decomposition of covariance matrix for a variance components model. The method introduced in this paper is based on a partial ordering of symmetric matrix and relation matrix. A method is also given for checking straightforwardly whether these distinct eigenvalues are linear dependent as functions of variance components. Some examples and applications to illustrate the results are presented.

© 2006 Elsevier Inc. All rights reserved.

AMS 1991 subject classification: 62J05

Keywords: Spectral decomposition; Variance component; Partial ordering

1. Introduction

The general variance components model for balanced data (see, e.g., [8,7]) is

\[ y = X\beta + \sum_{i=1}^{k} (1_{n_1}^{t_i1} \otimes 1_{n_2}^{t_i2} \otimes \cdots \otimes 1_{n_a}^{t_ia}) \xi_i, \]

where \( y \) is an \( n \times 1 \) vector of observations, \( X \) is a known \( n \times p \) matrix of rank \( p \), \( \beta \) is a vector of \( p \) parameters, \( 1_{n_i} \) is a vector of \( n_i \) elements equal to unity, with \( 1_{n_i}^0 = I_{n_i} \), and \( t_i = (t_{i1}, \ldots, t_{ia}) \)

\textsuperscript{*} This work was partially supported by the National Natural Science Foundation of China (Grant No. 10271010) and the Natural Science Foundation of Beijing (Grant No. 1032001).

\textsuperscript{*} Corresponding author.

E-mail addresses: shijh70@sina.com (S. Jian-Hong), wangsg@bjut.edu.cn (W. Song-Gui).
with \( t_{ij} = 0 \) or 1 for \( i = 1, 2, \ldots, k, j = 1, 2, \ldots, a \). \( \xi_t \) is a vector of \( q_i \) random effects, \( \xi_t \sim N(0, \sigma_i^2 I_q) \), \( i = 1, \ldots, k \), \( \text{Cov}(\xi_t, \xi_t) = 0 \) \((i \neq j)\). In particular, \( \xi_t \) is the random error term with \( t_k = (t_{k1}, \ldots, t_{ka}) = (0, \ldots, 0) \). \( \sigma^2 = (\sigma_1^2, \ldots, \sigma_k^2)' \) is a vector of \( k \) variance components. The parameters space of model (1.1) is \( R^p \otimes \Omega \), where \( \beta \in R^p \), \( \sigma^2 \in \Omega \), \( R^p \) is the \( p \)-Euclidean space, \( \Omega = \{ \sigma^2 : \sigma_1^2 \geq 0, \ldots, \sigma_k^2 \geq 0, \sigma_k^2 > 0 \} \).

With the above notations, the covariance matrix of \( y \), to be denoted by \( V \), is given by

\[
V = \sum_{i=1}^{k} \theta_i (\bar{J}_{n_1}^{h_1} \otimes \bar{J}_{n_2}^{h_2} \otimes \cdots \otimes \bar{J}_{n_a}^{h_a})
\]

where \( \theta_i = N_i \sigma_i^2, N_i = \prod_{j=1}^{a} n_j^{h_j} \) \((i = 1, \ldots, k)\). \( \bar{J}_{n_j} = (1/n_j) 1_{n_j} 1_{n_j}' \), \( I_{n_j} = I_{n_j} \) \((j = 1, \ldots, a)\). Define \( \theta = (\theta_1, \ldots, \theta_k)' \), \( K = \{ K_i = \bar{J}_{n_1}^{h_1} \otimes \bar{J}_{n_2}^{h_2} \otimes \cdots \otimes \bar{J}_{n_a}^{h_a}, i = 1, \ldots, k \} \). In the following text, the set \( K \) will be called the corresponding set of the covariance matrix \( V \).

When analyzing variance components model (1.1), it is useful to derive the expressions for \(|V|\), \( V^{-1} \) and the spectral decomposition of the covariance matrix \( V \) defined by (1.2), see, for example, Nerlove [5], Balestra [1], Fuller and Battese [2], Mazodier and Trognon [4]. Searle and Henderson [8] presented formulae for the eigenvalues, determinant and inverse of the covariance matrix \( V \). By means of some examples, Wansbeek and Kapteyn [14] obtained a straightforward way to obtain the spectral decomposition of the covariance matrix \( V \). We note that the number of distinct eigenvalues of the covariance matrix \( V \) and the linear relation of these different eigenvalues may be more important in studying the properties of estimations of variance components. However, this question has received limited consideration in literatures up to now. In Section 2 of this paper, we introduce a partial ordering and relation matrix, and provide a method for determining directly the number of the distinct eigenvalues of the covariance matrix \( V \) without calculating these eigenvalues. Based on this, a method for checking straightforwardly whether these distinct eigenvalues are linear dependent as functions of variance components is also given. In Section 3, a very simple procedure is suggested to obtain the spectral decomposition of covariance matrix by means of a relation matrix. To illustrate the results, two examples are presented. In Section 4, we apply the results derived in Sections 2 and 3 to the problem of studying the estimation of variance components.

### 2. Number and linear relation of distinct eigenvalues of covariance matrix

For the variance components model (1.1), as mentioned in the Introduction, the number and linear relation of distinct eigenvalues of covariance matrix are often useful in studying the estimation of variance components. In this section, we provide a method for determining the number and linear relation of distinct eigenvalues based on a partial ordering tool and a relation matrix. We first introduce a kind of partial ordering in the corresponding set \( K \) of the covariance matrix \( V \).

**Definition 2.1.** For \( K_i \) and \( K_j \) in \( K \), we say that \( K_i \) and \( K_j \) have relation \( \trianglelefteq \), denoted \( K_i \trianglelefteq K_j \), if and only if

\[
K_i K_j = K_i.
\]

If \( K_i \) and \( K_j \) have not relation \( \trianglelefteq \), we write \( K_i \not\trianglelefteq K_j \). By the properties of Kronecker product (see [13]) and the definition of partial ordering (see [3]), it can be easily verified that the relation
between elements of set \( K \) is a kind of partial ordering. Henceforth, we denote it by \(< K, \preceq >\) partial ordering set.

**Definition 2.2.** A \( k \times k \) matrix \( R = (r_{ij})_{k \times k} \) is called a relation matrix of partial ordering set \(< K, \preceq >\) if

\[
r_{ij} = \begin{cases} 
1 & \text{if } K_{t_i} \preceq K_{t_j}, \\
0 & \text{if } K_{t_i} \npreceq K_{t_j},
\end{cases} \quad K_{t_i}, K_{t_j} \in K, \quad i, j = 1, \ldots, k.
\]

**Definition 2.3.** For \( K_{t_i} \) in \( K \), denote \([K_{t_i}] = \{K_{t_j} : K_{t_i} \preceq K_{t_j}, K_{t_j} \in K\}\). The set \([K_{t_i}]\) is called the right partial ordering class of \( K_{t_i} \) in the set \( K \).

Verification for the succeeding lemmas are straightforward, and so the proofs are omitted. More details regarding verification, however, may be found in Shi [10].

**Lemma 2.1.** For any symmetric matrix \( A \), let \( \lambda_1, \ldots, \lambda_k \) be all the distinct eigenvalues of \( A \). Then there exist \( k \) symmetric and idempotent matrices \( M_1, \ldots, M_k \) such that

\[
(1) \quad M_i M_j = 0, \quad i \neq j, \quad (2) \quad \sum_{i=1}^{k} M_i = I, \quad (3) \quad A = \sum_{i=1}^{k} \lambda_i M_i,
\]

and the matrices \( M_1, \ldots, M_k \) (\( M_i \) is called the principal idempotent matrix of \( A \) corresponding to eigenvalue \( \lambda_i \)) are determined by matrix \( A \) uniquely.

**Lemma 2.2.** Let \( R \) be the relation matrix of the partial ordering set \(< K, \preceq >\), then \( R \) is a nonsingular matrix.

**Lemma 2.3.** For \( K_{t_i} \) and \( K_{t_j} \) in \( K \), \( K_{t_i} = K_{t_j} \) if and only if \( t_i = t_j \).

**Lemma 2.4.** Suppose \( K_{t_i} \preceq K_{t_j} \). For any \( r (1 \leq r \leq a) \), if \( t_{ir} = 0 \), then \( t_{jr} = 0 \).

**Lemma 2.5.** For \( K_{t_i} \) and \( K_{t_j} \) in \( K \), if \( K_{t_i}, K_{t_j} \in K \), then \( K_{t_i} K_{t_j} \preceq K_{t_i}, K_{t_i} K_{t_j} \preceq K_{t_j} \).

In the following, a set of matrices is called closed (not closed) if it is closed (not closed) with respect to ordinary matrix product.

2.1. Number and linear relation of distinct eigenvalues of covariance matrix when the corresponding set of the covariance matrix is closed

**Theorem 2.1.** For any \( K_{t_i} \) in \( K \), let \([K_{t_i}]\) be the right partial ordering class of \( K_{t_i} \), \( \mathcal{T}_{t_i} \) be the set of all subscripts to the elements of set \([K_{t_i}]\). Then \( \sum_{t_j \in \mathcal{T}_{t_i}} \theta_{t_j} \) is an eigenvalue of \( V \), and \( \sum_{t_j \in \mathcal{T}_{t_i}} \theta_{t_j} \neq \sum_{t_j \in \mathcal{T}_{t_i}} \theta_{t_j} \) if \( t_i \neq t_j \).

This theorem indicates that for any right partial ordering class there is an eigenvalue of \( V \) corresponding to it, and a different right partial ordering class corresponding to a different eigenvalue. Hence covariance matrix \( V \) has at least \( k \) eigenvalues. The proof of Theorem 2.1 is deferred to the Appendix.
Theorem 2.2. If the corresponding set $K$ of the covariance matrix $V$ is closed, then for any eigenvalue $\lambda$ of $V$ there exist a right partial ordering class $[K_{t_i}]$ such that $\lambda = \sum_{t_j \in T_{t_i}} \theta_{t_j}$, where $T_{t_i}$ is the set of all subscripts to the elements of set $[K_{t_i}]$.

Theorem 2.2 shows that when the corresponding set $K$ of the covariance matrix $V$ is closed, for any eigenvalue of $V$ there is a right partial ordering class corresponding to it. Therefore, covariance matrix $V$ has at most $k$ distinct eigenvalues. The proof of this theorem is presented in the Appendix.

Theorem 2.3. If $K$ is closed, then $V$ has only $k$ distinct eigenvalues. Furthermore, the $k$ components of vector

$$\lambda = (\lambda_{t_1}, \ldots, \lambda_{t_k})' = R\theta$$

are just all the distinct eigenvalues of $V$, where $R$ is the relation matrix of the partial ordering set $< K, \preceq >$.

Proof. The first part of Theorem 2.3 is a straightforward consequence of Theorems 2.1 and 2.2. From the definition of relation matrix and the result of Theorem 2.1 it is easy to see that, for any $j = 1, \ldots, k$, $\lambda_{t_j} = r_{j1}\theta_{t_1} + r_{j2}\theta_{t_2} + \cdots + r_{jk}\theta_{t_k}$ is just the eigenvalue of $V$ to which the right partial ordering class $[K_{t_j}]$ is corresponding. Thus, using the results of Theorems 2.1 and 2.2, the later result of Theorem 2.3 follows. □

A set of linear functions $c_1'\sigma_1^2, \ldots, c_a'\sigma_2^2$ of variance components is said to be linearly independent if the set of vectors $c_1, \ldots, c_a$ is linearly independent, otherwise it is linearly dependent. With such a definition we get the following.

Corollary 2.1. If $K$ is closed, then all distinct eigenvalues of $V$ are linearly independent functions of variance components $\sigma_1^2, \ldots, \sigma_k^2$.

The result follows from Lemma 2.2 and Theorem 2.3 immediately.

2.2. Number and linear relation of distinct eigenvalues of covariance matrix when the corresponding set of the covariance matrix is not closed

Define

$$\mathcal{K} = \{K_{t_i}: K_{t_i} = \bar{J}_{n_1}^{t_{i1}} \otimes \cdots \otimes \bar{J}_{n_a}^{t_{ia}}, t_i = (t_{i1}, \ldots, t_{ia}), t_{ij} = 0 \text{ or } 1, i = 1, \ldots, 2^a, j = 1, \ldots, a\}$$

It is obvious that $\mathcal{K}$ is closed and the corresponding set $K$ of covariance matrix $V$ is a subset of $\mathcal{K}$.

Definition 2.4. Let $A$ and $B$ be two subsets of $\mathcal{K}$. $A$ is called the least generating set of $B$ if $A$ satisfies the following conditions: (i) $A$ is closed; (ii) $B \subseteq A$; (iii) if $C$ is a closed subset of $\mathcal{K}$ and $B \subseteq C$, then $A \subseteq C$.

Since $\mathcal{K}$ is closed, it can be easily proved that the least generating set of any subset of $\mathcal{K}$ exists uniquely. In the rest of this section we will always assume that the corresponding set $K$
of covariance matrix $V$ is not closed and $K^*$ is the least generating set of $K$. Without loss of generality we assume that

$$K^* = \{K_1, \ldots, K_k, K_{k+1}, \ldots, K_{k+s}\},$$

where $K_1, \ldots, K_k$ are all the elements of $K$, and $K_{k+1}, \ldots, K_{k+s}$ are the additive elements and belong to set $K$. Obviously, $K^*$ is the corresponding set of the following covariance matrix:

$$V^* = \sum_{i=1}^{k} \theta_i K_i + \sum_{i=k+1}^{k+s} \theta_i K_i,$$

where $\theta_{k+1} \geq 0, \ldots, \theta_{k+s} \geq 0$.

Since $K^*$ is closed, by using Theorem 2.3 it follows that $V^*$ has only $k + s$ distinct eigenvalues, denoted $\lambda_{t_1}^*, \lambda_{t_2}^*, \ldots, \lambda_{t_{k+s}}^*$, and they are defined by

$$\lambda^* = (\lambda_{t_1}^*, \lambda_{t_2}^*, \ldots, \lambda_{t_{k+s}}^*) = R^* \theta^*,$$

where $R^*$ is the relation matrix of partial ordering set $< K^*, \preceq >$, and $\theta^* = (\theta_{t_1}, \theta_{t_2}, \ldots, \theta_{t_{k+s}})'$.

By (2.4) and (2.5), $V^*$ and $\lambda_{t_i}^*$, $i = 1, 2, \ldots, k + s$ are all the functions of $\theta_{t_1}, \ldots, \theta_{t_{k+s}}$, so in the following we also write $V^*$ as $V^*(\theta_{t_1}, \ldots, \theta_{t_{k+s}})$, $\lambda_{t_i}^*$ as $\lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_{k+s}})$. In addition, in the following $[K_{t_i}]^*$ represents the right partial ordering class of $K_{t_i}$ in set $K^*$.

**Lemma 2.6.** Let $K^*$ be the least generating set of $K$. Define

$$G = \{G : G \text{ is an element or a product of any } m \text{ elements of set } K, \ m = 2, \ldots, k\}.$$

Then $K^* = G$, that is, $G$ is the least generating set of $K$.

**Proof.** Since the elements in $K$ are all idempotent and commutative, hence set $G$ is closed and includes $K$. In addition, any closed subset of $K$ is also closed with respect to the limited product, that is, if $C$ is a closed subset of $K$ and $K \subseteq C$, then $G \subseteq C$. Thus, by the definition of least generating set it follows that $G$ is the least generating set of $K$. □

**Theorem 2.4.** For any $i = 1, \ldots, k + s$, $\lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0)$ is an eigenvalue of $V$.

**Theorem 2.5.** For any eigenvalue $\tau$ of $V$, there exists an eigenvalue $\lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_k})$ of $V^*$ such that $\tau = \lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0)$.

By Theorems 2.4 and 2.5, we conclude that $\lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0), i = 1, \ldots, k + s$ are all the eigenvalues of $V$, and so $V$ has at most $k + s$ eigenvalues. However, from Theorems 2.4 and 2.5, we cannot say that $V$ has only $k + s$ distinct eigenvalues. The proofs of Theorems 2.4, 2.5 and 2.6 are deferred to the Appendix.

**Theorem 2.6.** If $t_i \neq t_j, i, j = 1, \ldots, k + s$, then $\lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0) \neq \lambda_{t_j}^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0)$.

By Theorems 2.4, 2.5 and 2.6, we can easily get the following result.
Theorem 2.7. If $K$ is not closed and $K^*$ is the least generating set of $K$, then $V$ has only $k + s$ distinct eigenvalues. Furthermore, the $k + s$ components of vector

$$\tilde{\lambda} = (\tilde{\lambda}_t_1, \ldots, \tilde{\lambda}_{t_{k+s}})' = R^*\tilde{\theta}$$

are just all the distinct eigenvalues of $V$, where $R^*$ is the relation matrix of the partial ordering set $< K^*, \preceq, \tilde{\theta} = (\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0)'$.

Proof. The first part of Theorem 2.7 is a straightforward consequence of Theorems 2.4, 2.5 and 2.6. Note that $\tilde{\lambda}_{t_i} = \tilde{\lambda}_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0)'$ for $i = 1, 2, \ldots, k + s$, the latter result follows. □

Corollary 2.2. If $K$ is not closed and $K^*$ is the least generating set of $K$. Then among the eigenvalues of $V$, there exist only $k$ eigenvalues to be linearly independent functions of variance components $\sigma^2_{t_1}, \ldots, \sigma^2_{t_k}$.

This result follows from (2.6) immediately.

3. Spectral decomposition of covariance matrix

Based on the results of last section, the unique spectral decomposition of $V$ can be given by

$$V = \sum_{i=1}^{k} \lambda_{t_i} M_{t_i},$$

where $\lambda_{t_i}$, $i = 1, \ldots, k$ are all the distinct eigenvalues of $V$ given by (2.2), $M_{t_i}$’s are the principal idempotent matrices of $V$ corresponding to eigenvalue $\lambda_{t_i}$. In the following, we give a simple procedure for obtaining all principal idempotent matrices.

Assume that $K$ is closed with respect to ordinary matrix product, it follows from (1.2) and (3.1) that

$$V = \sum_{i=1}^{k} \theta_{t_i} K_{t_i} = \sum_{i=1}^{k} \tilde{\lambda}_{t_i} M_{t_i}.$$ Denote $K = (K_{t_1}', \ldots, K_{t_k}')', M = (M_{t_1}', \ldots, M_{t_k}')'$. Therefore $V$ can be rewritten as

$$V = (\theta' \otimes I_n)K = (\lambda' \otimes I_n)M.$$ Using (2.2) yields

$$(\theta' \otimes I_n)K = (\theta' \otimes I_n)(R' \otimes I_n)M.$$ Since above equality holds for all $\sigma^2_{t_1} \geq 0, \ldots, \sigma^2_{t_{k-1}} \geq 0, \sigma^2_{t_k} > 0$, thus

$$K = (R' \otimes I_n)M.$$ Since relation matrix $R$ is nonsingular, therefore

$$M = ((R')^{-1} \otimes I_n)K.$$ We summarize our observations in the following theorem.
Theorem 3.1. If the corresponding set $K$ of $V$ is closed, then the unique spectral decomposition of $V$ is

$$V = \sum_{i=1}^{k} \lambda_i M_i,$$

(3.4)

where $\lambda_i$ and $M_i$ are defined by (2.2) and (3.3), respectively.

Example 3.1. Two-way nested classification

$$y_{ijk} = \mu + \alpha_i + \beta_{j(i)} + \epsilon_{ijk}, \quad i = 1, \ldots, a, \quad j = 1, \ldots, b, \quad k = 1, \ldots, n,$$

where $\mu$ is a fixed effect, $\alpha_i$, $\beta_{j(i)}$ and $\epsilon_{ijk}$ are random effects, all assumed to be independently and normally distributed with zero mean and respective variances $\sigma^2_\alpha$, $\sigma^2_\beta$ and $\sigma^2_\epsilon$. Then the covariance matrix of $y = (y_{111}, \ldots, y_{11n}, \ldots, y_{ab1}, \ldots, y_{abn})'$ is given by

$$V = bn\sigma^2_\alpha (I_a \otimes \bar{J}_b \otimes \bar{J}_n) + n\sigma^2_\beta (I_a \otimes I_b \otimes \bar{J}_n) + \sigma^2_\epsilon (I_a \otimes I_b \otimes I_n).$$

(3.5)

By above notations, the corresponding set $K$ of covariance matrix $V$ in Eq. (3.5) is

$$K = \{K_{011}, K_{001}, K_{000}\}, \quad \theta = (\theta_{011}, \theta_{001}, \theta_{000})',$$

where $K_{011} = I_a \otimes \bar{J}_b \otimes \bar{J}_n$, $K_{001} = I_a \otimes I_b \otimes \bar{J}_n$, $K_{000} = I_a \otimes I_b \otimes I_n$, $\theta_{011} = bn\sigma^2_\alpha$, $\theta_{001} = n\sigma^2_\beta$, $\theta_{000} = \sigma^2_\epsilon$.

Since $K$ is closed, it follows from Theorem 2.3 that $V$ has only three linear-independent eigenvalues. Thus, we obtain that $V$ has explicit maximum likelihood estimate by the results in Szatrowski and Miller [11]. And this will be discussed again in Section 4.

The relation matrix and its inverse matrix are given by

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$
The solutions of any parameter of is given by immediately clear that are symmetric and idempotent matrices and satisfy are the best quadratic unbiased estimators of , respectively (see [12]). And it is immediately clear that and are independently chi-square distributed. Furthermore, if we let the estimate of eigenvalue be equal to their corresponding eigenvalue, i.e.,

\[
\lambda_{011} = \lambda_{001} = \sigma^2_e + n\sigma^2_\beta + bn\sigma^2_x,
\]

\[
\lambda_{000} = \frac{1}{ab(n - 1)} y'M_{000}y.
\]

Solving the above system of equations gives

\[
\hat{\sigma}^2_e = \hat{\lambda}_{000}, \quad \hat{\sigma}^2_\beta = (\hat{\lambda}_{001} - \hat{\lambda}_{000})/n, \quad \hat{\sigma}^2_x = (\hat{\lambda}_{011} - \hat{\lambda}_{001})/bn.
\]

The solutions \(\hat{\sigma}^2_e, \hat{\sigma}^2_\beta, \hat{\sigma}^2_x\) are nothing but the ANOVA estimators of the variance components, which can, of course, also be derived by other ways.

In the remainder of this section we consider the spectral decomposition of , where the corresponding set \(K\) of \(V\) is not closed.

In the following, we still use the notations and assumptions of Section 2, that is, \(K^*\) defined by (2.3) is the least generating set of \(K\) and \(V^*\) defined by (2.4) is the corresponding covariance matrix of \(K^*\).

Since \(K^*\) is closed, from Theorem 3.1 it follows that the unique spectral decomposition of \(V^*\) is given by

\[
V^* = V^*(\theta_1, \ldots, \theta_{k+s}) = \sum_{i=1}^{k+s} \lambda^*_i (\theta_1, \ldots, \theta_{k+s}) M^*_i,
\]

(3.6)

where \(\lambda^*_i (\theta_1, \ldots, \theta_{k+s}), i = 1, \ldots, k + s\) are defined by (2.5) and \(M^*_i, i = 1, \ldots, k + s\) are defined as

\[
(M^*_1, \ldots, M^*_{k+s})' = ((R^*)^{-1} \otimes I_n) K^*.
\]

(3.7)

where \(R^*\) has the same definition as in (2.5) and \(K^* = (K'_1, \ldots, K'_{k+s})'\).

If we set \(\theta_i = 0, i = k + 1, \ldots, k + s\) in (3.6), noting that \(M^*_i, i = 1, \ldots, k + s\) do not involve any parameter of \(\theta_i, i = 1, \ldots, k + s\), we obtain

\[
V = V^*(\theta_1, \ldots, \theta_k, 0, \ldots, 0) = \sum_{i=1}^{k+s} \lambda^*_i (\theta_1, \ldots, \theta_k, 0, \ldots, 0) M^*_i = \sum_{i=1}^{k+s} \lambda^*_i M^*_i.
\]

(3.8)

Noting that \(\lambda^*_i, i = 1, \ldots, k + s\) are all the distinct eigenvalues of \(V\) and \(M^*_i, i = 1, \ldots, k + s\) are symmetric and idempotent matrices and satisfy \(M^*_i M^*_j = 0, i \neq j\) and \(\sum_{i=1}^{k+s} M^*_i = I\), from
Theorem 3.2. If $\mathbf{K}$ is not closed and $\mathbf{K}^*$ is the least generating set of $\mathbf{K}$ and $\mathbf{V}^*$ is the corresponding covariance matrix of $\mathbf{K}^*$. Then the unique spectral decomposition of $\mathbf{V}$ is (3.8).

Example 3.2. Two-way crossed classification with interaction

$$y_{ijk} = \mu + x_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}, \quad i = 1, \ldots, a, \quad j = 1, \ldots, b, \quad k = 1, \ldots, n,$$

where $\mu$ is a fixed effect and $x_i, \beta_j, \gamma_{ij}$ and $\epsilon_{ijk}$ are random effects. It is assumed that $\text{Var}(x_i) = \sigma^2_x, \text{Var}(\beta_j) = \sigma^2_\beta, \text{Var}(\gamma_{ij}) = \sigma^2_\gamma$ and $\text{Var}(\epsilon_{ijk}) = \sigma^2_\epsilon$. Then the covariance matrix of $\mathbf{Y} = (y_{111}, \ldots, y_{11n}, \ldots, y_{ab1}, \ldots, y_{abn})'$ is

$$V = \mathbf{bn}\sigma^2_x(I_a \otimes \tilde{J}_b \otimes \tilde{J}_n) + an\sigma^2_\beta(J_a \otimes I_b \otimes \tilde{J}_n) + n\sigma^2_\gamma(I_a \otimes I_b \otimes \tilde{J}_n) + \sigma^2_\epsilon(I_a \otimes I_b \otimes I_n).$$

(3.9)

Now it is possible to derive the spectral decomposition of $\mathbf{V}$ defined by (3.9) according to Theorem 3.2. In this example

$$\mathbf{K} = \{K_{011}, K_{101}, K_{001}, K_{000}\}, \quad \theta = (\theta_{011}, \theta_{101}, \theta_{001}, \theta_{000})',$$

where $K_{011} = I_a \otimes \tilde{J}_b \otimes \tilde{J}_n, K_{101} = \tilde{J}_a \otimes I_b \otimes \tilde{J}_n, K_{001} = I_a \otimes I_b \otimes \tilde{J}_n, K_{000} = I_a \otimes I_b \otimes I_n, \theta_{011} = bn\sigma^2_x, \theta_{101} = an\sigma^2_\beta, \theta_{001} = n\sigma^2_\gamma, \theta_{000} = \sigma^2_\epsilon$.

It is readily verified that $\mathbf{K}$ is not closed and its least generating set is given by

$$\mathbf{K}^* = \{K_{011}, K_{101}, K_{001}, K_{000}, K_{111}\},$$

where $K_{111} = \tilde{J}_a \otimes \tilde{J}_b \otimes \tilde{J}_n$. The relation matrix of partial ordering set $< \mathbf{K}^*, \preceq >$ and its inverse are

$$R^* = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}, \quad R^{-1} = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 1 & 0 & 1
\end{pmatrix}.$$ 

By Theorem 2.7 we know that $\mathbf{V}$ has following five distinct eigenvalues:

$$\begin{pmatrix}
\gamma_{011} \\
\gamma_{101} \\
\gamma_{001} \\
\gamma_{000} \\
\gamma_{111}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
\theta_{011} \\
\theta_{101} \\
\theta_{001} \\
\theta_{000} \\
0
\end{pmatrix} = \begin{pmatrix}
bn\sigma^2_x + n\sigma^2_\gamma + \sigma^2_\epsilon \\
an\sigma^2_\beta + n\sigma^2_\gamma + \sigma^2_\epsilon \\
n\sigma^2_\gamma + \sigma^2_\epsilon \\
\sigma^2_\epsilon \\
bn\sigma^2_x + an\sigma^2_\beta + n\sigma^2_\gamma + \sigma^2_\epsilon
\end{pmatrix}.$$
From (3.7), the corresponding principal idempotent matrices are, respectively, given by
\[
\begin{pmatrix}
M_{011} & M_{101} & M_{001} & M_{000} \\
M_{101} & M_{111} \\
M_{001} & M_{000} \\
M_{000} & M_{000}
\end{pmatrix} =
\begin{pmatrix}
I_{abn} & 0 & 0 & -I_{abn} \\
0 & I_{abn} & 0 & -I_{abn} \\
-I_{abn} & -I_{abn} & I_{abn} & 0 \\
0 & 0 & -I_{abn} & I_{abn} \\
0 & 0 & 0 & I_{abn}
\end{pmatrix}
\begin{pmatrix}
K_{011} & K_{101} & K_{001} & K_{000} \\
K_{101} & K_{111} \\
K_{001} & K_{000} \\
K_{000} & K_{111}
\end{pmatrix} =
\begin{pmatrix}
E_a \otimes \bar{J}_b \otimes \bar{J}_n \\
\bar{J}_a \otimes E_b \otimes \bar{J}_n \\
E_a \otimes E_b \otimes \bar{J}_n \\
I_{a} \otimes I_{b} \otimes E_{N}
\end{pmatrix},
\]
where \( E_c = I_c - \bar{J}_c \). Thus, by Theorem 3.2, the spectral decomposition of \( V \) is
\[
V = (bn\sigma_x^2 + n\sigma_j^2 + \sigma_e^2) E_a \otimes \bar{J}_b \otimes \bar{J}_n + (an\sigma_{\beta_j}^2 + n\sigma_j^2 + \sigma_e^2) \bar{J}_a \otimes E_b \otimes \bar{J}_n + (n\sigma_j^2 + \sigma_e^2) E_a \otimes E_b \otimes \bar{J}_n + \sigma_e^2 I_a \otimes I_b \otimes E_n + (bn\sigma_x^2 + an\sigma_{\beta_j}^2 + n\sigma_j^2 + \sigma_e^2) \bar{J}_a \otimes \bar{J}_b \otimes \bar{J}_n.
\]

The method given in this section to obtain the spectral decomposition can be easily applied to any covariance matrix of a model for balanced data consisting of sums of Kronecker products in \( I \) and \( J \). Finally, the summary of our approach is as follows:

**Case 1**: The corresponding set \( K \) of covariance matrix \( V \) is closed.
1. Write out the corresponding set \( K \) and vector \( \theta \) according to the expression of the covariance matrix \( V \).
2. Obtain the relation matrix \( R \) of the partial ordering set \( < K, \preceq > \) and its inverse \( R^{-1} \).
3. Calculate all the distinct eigenvalues of \( V \) by (2.2) and the corresponding principal idempotent matrices by (3.3).
4. Inserting these results in (3.4) yields the spectral decomposition of \( V \).

**Case 2**: The corresponding set \( K \) of covariance matrix \( V \) is not closed.
1. Write out the corresponding set \( K \) of the covariance matrix \( V \).
2. Get the least generating set \( K^* \) of \( K \) and write out the vector \( \hat{\theta} \).
3. Obtain the relation matrix \( R^* \) of the partial ordering set \( < K^*, \preceq > \) and its inverse \( R^{*-1} \).
4. Calculate all the distinct eigenvalues of \( V \) by (2.6) and the corresponding principal idempotent matrices by (3.7).
5. Inserting these results in (3.8) gives the spectral decomposition of \( V \).

**4. Applications**

In this section, we will apply the results derived in Sections 2 and 3 to the problem of studying the estimation of variance components of the variance components model for balanced data.

**4.1. Explicit solution for MLE**

If \( X \beta \) in model (1.1) can be rewritten in the form \( X_1\beta_1 + X_2\beta_2 + \cdots + X_s\beta_s \) and matrix \( X_i \) has similar form to the design matrices of the random effects \( \xi_i \), that is, \( X_i \) can be also expressed as a Kronecker product of identity matrix and column vector of ones, then the model (1.1) is referred as a balanced mixed model of the analysis of variance (see, e.g., [11]).

Theorem 3 in Szatrowski and Miller [11] has presented a procedure for checking whether or not explicit maximum likelihood estimates exist for the variance components in the balanced mixed
model of the analysis of variance. We can yield another simple method by applying the results in Section 2.

Theorem 4.1. For the balanced mixed model of the analysis of variance, when y is distributed according to an n-dimensional normal distribution, V has explicit maximum likelihood estimates for the model without the variance components constraint (the requirement $\sigma_i \geq 0$ is referred as the “variance components constraint”) if and only if the corresponding set $K$ of covariance matrix V is closed.

Theorem 4.1 can be obtained by combing Corollary 2.1 with the Theorem 2 in Szatrowski and Miller [11]. By Theorem 4.1, it is obvious that the covariance matrix V of Example 3.1 has explicit maximum likelihood estimates without the variance components constraint but the covariance matrix V of Example 3.2 has not.

For the general variance components model for balanced data (1.1), we can also give a simple procedure for checking whether or not explicit maximum likelihood estimates exist for the variance components.

Theorem 4.2. Assume that y distributed according to an n-dimensional normal distribution in model (1.1). If $P_XV = VP_X$ and the corresponding set $K$ of covariance matrix V is closed, then V has explicit maximum likelihood estimates for the model without the variance components constraint, where $P_X = X(X'X)^{-1}X'$.

Note that when $P_XV = VP_X$, the maximum likelihood estimates for mean $X\beta$ is equal to $X(X'X)^{-1}X'y$. Then we can obtain Theorem 4.2. By Corollary 2.1 and Theorem 2 in Szatrowski and Miller [11].

4.2. Unbiased nonnegative estimates

Pukelsheim [6] provided a general criterion for the existence of unbiased nonnegative quadratic estimates for a linear combination of variance components. In this section, by the results obtained in Section 3, we will transform the criterion to a form that can be easily applied in practice.

Theorem 4.3. Assume that $P_XV = VP_X$ and the corresponding set $K$ of covariance matrix V is closed in model (1.1). Then there exists an unbiased nonnegative quadratic estimate for a linear function of variance components $\varphi = c'\sigma^2$ if and only if $\varphi = c'\sigma^2 = l_1\lambda_{t_1} + \cdots + l_k\lambda_{t_k}$, where $l_i \geq 0, i = 1, \ldots, k$ and $\lambda_{t_i}, i = 1, \ldots, k$ are the all distinct eigenvalues of V.

Proof. Let

$$B = \left\{ \sum_{i=1}^{k} b_{t_i} N_X K_{t_i} N_X : b_{t_1}, \ldots, b_{t_k} \in \mathbb{R} \right\},$$

where $N_X = I - P_X$. By the assumptions that $P_XV = VP_X$ and $K$ is closed, it follows that B is a commutative $k$-dimensional quadratic subspace of symmetric matrices (see [9]).

By Theorem 3.1, the spectral decomposition of covariance matrix V is

$$V = \sum_{i=1}^{k} \lambda_{t_i} M_{t_i}.$$
Hence

\[ N_X M_{t_1} N_X, N_X M_{t_2} N_X, \ldots, N_X M_{t_k} N_X \]

is a basis of pairwise orthogonal projectors for \( B \). Consequently, it follows from Corollary 3 in Pukelsheim [6] that \( \varphi = c' \sigma^2 \) has an unbiased nonnegative quadratic estimate if and only if all components of \( (DR')^{-1} c \) are nonnegative, where \( D = \text{diag}(N_{t_1}, \ldots, N_{t_k}) \) and \( R \) is as specified in (2.2). If we let \( (DR')^{-1} c = l \), then from Theorem 2.3 it follows that

\[ \varphi = c' \sigma^2 = l'R D \sigma^2 = l' \lambda, \]

where \( \lambda = (\lambda_{t_1}, \ldots, \lambda_{t_k})' \). Thus the result follows. \( \square \)

As an application of Theorem 4.3, we now consider Example 3.1 again. Note that the assumptions of Theorem 4.3 are satisfied for two-way nested random effects model in Example 3.1. Hence, by Theorem 4.3, it follows that the necessary and sufficient conditions for a function

\[ \varphi = c_1 \sigma_x^2 + c_2 \sigma_R^2 + c_3 \sigma_e^2 = l_1 \lambda_{011} + l_2 \lambda_{001} + l_3 \lambda_{000}. \]

With the results obtained in Example 3.1, that is, \( \lambda_{011} = \sigma_x^2 + n \sigma_R^2 + bn \sigma_x^2, \lambda_{001} = \sigma_e^2 + n \sigma_R^2, \lambda_{000} = \sigma_e^2 \), we obtain

\[
\begin{pmatrix}
l_1 \\
l_2 \\
l_3
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{bn} & 0 & 0 \\
-\frac{1}{bn} & \frac{1}{n} & 0 \\
0 & -\frac{1}{n} & 1
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix}.
\]

Hence \( \varphi = c_1 \sigma_x^2 + c_2 \sigma_R^2 + c_3 \sigma_e^2 \) exists an unbiased nonnegative quadratic estimate if and only if \( bn c_3 \geq bc_2 \geq c_1 \geq 0 \).

Acknowledgments

We would like to thank the editor and the referees for their many constructive comments and helpful suggestions which helped to improve an earlier version of this article.

Appendix A. Proofs

Proof of Theorem 2.1. Denote \( \mathcal{T} \) the set of all subscripts to the elements of set \( K \), and define \( \overline{\mathcal{T}}_{t_i} = \mathcal{T} - \mathcal{T}_{t_i} \). Based on the definition of \( \mathcal{T}_{t_i} \) and \( \overline{\mathcal{T}}_{t_i} \), the covariance matrix \( V \) defined by (1.2) can be rewritten as

\[
V = \sum_{j=1}^{k} \theta_{t_j} K_{t_j} = \sum_{t_j \in \mathcal{T}_{t_i}} \theta_{t_j} K_{t_j} + \sum_{t_j \in \overline{\mathcal{T}}_{t_i}} \theta_{t_j} K_{t_j}.
\]
For the fixed $t_i$, define matrix $E_{t_i} = E_{n_1}^{t_i} \otimes E_{n_2}^{t_i} \otimes \cdots \otimes E_{n_a}^{t_i}$, where $E_{n_j}^0 = I_{n_j} - \bar{J}_{n_j}$, $E_{n_j}^1 = \bar{J}_{n_j}$. It is obvious that matrix $E_{t_i}$ is a nonzero matrix.

Consider the case where $t_j \in \bar{T}_i$. It is easy to see that $K_{t_i} \preceq K_{t_j}$. Hence from Lemma 2.4 it follows that $\tilde{J}_{n_r}^{t_j} E_{n_r}^{t_j} = E_{n_r}^{t_j}$ for $r = 1, \ldots, a$. Thus

$$
\left( \sum_{t_j \in \bar{T}_i} \theta_{t_j} K_{t_j} \right) E_{t_i} = \left( \sum_{t_j \in \bar{T}_i} \theta_{t_j} (\tilde{J}_{n_1}^{t_j} \otimes \tilde{J}_{n_2}^{t_j} \otimes \cdots \otimes \tilde{J}_{n_a}^{t_j}) \right) E_{t_i}
$$

$$
= \sum_{t_j \in \bar{T}_i} \theta_{t_j} \left( \tilde{J}_{n_1}^{t_j} E_{n_1}^{t_j} \otimes \tilde{J}_{n_2}^{t_j} E_{n_2}^{t_j} \otimes \cdots \otimes \tilde{J}_{n_a}^{t_j} E_{n_a}^{t_j} \right)
$$

$$
= \sum_{t_j \in \bar{T}_i} \theta_{t_j} \left( E_{n_1}^{t_j} \otimes E_{n_2}^{t_j} \otimes \cdots \otimes E_{n_a}^{t_j} \right) = \left( \sum_{t_j \in \bar{T}_i} \theta_{t_j} \right) E_{t_i}. \tag{A.2}
$$

Secondly, if $t_j \in \bar{T}_i$, then $K_{t_i}$ and $K_{t_j}$ do not satisfy relation $\preceq$. Hence from the definition of partial ordering $\preceq$ it follows that for any $t_j \in \bar{T}_i$ there exist $r$ ($1 \leq r \leq a$) such that $t_{ir} = 0$ and $t_{jr} = 1$, so $\tilde{J}_{n_r}^{t_j} E_{n_r}^{t_j} = \bar{J}_{n_r}$, $(I_{n_r} - \bar{J}_{n_r}) = 0$. Thus

$$
\left( \sum_{t_j \in \bar{T}_i} \theta_{t_j} K_{t_j} \right) E_{t_i} = \left( \sum_{t_j \in \bar{T}_i} \theta_{t_j} \left( \tilde{J}_{n_1}^{t_j} \otimes \tilde{J}_{n_2}^{t_j} \otimes \cdots \otimes \tilde{J}_{n_a}^{t_j} \right) \right) E_{t_i}
$$

$$
= \sum_{t_j \in \bar{T}_i} \theta_{t_j} \left( \tilde{J}_{n_1}^{t_j} E_{n_1}^{t_j} \otimes \tilde{J}_{n_2}^{t_j} E_{n_2}^{t_j} \otimes \cdots \otimes \tilde{J}_{n_a}^{t_j} E_{n_a}^{t_j} \right) = 0. \tag{A.3}
$$

It follows from (A.2) and (A.3) that $V E_{t_i} = \left( \sum_{t_j \in \bar{T}_i} \theta_{t_j} \right) E_{t_i}$, so $\sum_{t_j \in \bar{T}_i} \theta_{t_j}$ is an eigenvalue of $V$.

It remains to prove $\sum_{t_j \in \bar{T}_i} \theta_{t_j} \neq \sum_{t_j \in \bar{T}_i} \theta_{t_j}$ if $t_i \neq t_j$. It is easy to see that we need only to prove $[K_{t_i}] \neq [K_{t_j}]$ if $t_i \neq t_j$.

From the definition of right partial ordering class it is clear that $K_{t_i} \in [K_{t_i}]$ and $K_{t_j} \in [K_{t_j}]$. If $K_{t_i} \not\subseteq [K_{t_i}]$, then $[K_{t_i}] \neq [K_{t_i}]$ holds. Otherwise, if $K_{t_i} \subseteq [K_{t_i}]$, then since the partial ordering is not symmetric and $t_i \neq t_j$ it follows that $K_{t_i} \not\subseteq [K_{t_i}]$, and so $[K_{t_i}] \neq [K_{t_i}]$ holds. The proof is completed. \(\square\)

**Proof of Theorem 2.2.** From Searle and Henderson [8] we know that any eigenvalue of $V$ is a sum of some elements of set $\{\theta_{t_1}, \theta_{t_2}, \ldots, \theta_{t_k}\}$. Thus, without loss of generality let $\tau = \sum_{t_j \in \bar{T}_i} \theta_{t_j}$ be an eigenvalue of $V$, where $\bar{T}_i$ is a subset of $T$ defined in Theorem 2.1. To prove Theorem 2.2 we need only to verify that the set $\{K_{t_j}: t_j \in \bar{T}_i\}$ is a right partial ordering class.

Define $\bar{T}_i = T - \bar{T}_i$, then the matrix $V$ can be rewritten as

$$
V = \sum_{j=1}^{k} \theta_{t_j} K_{t_j} = \sum_{t_j \in \bar{T}_i} \theta_{t_j} K_{t_j} + \sum_{t_j \in \bar{T}_i} \theta_{t_j} K_{t_j}. \tag{A.4}
$$
Moreover, we suppose that vector $\xi(\neq 0)$ is an eigenvector of $V$ associated with eigenvalue $\tau$, that is, $V\xi = \tau\xi$. Thus, it follows from (A.4) that

$$
\sum_{t_j \in T_1} \theta_{t_j} K_{t_j} \xi + \sum_{t_j \in \overline{T}_1} \theta_{t_j} K_{t_j} \xi = \sum_{t_j \in T_1} \theta_{t_j} \xi. \tag{A.5}
$$

Since (A.5) holds for all $\sigma_1^2 \geq 0, \ldots, \sigma_{k-1}^2 \geq 0, \sigma_k^2 > 0$ and $\xi$ is independent of $\sigma^2$'s (see [8]), so we have

$$
K_{t_j} \xi = \begin{cases} 
\xi & \text{if } t_j \in T_1, \\
0 & \text{if } t_j \in \overline{T}_1.
\end{cases} \tag{A.6}
$$

With the assumption that $K$ is a closed set it follows that $\prod_{t_j \in T_1} K_{t_j} \in K$. If $\prod_{t_j \in \overline{T}_1} K_{t_j} \notin \{ K_{t_j} : t_j \in T_1 \}$, then there must exists $t_b \in \overline{T}_1$ such that $\prod_{t_j \in \overline{T}_1} K_{t_j} = K_{t_b}$, and so $\prod_{t_j \in T_1} K_{t_j} \xi = K_{t_b} \xi$. By (A.6) it follows that $\xi = 0$. It is contradictory with the assumption that $\xi \neq 0$. Thus, there exists a $t_c \in T_1$ such that $\prod_{t_j \in T_1} K_{t_j} = K_{t_c}$, that is, $K_{t_c} \in \{ K_{t_j} : t_j \in T_1 \}$. In the following we will prove that $\{ K_{t_j} : t_j \in T_1 \} = [K_{t_c}]$, that is, the set $\{ K_{t_j} : t_j \in T_1 \}$ is just the right partial ordering class of $K_{t_c}$.

Since $K_{t_c}$, $t_j \in T$ is idempotent matrix and satisfies commutativity with respect to ordinary matrix product, so $K_{t_c} K_{t_j} = K_{t_c}$ for any $t_j \in T_1$. Thus, from the definition of right partial ordering class it follows that $\{ K_{t_c} : t_j \in T_1 \} \subseteq [K_{t_c}]$. It remains to be proved that $K_{t_c}$ and $K_{t_j}$ do not satisfy relation $\preceq$ for any $t_j \in \overline{T}_1$.

Suppose that there exists a $t_d \in \overline{T}_1$ such that $K_{t_c} \preceq K_{t_d}$. By (A.6), we have $K_{t_c} \xi = \xi$ and $K_{t_d} \xi = 0$. Thus, using the definition of the relation $\preceq$ it follows that

$$
\xi = K_{t_c} \xi = K_{t_c} K_{t_d} \xi = 0.
$$

This is contradictory to the assumption that $\xi \neq 0$, so $K_{t_j} \notin [K_{t_c}]$ for any $t_j \in \overline{T}_1$. Hence $\{ K_{t_j} : t_j \in T_1 \} = [K_{t_c}]$. The proof of Theorem 2.2 is completed. $\square$

**Proof of Theorem 2.4.** For any $t_i$, since $\lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_{k+s}})$ is an eigenvalue of $V^*(\theta_{t_1}, \ldots, \theta_{t_{k+s}})$, thus

$$
V^*(\theta_{t_1}, \ldots, \theta_{t_{k+s}}) \xi = \lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_{k+s}}) \xi, \tag{A.7}
$$

where $\xi$ is an eigenvector of $V^*(\theta_{t_1}, \ldots, \theta_{t_{k+s}})$ corresponding to eigenvalue $\lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_{k+s}})$. Let $\theta_{t_{k+1}} = 0, \ldots, \theta_{t_{k+s}} = 0$ in (A.7), we get

$$
V^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0) \xi = \lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0) \xi.
$$

Note that $V^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0) = V$, hence $\lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0)$ is an eigenvalue of $V$. $\square$

**Proof of Theorem 2.5.** Suppose that there exists an eigenvalue $\tau$ of $V$ which is not equal to any of $\lambda_{t_i}^*(\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0)$, $i = 1, 2, \ldots, k+s$. Let $\xi$ be an eigenvector of $V$ corresponding to $\tau$, then we have

$$
V \xi = \tau \xi.
$$
and so

\[ V^* \zeta = V \zeta + \sum_{i=k+1}^{k+s} \theta_i K_{t_i} \zeta = \tau \zeta + \sum_{i=k+1}^{k+s} \theta_i K_{t_i} \zeta. \]  

(A.8)

In addition, from the proof of Theorem 2.2 we know that (A.6) is always true whether the corresponding set \( K \) of covariance matrix \( V \) is closed with respect to general matrix product or not. And by Lemma 2.6, for any \( i = k + 1, \ldots, k + s \), \( K_{t_i} \) can be expressed as product of some matrices of \( K_{t_i}, i = 1, \ldots, k \). Therefore, \( K_{t_i} \zeta = \zeta \) or 0 for any \( i = k + 1, \ldots, k + s \). Using this fact, (A.8) can be rewritten as

\[ V^* \zeta = \tau \zeta + \tau^* (\theta_{k+1}, \ldots, \theta_{k+s}) \zeta, \]  

(A.9)

where \( \tau^* (\theta_{k+1}, \ldots, \theta_{k+s}) \) is a sum of some parameters of \( \theta_{k+1}, \ldots, \theta_{k+s} \). From (A.9) it follows that \( \tau + \tau^* (\theta_{k+1}, \ldots, \theta_{k+s}) \) is an eigenvalue of \( V^* \), and with the assumption we get \( \tau + \tau^* (\theta_{k+1}, \ldots, \theta_{k+s}) \) is not equal to any of \( \lambda_i^* (\theta_{t_1}, \ldots, \theta_{t_m}), i = 1, 2, \ldots, k + s \). Thus we get \( V^* \) has at least \( k + s + 1 \) distinct eigenvalues, which is contradictory with the result that \( V^* \) has only \( k + s \) distinct eigenvalues. Hence, for any eigenvalue \( \tau \) of \( V \), there exists an eigenvalue \( \lambda_i^* (\theta_{t_1}, \ldots, \theta_{t_k}, 0, \ldots, 0) \).  

Proof of Theorem 2.6. It is clear that to prove Theorem 2.6 we need only to prove that \( \{K_i\}^* \cap K \neq \{K_i\}^* \cap K \), for any \( i \neq j, i, j, 1, \ldots, k + s \).

Suppose that there exist \( a, b \) (\( a \neq b \)) among \( 1, 2, \ldots, k + s \) such that \( \{K_i\}^* \cap K = \{K_i\}^* \cap K \). For any \( A \subseteq \{K_i\} \), if \( A \subseteq \{K_i\}^* \cap K \), then from the above assumption it follows that \( A \subseteq \{K_i\}^* \). On the other hand, if \( A \neq \{K_i\}^* \cap K \), then \( A \) is among \( K_{t_{k+1}}, \ldots, K_{t_{k+s}} \). Thus, from Lemma 2.6 we get that there exist \( c_1, c_2, \ldots, c_m \) among \( 1, 2, \ldots, k \) such that \( A = K_{t_{k+1}} K_{t_{k+2}} \cdots K_{t_{k+s}} \). Since \( K_{t_i} \leq A \), and \( A \leq K_{t_i} \) for any \( c_i \) among \( c_1, c_2, \ldots, c_m \), it follows that \( K_{t_{k+1}} \leq [K_{t_{i+c}}] \leq A \). Thus, from the assumption that \( K_{t_{i+c}} \subseteq [K_{t_{k+1}}] \), we obtain that \( K_{t_{i+c}} = K_{t_{i+c}} = c_1, c_2, \ldots, c_m \) belong to \( [K_{t_{k+1}}] \), and so \( c_i = c_1, c_2, \ldots, c_m \). Using this result we get \( K_{t_{i+c}} = A = K_{t_{i+c}} K_{t_{j+1}} K_{t_{j+2}} \cdots K_{t_{k+s}} = K_{t_{i+c}} \), so from the definition of right partial ordering class it follows that \( A \) also belongs to \( [K_{t_{k+1}}] \). Thus, we get \( [K_{t_{k+1}}] \subseteq [K_{t_{k+1}}] \). On the other hand, we can also derive \( \{K_i\}^* \Sigma [K_{t_{k+1}}] \) by a similar manner. Hence it follows that \( \{K_i\}^* = [K_{t_{k+1}}] \). However, it is contradictory with the fact that \( \{K_i\}^* \neq [K_{t_{k+1}}] \) if \( t_i \neq t_{j} \). Thus, the proof is completed.

References