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Numerical Computations for Singular Semilinear Elliptic Boundary Value Problems

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Abstract—The paper studies a class of Dirichlet problems with homogeneous boundary conditions for singular semilinear elliptic equations in a bounded smooth domain in $\mathbb{R}^n$. A numerical method is devised to construct an approximate Green's function by using radial basis functions and the method of fundamental solutions. An estimate of the error involved is also given. A weak solution of the above given problem is a solution of its corresponding nonlinear integral equation. A computational method is given to find the minimal weak solution U, and the critical index λ^* (such that a weak solution U exists for $\lambda < \lambda^*$, and U does not exist for $\lambda > \lambda^*$). © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $\partial\Omega$ be its boundary, and q be a real number greater than 2. We consider the following singular semilinear elliptic boundary value problem:

$$-\Delta U = \lambda g(x)h(U), \qquad \text{in } \Omega,\ U = 0,\ \text{on } \partial\Omega, \tag{1}$$

where $g \in L^q(\Omega)$ is positive and may have singularities in Ω, and $h(U) \in C^1([0,c))$ for some positive constant c such that $h(0) > 0$, $h' \geq 0$, and $\lim_{U\to c^-} h(U) = \infty$. Let $G(x;\xi)$ denote Green's function of $-\Delta U = 0$ in Ω corresponding to the boundary condition $U = 0$ on $\partial\Omega$. Then,

$$U(x) = \lambda \int_\Omega G(x;\xi)g(\xi)h(U(\xi))\,d\xi. \tag{2}$$

A weak solution of the Dirichlet problem (1) is a solution of the nonlinear integral equation (2). We call λ^* the critical index of problem (1) if a weak solution U exists for $\lambda < \lambda^*$, and U does not exist for $\lambda > \lambda^*$.

When $g(x) \equiv 1$, computational methods for λ^* have been given by Chan [1] using Green's function for elliptic domains, Chan and Chen [2] using fundamental solutions (cf. [3]) with the dual reciprocity method, and Chan and Ke [4] using the finite-difference method for rectangular domains.

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Due to the singularity of $g(x)$, none of the methods used above works for problem (1). Our main purpose here is to introduce a numerical method to compute Green's function and the critical index λ^*. In Section 2, we introduce a numerical method to construct an approximate Green's function by using radial basis functions and the method of fundamental solutions. An estimate of the error involved is also given. In Section 3, we give examples to illustrate the errors involved in computing the approximate Green's function, the computation of the minimal weak solution U, and the critical index λ^*.

As an application of our study here, we note that problem (1) is the steady-state of the following singular (quenching) problem:

$$\begin{aligned} u_t - \Delta u &= \lambda g(x)h(u), && \text{in } \Omega, \\ u(x,0) &= 0, && \text{on } \bar{\Omega},\ u(x,t) = 0,\ \text{on } \partial\Omega \times (0,T), \end{aligned}$$

where $T \leq \infty$, and $\bar{\Omega}$ denotes the closure of Ω. A solution u is said to quench if there exists a finite time T such that

$$\sup_{x\in\Omega} u(x,t) \to c^-, \qquad \text{as } t \to T.$$

The critical index λ^* corresponds to the size of the domain such that a solution u does not quench for $\lambda < \lambda^*$, and u quenches at some finite time T for $\lambda > \lambda^*$.

2. NUMERICAL METHOD

We note that

$$G(x;\xi) = \gamma(x;\xi) + \Gamma(x;\xi),$$

where $\gamma(x;\xi)$ is the fundamental solution of the operator Δ, and $\Gamma(x;\xi)$ is the solution of the following problem:

$$-\Delta\Gamma(x;\xi) = 0, \qquad \text{in } \Omega,\ \Gamma(x;\xi) = -\gamma(x;\xi),\ \text{on } \partial\Omega. \tag{3}$$

Since $\gamma(x;\xi)$ is known, we use the method of fundamental solutions to construct $\Gamma(x;\xi)$. For any fixed $\xi \in \Omega$,

$$\Gamma(x;\xi) = \int_{\partial B} \sigma(\eta;\xi)\gamma(x;\eta)\, d\eta,$$

where $\sigma(\eta;\xi)$ is an unknown source density function, and B is an arbitrary domain containing Ω with ∂B denoting the boundary of B (cf. [5]). Let us approximate $\sigma(\eta;\xi)$ by

$$\sigma_N(\eta;\xi) = \sum_{k=1}^{N} c_k(\xi)\delta(\eta - \eta_k), \qquad \text{for } \eta \in \partial B,$$

where $\{\eta_k\}_{k=1}^N$ is a set of points on the boundary ∂B. Hence,

$$\Gamma(x;\xi) \simeq \sum_{k=1}^{N} c_k(\xi)\gamma(x;\eta_k).$$

By selecting a set of points $\{x_j\}_{j=1}^N$ on $\partial\Omega$, we have

$$\sum_{k=1}^{N} c_k(\xi)\gamma\left(x_j;\eta_k\right) = -\gamma\left(x_j;\xi\right), \qquad j = 1,2,3,\ldots,N.$$

To construct Green's function, let us choose a set of points $\{\xi_i\}_{i=1}^m$ in Ω, and compute

$$\left\{c_k\left(\xi_i\right)\right\}_{k=1}^N, \qquad \text{for } i = 1,2,3,\ldots,m.$$

Once $\{\tilde{\Gamma}(x;\xi_i)\}_{i=1}^m$ is known, we construct $\tilde{\Gamma}(x;\xi)$ by interpolation. Since $\tilde{\Gamma}(x;\xi)$ is an approximation, it need not satisfy (3). Thus, there exists a function $\varepsilon(x;\xi)$ such that

$$-\Delta\tilde{\Gamma}(x;\xi) = \varepsilon(x;\xi), \qquad \text{in } \Omega,\ \tilde{\Gamma}(x;\xi) = -\gamma(x;\xi),\ \text{on } \partial\Omega.$$

Let $w = \tilde{\Gamma} - \Gamma$. We have

$$-\Delta w = \varepsilon(x;\xi), \qquad \text{in } \Omega,\ w = 0,\ \text{on } \partial\Omega.$$

Thus,

$$\begin{aligned} |w(x;\xi)| &= \left| \int_\Omega G(x;\eta)\varepsilon(\eta;\xi)\,d\eta \right| \\ &\le \|\varepsilon\|_\infty \int_\Omega G(x;\eta)\,d\eta. \end{aligned}$$

From

$$\int_{\Omega\times\Omega} w^2\,dx\,d\xi \le \|\varepsilon\|_\infty^2 \int_{\Omega\times\Omega} \left(\int_\Omega G(x;\eta)\,d\eta \right)^2 dx\,d\xi,$$

we have an estimate of the error: $\|w\|_2 \le C_1\|\varepsilon\|_\infty$, where C_1 is a constant depending only on Ω.

For the linear problem,

$$-\Delta U = p(x), \qquad \text{in } \Omega,\ U = 0,\ \text{on } \partial\Omega,$$

once $\tilde{\Gamma}(x;\xi)$ is known, we can use (2) to compute an approximation $\tilde{U}(x)$ of $U(x)$. If $\tilde{G}(x;\xi)$ is an approximation of $G(x;\xi)$, then

$$\begin{aligned} \left| U(x) - \tilde{U}(x) \right| &= \left| \int_\Omega \left[G(x;\eta) - \tilde{G}(x;\eta) \right] p(\eta)\,d\eta \right| \\ &= \left| \int_\Omega w(\eta)p(\eta)\,d\eta \right| \\ &\le \|p\|_2\,\|w\|_2 \\ &\le C_2\,\|p\|_2\,\|\varepsilon\|_\infty \end{aligned}$$

for some positive constant C_2.

For the nonlinear case, we have the successive iterations:

$$\begin{aligned} U^{(0)}(x) &= 0, \\ U^{(n+1)}(x) &= \lambda \int_\Omega G(x;\eta)g(\eta)h\left(U^{(n)}(\eta)\right)\,d\eta, \qquad n = 0,1,2,\ldots. \end{aligned}$$

Since g and h are positive in Ω, and h is a nondecreasing function, it follows that $\{U^{(n)}\}$ is a nondecreasing sequence. If $\{U^{(n)}\}$ converges to a function $U(x)$ pointwise, then by the monotone convergence theorem, $U(x)$ is a solution of the integral equation (2). Because the construction starts with a lower solution, it follows that U is the minimal solution.

To construct $U^{(n)}$, we use the following algorithm. Let $n = 0$, and we choose a set of points $\{\xi_i\}_{i=1}^m \subset \Omega$.

(1) We construct Green's function $\tilde{G}(x;\eta)$ according to the above algorithm.
(2) For $n = n + 1$, we compute $\{U^{(n)}(\xi_i)\}_{i=1}^m$, and construct $U^{(n)}(x)$ in Ω.
(3) We compute $\delta = \|U^{(n)} - U^{(n-1)}\|_\infty$, and if δ is less than a given tolerance, we stop; otherwise, we repeat Step (2).

3. EXAMPLES

We give several two-dimensional numerical examples by using Meschach (a linear algebra package in C, developed by School of Mathematical Sciences, Australian National University), GNU C, and Mathematica. We also use the thinplate spline interpolation to construct the approximate Green's function and the function $U^{(n)}$ from the discrete data. Let

$$s\left(x^{(j)}\right) = f_j, \qquad j = 1, 2, 3, \ldots, N,$$

where $\{x^{(j)}\}_{j=1}^N$ is a given set of points in $\mathbb{R}^2$, and $\{f_j\}_{j=1}^N$ is a given set of real numbers. In the thinplate spline interpolation, we seek a smooth function $s(\cdot) : \mathbb{R}^2 \mapsto \mathbb{R}$ by minimizing the following integral:

$$I(s) = \int_{\mathbb{R}^2} \left[\left(\frac{\partial^2 s}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 s}{\partial x_1 \, \partial x_2} \right)^2 + \left(\frac{\partial^2 s}{\partial x_2^2} \right)^2 \right] dx.$$

Duchon [6] showed that $s(\cdot)$ has the form

$$s(x) = \sum_{j=1}^{N} \beta_j \left\| x - x^{(j)} \right\|_2^2 \ln \left\| x - x^{(j)} \right\|_2 + p(x), \qquad x \in \mathbb{R}^2,$$

where $\{\beta_j\}_{j=1}^N$ is a set of real coefficients, $p(x)$ is a linear polynomial, and $I(s)$ is finite if and only if

$$\sum_{j=1}^{N} \beta_j = \sum_{j=1}^{N} \beta_j x_1^{(j)} = \sum_{j=1}^{N} \beta_j x_2^{(j)} = 0.$$

Thus, to find the interpolation, we only need to solve

$$\begin{pmatrix} A & P \\ P^{\mathsf{T}} & 0 \end{pmatrix} \begin{pmatrix} \beta \\ a \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

where

$$A = \left(\left\| x^{(i)} - x^{(j)} \right\|_2^2 \ln \left\| x^{(i)} - x^{(j)} \right\|_2 \right),$$

$$P = \begin{pmatrix} 1 & x_1^{(1)} & x_2^{(1)} \\ & \vdots & \\ 1 & x_1^{(N)} & x_2^{(N)} \end{pmatrix},$$

$$f = (f_1, f_2, f_3, \ldots, f_N)^{\mathsf{T}},$$

$$\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_N)^{\mathsf{T}},$$

$$a = (a_1, a_2, a_3).$$

3.1. Green's Function

For illustration, let us consider a unit disk,

$$\Omega = \left\{ (x_1, x_2) : x_1^2 + x_2^2 < 1 \right\}.$$

Its Green's function is given by

$$G(x_1, x_2; \xi_1, \xi_2) = \frac{1}{2\pi} \ln \left(\frac{\rho R_2}{R_1} \right),$$

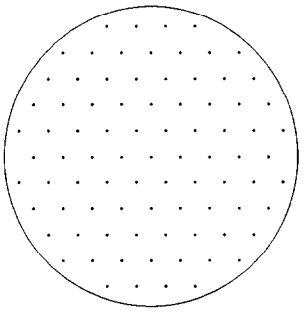

Figure 1. 85 grid points in a unit disk.

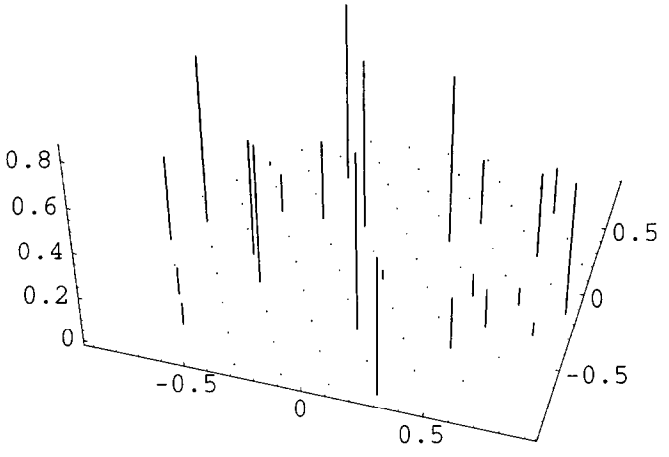

Figure 2. Errors of the approximated Green's function with magnification factor 10^3.

where ρ is the distance between $(0,0)$ and (ξ_1,ξ_2), R_1 is the distance between (ξ_1,ξ_2) and (x_1,x_2), and R_2 is the distance between $(\xi_1,\xi_2)/\rho^2$ and (x_1,x_2). Let us take 50 points on the boundary $\partial\Omega$ and 85 points in Ω. See Figure 1.

We choose ∂B to be $x_1^2+x_2^2=4$. Figure 2 shows the accumulated errors of 100 random points in Ω for some fixed (ξ_1,ξ_2).

3.2. Linear Equation

Let us consider

$$-\Delta U = \frac{\left|\ln\left[(x_1-0.5)^2+x_2^2\right]\right|}{4}$$

in the ellipse domain,

$$\Omega = \left\{(x_1,x_2) : \frac{x_1^2}{2}+x_2^2<1\right\}.$$

As shown in Figure 3, we set up the grid with 100 boundary points on $\partial\Omega$ and 125 interior points in Ω.

We choose ∂B to be $x_1^2/8+x_2^2/4=1$. With this setting, the solution is shown in Figure 4.

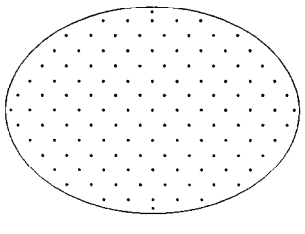

Figure 3. 125 grid points in an ellipse.

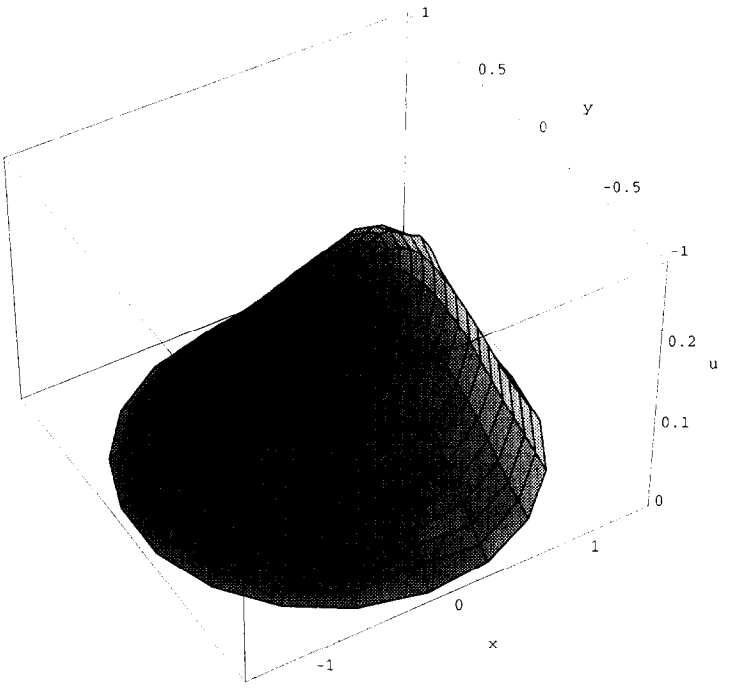

Figure 4. Linear problem.

3.3. Nonlinear Equation

Let us consider

$$-\Delta U = \lambda \frac{\left|\ln\left[(x_1 - 0.5)^2 + x_2^2\right]\right|}{1 - U}$$

in the elliptic domain,

$$\Omega = \left\{(x_1, x_2) : \frac{x_1^2}{2} + x_2^2 < 1\right\},$$

with the same grid setting as in the previous example. For $\lambda = 1$, we have the minimal solution U as shown in Figure 5.

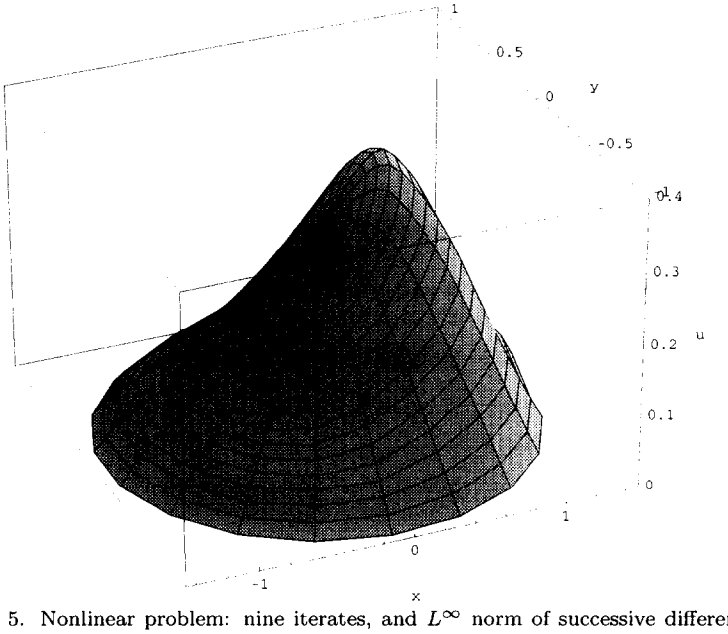

Figure 5. Nonlinear problem: nine iterates, and L^∞ norm of successive difference = 0.000096.

3.4. Critical Index

Let us consider the problem

$$-\Delta U = \lambda \frac{\left|\ln\left|(x_1 - 0.5)^2 + x_2^2\right|\right|}{1 - U}, \qquad \text{in } \Omega,\ U = 0,\ \text{on } \partial\Omega.$$

Let U_1 be the solution of

$$-\Delta U = \left|\ln\left|(x_1 - 0.5)^2 + x_2^2\right|\right|, \qquad \text{in } \Omega,\ U = 0,\ \text{on } \partial\Omega.$$

We note that $\lambda^{(0)} = 1/\|U_1\|_\infty$ is an upper bound of λ^*. Since 0 is a lower bound of λ^*, we develop the following bisection procedure to determine the critical index λ^*.

STEP 1. The first estimate of λ^* is $\lambda^{(1)} = \lambda^{(0)}/2$.

STEP 2. For each approximation $\lambda^{(n)}$, we compute the sequence $\{U^{(i)}\}$ defined by $U^{(0)} = 0$ on $\bar{\Omega}$, and for $i \geq 1$,

$$-\Delta U^{(i)} = \lambda^{(n)} \frac{\left|\ln\left|(x_1 - 0.5)^2 + x_2^2\right|\right|}{1 - U^{(i-1)}}, \qquad \text{in } \Omega,\ U^{(i)} = 0,\ \text{on } \partial\Omega.$$

If $\{U^{(i)}\}$ is bounded by 1 and converges, then we take this value $\lambda^{(n)}$ as a lower bound of λ^*; otherwise, it is an upper bound of λ^*.

STEP 3. For $n = n + 1$, if $|\lambda^{(n-1)} - \lambda^{(n-2)}| < \epsilon$ (a given tolerance), then $(1/2)(\lambda^{(n-1)} + \lambda^{(n-2)})$ is accepted as the final estimate of the critical index, and we stop; otherwise, we go to the next step.

STEP 4. We update $\lambda^{(n)}$ by the following criterion: if the sequence $\{U^{(i)}\}$ in Step 2 is bounded by 1 and converges, then

$$\lambda^{(n)} = \lambda^{(n-1)} + \frac{1}{2}\left|\lambda^{(n-1)} - \lambda^{(n-2)}\right|;$$

otherwise,

$$\lambda^{(n)} = \lambda^{(n-1)} - \frac{1}{2}\left|\lambda^{(n-1)} - \lambda^{(n-2)}\right|.$$

STEP 5. We go back to Step 2.

Applying the above algorithm, we obtain 1.205 for the critical index λ^*. We remark that the above bisection method works for any positive $g(x) \in L^q(\Omega)$ with $q > 2$.

We note that Green's function needs to be computed only once, even for the nonlinear problem. Once Green's function is obtained, the domain integration required for obtaining $U(x)$ (for the linear case or in each successive iterate for the nonlinear case) is computationally intensive. However, from the representation formula, the computations of $U(x)$ at different points x are independent of each other. Thus, a parallel algorithm can be used to compute $U(x)$ at different x at the same time. Such a parallelization is sometimes called an "embarrassing parallelization". In the iterative procedure for solving the semilinear problem, parallel computations can be used to integrate the integrals at different x in each iterate since each iterate corresponds to a linear problem. We also remark that these algorithms are not only natural, but also excellent for distributed computations on a network cluster such as PVM and MPI, because they require only a very small amount of communication between subtasks.

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