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Tree-visibility orders

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Abstract

We introduce a new class of partially ordered sets, called tree-visibility orders, containing interval orders, duals of generalized interval orders and height one orders. We give a characterization of tree-visibility orders by an infinite family of minimal forbidden suborders. Furthermore, we present an efficient recognition algorithm for tree-visibility orders. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

The motivation of this work is to extend the class of interval orders in a fashion similar to the extension of interval graphs to chordal graphs (for more details on these graph classes we refer to [7]). A survey about two other generalizations of interval orders, one allowing intervals to overlap with a given ratio and the second dealing with intervals of partial but no more total order, has been done by Bogart [1]. Another generalization of interval orders dealing with convex subsets of partial but no more total order, has been introduced by Müller and Rampon [9]. This generalization is close to the one presented by Bogart [1] when convex subsets are restricted to be intervals. Generalized interval orders form a class of orders considered by Faigle et al. [3], extending the successor set inclusion property of interval orders.

We have chosen the characterization of chordal graphs as intersection graphs of subtrees of a tree and the 'visibility definition' of interval orders for extending interval orders. The combination of these two concepts leads to a class of partially ordered sets defined via visibility in a rooted directed tree. Thus tree-visibility orders are exactly the class of orders defined (in the sense of Müller and Rampon [9]) by convex subsets of partial orders whose transitive reduction is an in-rooted directed tree. By definition, the

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tree-visibility orders contain all interval orders. Furthermore, they also contain the dual order of any generalized interval order and all height one orders. Our major contributions are a characterization of tree-visibility orders by an infinite family of minimal forbidden suborders and an efficient recognition algorithm for tree-visibility orders.

As this is the case for graph classes, we are convinced that there must be more interesting classes of orders worth to be studied. In fact we are sure that tree-visibility orders form an interesting new class of orders and we hope that they have nice structural properties still to be established. The tree model of a tree-visibility order is likely to support the design of efficient algorithms. For example, PRECEDENCE CONSTRAINT 3-PROCESSOR SCHEDULING (problem [OPEN8] of [5]) might be solvable by a polynomial time algorithm when restricted to tree-visibility orders. Recall that PRECEDENCE CONSTRAINT 3-PROCESSOR SCHEDULING is one of the remaining problems in the list of 12 open problems in [5] for which the algorithmic complexity is still unknown. For relations to coding of orders we refer the reader to Section 6.

2. Preliminaries

Most of the terminology on partially ordered sets (for short orders), used in this paper, can be found in the book of Trotter [10]. For graph theoretic notions we refer to Bondy and Murty [2].

We mention some order-theoretic definitions. Let $P = (V(P), \prec_P)$ be an order. Two elements $u, v \in V$ are *comparable*, denoted by $u \sim_P v$, if $u \prec_P v$ or $v \prec_P u$. If $u \neq v$ and neither $u \prec_P v$ nor $v \prec_P u$ then u and v are *incomparable*, denoted by $u \parallel_P v$. The *comparability graph* of an order P, denoted by $G(P)$, is an undirected graph with vertex set V and two vertices $u, v \in V$ are joined by an edge if and only if u and v are comparable. The undirected graph $\overline{G(P)} = (V(P), E)$ with $E = \{\{x,y\} : x \parallel_P y\}$ is the complement of the comparability graph of P and it is called the *cocomparability graph* of P.

We denote the set of all maximal (respectively, minimal) elements of P by MAX(P) (respectively, MIN(P)). $\mathrm{Pred}(x) := \{y \in V(P) : y \prec_P x\}$ and $\mathrm{Succ}(x) := \{y \in V(P) : x \prec_P y\}$ are the predecessor set and the successor set, respectively, of an element $x \in V(P)$. An element $x \in \mathrm{MAX}(P)$ is said to be *universal* if its predecessor set $\mathrm{Pred}(x) := \{y \in V(P) : y \prec_P x\}$ is equal to $V(P) - \mathrm{MAX}(P)$. Hence a maximal element x is universal if it is comparable to all elements of P except the maximal ones.

The *height* of P is the number of elements of a maximum size chain minus one. Given any subset $A \subseteq V(P)$ the *suborder of P induced by A* is the order denoted $P[A]$ such that $V(P[A]) = A$ and for any $a, b \in A$ we have $a \prec_{P[A]} b$ if and only if $a \prec_P b$. For short we denote by $P - A$ the suborder $P[V \setminus A]$.

The *dual* poset of P is the poset P^* such that $V(P) = V(P^*)$ and for any $a, b \in V(P)$ we have $a \prec_P b$ if and only if $b \prec_{P^*} a$.

A class $\mathscr{P}$ of orders is *hereditary* if $P \in \mathscr{P}$ implies that any suborder P' of P belongs to $\mathscr{P}$. Many interesting classes of orders are hereditary, as e.g. interval orders

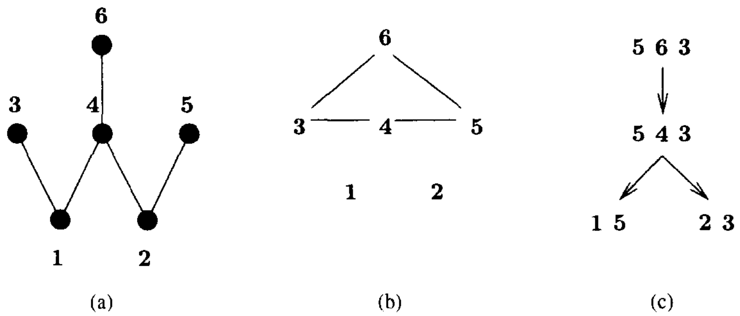

Fig. 1. In (a) a tree-visibility order P is given by its Hasse diagram. In (b) the forcing graph (see Section 3) of P, that is not chordal, is depicted. In (c) we give a visibility tree for P. The nodes of $\boldsymbol{T}$ are labeled in such a way that for any $x \in V(P)$ the subtree $\boldsymbol{T}_x$ is induced by all nodes of $\boldsymbol{T}$ having label x.

and two dimensional orders. If a class $\mathscr{P}$ is hereditary then it can be characterized by the (possibly infinite) list of all its minimal forbidden suborders, where Q is a *minimal forbidden suborder* of the class $\mathscr{P}$ if $Q \notin \mathscr{P}$ but any proper suborder of Q belongs to $\mathscr{P}$. Then an order P belongs to the hereditary class $\mathscr{P}$ if and only if none of the minimal forbidden suborders of $\mathscr{P}$ is contained as a suborder in P. This nice type of characterization is certainly a very powerful tool for studying structural properties of orders as well as for applications of certain classes of orders.

For an order P depicted by its Hasse diagram (Fig. 1) we assume that given two elements a, b we have $a \prec_P b$ if they are connected by an edge and a is below b.

Now we introduce a new class of orders extending the class of interval orders. Notice that we assume that in a rooted directed tree each edge is directed away from the root.

Definition 1. An order P is a *tree-visibility order* if there exists a rooted directed tree $\boldsymbol{T} = (V(\boldsymbol{T}), E(\boldsymbol{T}))$ and a one-to-one mapping from $V(P)$ to a family $\langle \boldsymbol{T}_x,\ x \in V(P)\rangle$ of directed rooted subtrees of $\boldsymbol{T}$ such that $u \prec_P v$ if and only if

(i) $V(\boldsymbol{T}_u) \cap V(\boldsymbol{T}_v) = \emptyset$, and

(ii) there are $x_v \in V(\boldsymbol{T}_v)$ and $x_u \in V(\boldsymbol{T}_u)$ such that there is a directed path from x_v to x_u in $\boldsymbol{T}$.

The rooted directed tree $\boldsymbol{T}$ is said to be a *visibility tree* of P and the tuple $(\boldsymbol{T}, \langle \boldsymbol{T}_x, x \in V(P)\rangle)$ is said to be a *tree-visibility model* of P.

Of course several elements of a tree-visibility order P may be associated to the same subtree of a visibility tree $\boldsymbol{T}$ of P.

Remark 1. Condition (*ii*) can be replaced by the condition

(ii') there is a $x_v \in V(\boldsymbol{T}_v)$ such that for any $x_u \in V(\boldsymbol{T}_u)$ there is a directed path from x_v to x_u in $\boldsymbol{T}$.

Remark 2. Let P be a tree-visibility order and $(\boldsymbol{T}, \langle \boldsymbol{T}_x,\ x \in V(P)\rangle)$ a tree-visibility model of P. Then

(i) for any $u \in V(P)$ the (unique) directed path $\mathscr{P}(u)$ from the root of $\boldsymbol{T}$ to the root of $\boldsymbol{T}_u$ contains the root of the subtree $\boldsymbol{T}_v$ for any $v \in V$ with $u \prec_P v$.

(ii) for any pair $u, v \in V(P)$ we have $u \prec_P v$ if and only if $V(\boldsymbol{T}_u) \cap V(\boldsymbol{T}_v) = \emptyset$ and the root of $\boldsymbol{T}_v$ is an ancestor of the root of $\boldsymbol{T}_u$ in the visibility tree $\boldsymbol{T}$.

Clearly, a tree-visibility order may have various visibility trees and, a tree-visibility order P may have different tree-visibility models $(\boldsymbol{T}, \langle \boldsymbol{T}_x,\ x \in V(P)\rangle)$ for a fixed visibility tree $\boldsymbol{T}$. Hence it is natural to look for tree-visibility models that are minimal in a certain sense.

Remark 3. Let $(\boldsymbol{T}, \langle \boldsymbol{T}_x,\ x \in V(P)\rangle)$ be a tree-visibility model of order P and let P' be a suborder of P induced by the set $A \subseteq V(P)$. Then $(\boldsymbol{T}, \langle \boldsymbol{T}_x,\ x \in A\rangle)$ is a tree-visibility model of P'.

Therefore the class of tree-visibility orders is hereditary.

Our characterization of tree-visibility orders, given in Section 4, directly implies that height one orders as well as interval orders form subclasses of the class of tree-visibility orders. To enhance the familiarity of the reader with tree-visibility models, we show how to obtain a tree-visibility model for these well-known classes of orders.

Height one orders. Let $A = \{a_1, a_2, \dots, a_r\}$, $r \geq 1$, be the set of minimal elements of the height one order P and let $B = \{b_1, b_2, \dots, b_s\}$, $s \geqslant 0$, be $V(P) \setminus A$.

We construct a visibility tree $\boldsymbol{T}$ of P as follows. The vertex set of $\boldsymbol{T}$ is $V(\boldsymbol{T}) = \{u\} \cup \{v_1, v_2, \dots, v_r\}$. The edge set of $\boldsymbol{T}$ is $E(\boldsymbol{T}) = \{(u, v_i) : i = 1, 2, \dots, r\}$. The subtrees $\boldsymbol{T}_x$ are induced subtrees of $\boldsymbol{T}$, hence it suffices to give their vertex sets. For any $a_i \in A$ we take $V(T_{a_i}) = \{v_i\}$ and for any $b_j \in B$ we take $V(T_{b_j}) = \{u\} \cup \{v_i :\ a_i \parallel_P b_j\}$.

Interval orders. The visibility tree $\boldsymbol{T}$ is a directed path for which the vertices correspond to the endpoints of the intervals in the interval model of the interval order P. The subtree $\boldsymbol{T}_x$ associated to the element x of P is a directed subpath and consists of all vertices associated to interval endpoints r with $a(x) \leqslant r \leqslant b(x)$, where $a(x)$ (respectively $b(x)$) denotes the left endpoint (respectively right endpoint) of the interval associated to x.

3. Chordal sandwich graphs

In this section we derive a necessary condition for an order to be a tree-visibility order.

Lemma 2. *Let P be a tree-visibility order and $(\boldsymbol{T}, \langle \boldsymbol{T}_x,\ x \in V(P)\rangle)$ a tree-visibility model of P. Then $V(\boldsymbol{T}_x) \cap V(\boldsymbol{T}_y) \neq \emptyset$ for any pair of incomparable elements $x, y \in V(P)$ having a common predecessor z.*

Proof. Let x and y be two incomparable elements with a common predecessor z. Thus the root of $\boldsymbol{T}_x$ and the root of $\boldsymbol{T}_y$ occur on the unique directed path $\mathscr{P}(z)$ from the root of $\boldsymbol{T}$ to the root of $\boldsymbol{T}_z$. Hence there is either a directed path from the root of $\boldsymbol{T}_x$ to the root of $\boldsymbol{T}_y$ or vice versa. Hence x and y have $V(\boldsymbol{T}_x) \cap V(\boldsymbol{T}_y) \neq \emptyset$ since they are incomparable. □

Lemma 2 leads to the following concept of a forcing graph which is helpful when studying tree-visibility orders.

Definition 3. Let P be an order. The undirected graph $G = (V(P), E)$ with $E = \{\{x, y\} : x \parallel_P y$ for which x and y have a common predecessor$\}$ is called the *forcing graph* of P.

Thus the forcing graph of an order P is a subgraph of the cocomparability graph of P. We are going to show that the existence of a tree-visiblity model for an order P requires that there exists a chordal sandwich graph between the forcing graph and the cocomparability graph of P. The concept of a sandwich graph has been introduced and extensively studied by Golumbic et al. [8].

Definition 4. A graph G is a *spanning subgraph* of the graph G' if both graphs have the same vertex set and G is a subgraph of G' (i.e. $E(G) \subseteq E(G')$).

Let $G = (V, E)$ be a spanning subgraph of the graph $G' = (V, E')$. Then $H = (V, E(H))$ is said to be a *sandwich graph* for (G, G') if G is a spanning subgraph of H and H is a spanning subgraph of G' (i.e. $E(G) \subseteq E(H) \subseteq E(G')$).

Now we are able to formulate our necessary condition for tree-visibility orders.

Theorem 5. *Let P be an order with forcing graph G and cocomparability graph $G' = \overline{G(P)}$. If P is a tree-visibility order then there exists a chordal sandwich graph H for (G, G').*

Proof. Let P be a tree-visibility order and $(\boldsymbol{T}, \langle \boldsymbol{T}_x, x \in V(P) \rangle)$ a tree-visibility model of P. Let T be the underlying undirected tree of $\boldsymbol{T}$ and for any $x \in V(P)$ let T_x be the underlying undirected tree of $\boldsymbol{T}_x$. Hence $\langle T_x : x \in V \rangle$ is a family of subtrees of the tree T. Let $H = (V(P), E(H))$ be the vertex intersection graph of the subtrees T_x, $x \in V(P)$, i.e., $u, v \in V(P)$ are adjacent in H if and only if $V(T_u) \cap V(T_v) \neq \emptyset$. Hence H is a chordal graph since it is the intersection graph of subtrees of a tree [6]. The forcing graph G of P is a spanning subgraph of H, since $\{u, v\} \in E(G)$ implies $V(\boldsymbol{T}_u) \cap V(\boldsymbol{T}_v) \neq \emptyset$ by Lemma 2, hence $\{u, v\} \in E(H)$. H is a spanning subgraph of the cocomparability graph G' of P since $\{u, v\} \in E(H)$ implies $V(\boldsymbol{T}_u) \cap V(\boldsymbol{T}_v) \neq \emptyset$ thus $u \parallel_P v$, by Definition 1. Consequently, H is a chordal sandwich graph for (G, G'). □

4. A characterization of tree-visibility orders

The aim of this section is to determine all minimal forbidden suborders for the class of tree-visibility orders.

Definition 6. Let $k \geqslant 1$. The order Q_k is defined as follows. The groundset of Q_k is $V(Q_k) = \{a_1, a_2, \ldots, a_k\} \cup \{b_1, b_2, \ldots, b_{k+1}\} \cup \{c_1, c_2\}$. Furthermore, $a_i \prec_{Q_k} b_j$ if and only if $j \in \{i, i+1\}$, $a_i \prec_{Q_k} c_j$ for all $i \in \{1, 2, \ldots k\}$ and $j \in \{1, 2\}$, $b_i \prec_{Q_k} c_1$ for $i \in \{1, 2, \ldots k\}$ and $b_i \prec_{Q_k} c_2$ for $i \in \{2, 3, \ldots k+1\}$. (See Fig. 2.)

Theorem 7. *The order Q_k is a minimal forbidden suborder for the class of tree-visibility orders for all $k \geqslant 1$.*

Proof. First we show that for any $k \geqslant 1$ the order Q_k is not a tree-visiblity order. Assume that Q_k would be a tree-visibility order for some $k \geqslant 1$. We consider the forcing graph G of Q_k. The graph G is not chordal, since it contains the chordless cycle $C = (b_1, b_2, \ldots, b_{k+1}, c_1, c_2, b_1)$. By Theorem 5, there is a chordal sandwich graph H for the pair (G, G') where G' is the cocomparability graph of Q_k. Taking the vertices c_1 and c_2 of the cycle C there is no vertex b_j, $j \in \{1, 2, \ldots, k+1\}$, in the cycle C adjacent to c_1 and c_2 in H since the only neighbors of c_1 in G' are c_2 and b_{k+1} and the only neighbors of c_2 in G' are c_1 and b_1. Take c_1 and c_2 and the vertices of a shortest path between c_1 and c_2 in the graph obtained from $H[C]$, the graph induced in H by the vertices of C, by deleting the edge $\{c_1, c_2\}$. This vertex set induces a chordless cycle of length at least 4 in H. Hence, H is not chordal. Consequently, by Theorem 5, Q_k is not a tree-visibility order for all $k \geqslant 1$.

Now it is a matter of routine to construct a tree-visibility model for any proper suborder $Q_k - \{x\}$, $x \in V(Q_k)$, of Q_k and any $k \geqslant 1$ (see Fig. 3 for a tree-visibility model of $Q_4 - \{b_3\}$). □

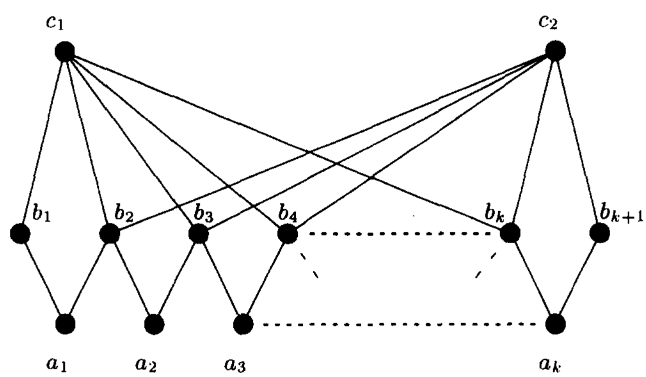

Fig. 2. The order Q_k Hasse diagram.

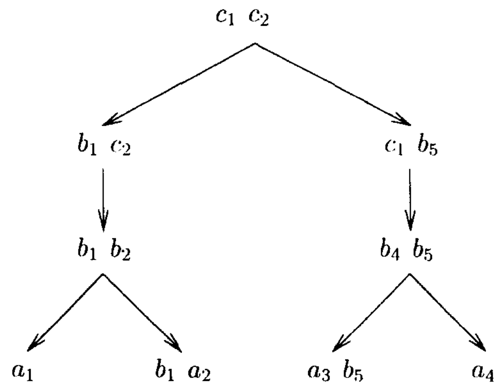

Fig. 3. The tree visibility model produced by the algorithm TREE-VISIBILITY of Section 5 for the order $Q_4 - \{b_3\}$.

For showing that the set of all orders Q_k, $k \geqslant 1$, is exactly the set of all minimal forbidden suborders for the class of tree-visibility orders, we present Proposition 8 and Lemma 9. Both of them are also crucial for the recognition algorithm for tree-visibility orders, that we present in the next section.

We shall need some more concepts for the proof of the main theorem. An order P is said to be *connected* if its comparability graph $G(P)$ is connected. Let u and v be elements of a connected order P. Then there is a shortest u,v-path $(u = x_0, x_1, \ldots, x_r = v)$ in $G(P)$ such that the internal vertices $x_1, \ldots, x_{r-1}$ of the path are alternately minimal and maximal elements of P. Such a u,v-path is said to be *normalized*. To see that a normalized path exists for any pair $u, v \in V(P)$, take any shortest u,v-path $(u = y_0, y_1, \ldots, y_r = v)$ in $G(P)$. Then either $y_{i-1} \prec_P y_i$ and $y_{i+1} \prec_P y_i$, or $y_i \prec_P y_{i-1}$ and $y_i \prec_P y_{i+1}$ for any $i \in \{1, \ldots, r-1\}$. If $y_i \notin \mathrm{MAX}(P)$ in the first case then replace it by a maximal element y_i' that is a successor of y_i. If $y_i \notin \mathrm{MIN}(P)$ in the second case then replace it by a minimal element y_i' that is a predecessor of y_i. This leads to a normalized path between u and v.

In the remainder of the paper we consider only normalized paths $(u = x_0, x_1, \ldots, x_r = v)$ between maximal elements of an order. Thus in any path $(u = x_0, x_1, \ldots, x_r = v)$, x_i is a maximal element if i is even and x_i is a minimal element if i is odd. Moreover, r is even.

Proposition 8. *Let P be a connected order with no universal element such that $P - \mathrm{MAX}(P)$ is connected. Then there is a $k \geqslant 1$ such that P contains Q_k as a suborder.*

Proof. Let P be an order that fulfills the assumptions of the theorem. We denote the connected suborder $P - \mathrm{MAX}(P)$ by P'. We say that a maximal element x of P has a *private predecessor* p_x if $p_x \prec_P x$ and $p_x \parallel_P y$ for all $y \in (\mathrm{MAX}(P) \setminus \{x\})$.

Case 1: *P has a maximal element x with a private predecessor p_x*. W.l.o.g. p_x is a maximal element of P'. Since x is not a universal element of P, there are elements $u \in \mathrm{MAX}(P')$ with $u \parallel_P x$. We choose $t \in \mathrm{MAX}(P') \cap \{u : u \parallel_P x\}$ such that the length of a shortest path between t and p_x in $G(P')$, is minimum among all elements $u \in \mathrm{MAX}(P')$ that are incomparable to x. Let $(p_x = x_0, x_1, \ldots, x_{2s} = t)$, $s \geqslant 1$, be a normalized p_x, t-path in P'. Clearly the set $A = \{p_x = x_0, x_1, \ldots, x_{2s} = t\}$ induces a fence in P'. Furthermore $x_{2i} \in \mathrm{MAX}(P')$ for all $i \in \{0, 1, \ldots, s\}$. By the choice of t we have $x_{2i} \prec_P x$ for all $i \in \{0, 1, \ldots, s-1\}$.

Since t is not a maximal element of P, there is a $y \in \mathrm{MAX}(P)$ with $t \prec_P y$. Furthermore, there is a j with $x_{2j} \parallel_P y$ since p_x is a private predecessor of $x = x_0$ implying $x \parallel_P y$. Now let j be the largest subscript such that $x_{2j} \parallel_P y$. Then the set $\{x, x_{2j}, x_{2j+1}, \ldots, x_{2s} = t, y\}$ induces a Q_{s-j} in P.

Case 2: *No maximal element of P has a private predecessor.* We choose among all elements of $\mathrm{MAX}(P')$ an element w having a successor set of minimum cardinality. Then let $R \subseteq \mathrm{MAX}(P)$ be a subset of $\mathrm{Succ}(w)$ containing all but one of the successors of w in P. Let x be the only successor of w in P not belonging to R. Notice that $R \neq \emptyset$, otherwise w would be a private predecessor of the maximal element x of P. By the choice of R, every maximal element of P' belongs to $P - R$ and has at least one successor in $P - R$. Thus, the maximal elements of the order $P - R$ are exactly the elements of $\mathrm{MAX}(P) \setminus R$. Hence the order $(P - R) - (\mathrm{MAX}(P - R))$ is exactly P' and thus connected. Furthermore, $P - R$ has no universal element since any universal element $u \in \mathrm{MAX}(P - R)$ of $P - R$ had to fulfill $\mathrm{MAX}(P') \subseteq \mathrm{Pred}(u)$ which would imply that u is universal in P, a contradiction. Finally w is private predecessor of x in $P - R$.

Altogether, $P - R$ fulfills the assumptions of Case 1. Hence, $P - R$ has a Q_k for some $k \geqslant 1$ as a suborder. Hence Q_k is also a suborder of P. □

We will need the following technical lemma.

Lemma 9. *Let P be a tree-visibility order. Then for any tree-visibility model $(\boldsymbol{T}, \langle \boldsymbol{T}_x, x \in V(P)\rangle)$ of P there is another tree-visibility model of P on the same visibility tree $\boldsymbol{T}$, say $(\boldsymbol{T}, \langle \boldsymbol{T}'_x,\ x \in V(P)\rangle)$, such that the root of $\boldsymbol{T}$ belongs to $\boldsymbol{T}'_x$ for all $x \in \mathrm{MAX}(P)$.*

Proof. Let $R(\boldsymbol{T})$ be the root of $\boldsymbol{T}$, and let $L(\boldsymbol{T}) = \{x \in V(P),\ R(\boldsymbol{T}) \in V(\boldsymbol{T}_x)\}$. Clearly $L(\boldsymbol{T}) \subseteq \mathrm{MAX}(P)$. Suppose $L(\boldsymbol{T}) \neq \mathrm{MAX}(P)$. For all $x \in \mathrm{MAX}(P) \setminus L(\boldsymbol{T})$, let $A(x)$ be the set of elements of $\boldsymbol{T}$ which do not belong to the maximal subtree of $\boldsymbol{T}$ rooted in the root of $\boldsymbol{T}_x$. For any such x, the root of $\boldsymbol{T}$ belongs to $A(x)$ and since x is a maximal element of P any element y of P such that $V(\boldsymbol{T}_y) \cap A(x) \neq \emptyset$ is incomparable to x in P. Then $(\boldsymbol{T}, \langle \boldsymbol{T}'_x,\ x \in V(P)\rangle)$, where $\boldsymbol{T}'_x$ is the subtree of $\boldsymbol{T}$ induced by $V(\boldsymbol{T}_x) \cup A(x)$ if $x \in \mathrm{MAX}(P) \setminus L(\boldsymbol{T})$ and $\boldsymbol{T}'_x = \boldsymbol{T}_x$ otherwise, is a tree-visibility model for P fulfilling the claimed property. □

This leads to the major theorem of this paper, that gives a characterization of tree-visibility orders by an infinite family of forbidden suborders of height two.

Theorem 10. *An order P is a tree-visibility order if and only if it does not contain an order Q_k, $k \geqslant 1$, (see Fig. 2) as a suborder.*

Proof. By Theorem 7, it remains to show that every order P, which does not contain an order Q_k as a suborder, has a tree-visibility model. We prove this claim by induction on the number of elements. Trivially, a one element order has a tree-visibility model.

Now let P be an order on at least two elements which does not have a Q_k as a suborder. Clearly, none of the suborders of P has an order Q_k as suborder.

Case 1: *P is not connected.* Let the orders $P_1, \ldots, P_l$, $l \geqslant 2$, be the connected components of P. By induction, any order P_i, $i \in \{1, 2, \ldots, l\}$, has a tree-visibility model $(\boldsymbol{T}^i, \langle \boldsymbol{T}^i_x,\ x \in V(P_i)\rangle)$. Let $\boldsymbol{T}$ be the rooted directed tree obtained from all the $\boldsymbol{T}^i$'s by adding a new root N_R such that its sons are the roots of the T^i's. Then $(\boldsymbol{T}, \bigcup_{i \in \{1,\ldots,k\}} \langle \boldsymbol{T}^i_x,\ x \in V(P_i)\rangle)$ is a tree-visibility model of P.

Case 2: *P is connected.*

Case 2.1: *P has a universal element.* Let u be a universal element of P. By induction and by Lemma 9, $P' = P - \{u\}$ has a tree-visibility model $(\boldsymbol{T}', \langle \boldsymbol{T}'_x,\ x \in V(P')\rangle)$ such that the root of $\boldsymbol{T}'$ belongs to $\boldsymbol{T}'_x$ for all $x \in \mathrm{MAX}(P - \{u\})$. Let $\boldsymbol{T}$ be the rooted directed tree obtained by adding a new root N_R to $\boldsymbol{T}'$ such that the unique son of N_R is the root of $\boldsymbol{T}'$. Then $(\boldsymbol{T}, \langle \boldsymbol{T}_x,\ x \in V(P)\rangle)$ with (*i*) $\boldsymbol{T}_x = \boldsymbol{T}'_x$ if $x \in V(P) \setminus \mathrm{MAX}(P)$, (*ii*) $\boldsymbol{T}_x$ is the subtree of $\boldsymbol{T}$ induced by $V(\boldsymbol{T}'_x) \cup \{N_R\}$ if $x \in \mathrm{MAX}(P) \setminus \{u\}$, and (*iii*) $\boldsymbol{T}_u$ is the subtree of $\boldsymbol{T}$ induced by $\{N_R\}$, is a tree-visibility model for P.

Case 2.2: *P has no universal element.* Proposition 8 implies that $P - \mathrm{MAX}(P)$ is not connected. Let the orders $K_1, \ldots, K_l$, $l \geqslant 2$, be the connected components of $P - \mathrm{MAX}(P)$, and let P_i, $i \in \{1, 2, \ldots, l\}$, be the suborder of P induced by $V(K_i) \cup M_i$, where M_i is the set of those elements of $\mathrm{MAX}(P)$ having at least one predecessor in $V(K_i)$. By induction and by Lemma 9, every order P_i has a tree-visibility model $(\boldsymbol{T}^i, \langle \boldsymbol{T}^i_x,\ x \in V(P_i)\rangle)$ such that the root of $\boldsymbol{T}^i$ belongs to $\boldsymbol{T}^i_x$ for all $x \in \mathrm{MAX}(P_i)$. Let $\boldsymbol{T}$ be the rooted directed tree obtained from all the $\boldsymbol{T}^i$'s by adding a root N_R such that its sons are the roots of the T^i's. Then $(\boldsymbol{T}, \langle \boldsymbol{T}_x,\ x \in V(P)\rangle)$ with (i) $\boldsymbol{T}_x = \boldsymbol{T}^i_x$ if $x \in V(P_i) \setminus \mathrm{MAX}(P)$, and (ii) $\boldsymbol{T}_x$ is the subtree of $\boldsymbol{T}$ induced by $\{N_R\} \cup \{V(T^i_x),\ x \in V(P_i)\} \cup \{V(T^i),\ x \notin V(P_i)\}$ if $x \in \mathrm{MAX}(P)$, is a tree-visibility model for P. $\square$

Since none of the minimal forbidden suborders is an interval order or an order of height one, Theorem 10 immediately implies

Corollary 11. *The class of height one orders as well as the class of interval orders are proper subclasses of the class of tree-visibility orders.*

Faigle, Schrader and Turán introduced the generalized interval orders in [3]. A linear time recognition algorithm for generalized interval orders has been given by Garbe [4].

Definition 12. An order P is said to be a *generalized interval order* if for all $x, y \in V(P)$ either $\mathrm{Succ}(x) \cap \mathrm{Succ}(y) = \emptyset$, $\mathrm{Succ}(x) \subseteq \mathrm{Succ}(y)$ or $\mathrm{Succ}(y) \subseteq \mathrm{Succ}(x)$.

Since $\mathrm{Pred}(c_1) \cap \mathrm{Pred}(c_2) \neq \emptyset$ for all Q_k, $k \geqslant 1$, and since any height one order is a tree-visibility order, Theorem 10 implies

Corollary 13. *The class of the duals of generalized interval orders is a proper subclass of the tree-visibility orders.*

5. Recognition algorithm

In this section we present an efficient algorithm to recognize tree-visibility orders. Our algorithm TREE-VISIBILITY(P) works by recursive calls of a subroutine TREE-VISIBILITY (K, N, INC).

It is started by calling TREE-VISIBILITY($P, R, \emptyset$), where R is a reference variable pointing to the future root of the eventual tree-visibility model of the given order P. The algorithm computes a visibility tree $\boldsymbol{T}$ of P, if there is one, by assigning to each node N of $\boldsymbol{T}$ a label set that is going to be the set of all those vertices u for which $\boldsymbol{T}_u$ contains the node N.

```
Subroutine TREE-VISIBILITY(K, N, INC)
  K:   /* Current order.                                        */
  N:   /* Father of the root of the subtree representing K.     */
  INC: /* Set of all elements of the label set of node N that   */
       /* are incomparable to all elements of the order K.      */
Begin
  MAX(K) := {x : x maximal in P};
  Compute the connected components K_1, K_2, ..., K_l of K − MAX(K);
  Create a node C in T with father N and label set INC ∪ MAX(K);
  If K − MAX(K) has exactly one connected component
  Then
    U(K) := {u : u universal in K};
    If U(K) = ∅
    Then
      EXIT; output "K is not a tree-visibility order."
    Else
      TREE-VISIBILITY(K − U(K); C; INC);
    EndIf
  Else/* The subroutine terminates if K − U(K) = ∅ */
    For all connected components K_i = (V(K_i), ≺_P) of K − MAX(K)
```

Do
$M_i := \left(\bigcup_{x\in V(K_i)} \text{Succ}(x)\right) \cap \text{MAX}(K)$;
$L_i := \{x \in M_i : V(K_i) \setminus \text{Pred}(x) \neq \emptyset\}$;
TREE-VISIBILITY($K[V(K_i) \cup L_i]; C$; INC $\cup$ ($\text{MAX}(K) \setminus M_i$));
EndFor
EndIf
End;

The algorithm TREE-VISIBILITY(P) terminates in two different ways. Either it outputs 'P is not a tree-visibility order' or it terminates with a tree-visibility model of the given order P, that is constructed as a tree T with a label set assigned to each node of T.

Theorem 14. *Given an order P, the algorithm TREE-VISIBILITY(P) decides whether P is a tree-visibility order. If so, the algorithm computes a tree-visibility model of P such that the number of nodes in the visibility tree is at most* $|V(P)|$. *The running time of the algorithm is* $\mathcal{O}(nm)$, *where n denotes the number of elements of P and m denotes the number of edges in the comparability graph of P.*

Proof. Suppose the algorithm terminates with the output 'P is not a tree-visibility order'. Hence a recursive call of TREE-VISIBILITY(K, N, INC) found a connected suborder K of P such that $K - \text{MAX}(K)$ is connected and has no universal element. Therefore K contains a suborder Q_k for some $k \geqslant 1$ by Proposition 8. Consequently there is a Q_k that is a suborder of P, thus P is not a tree-visibility order by Theorem 7 and the algorithm is correct in this case.

Otherwise, the algorithm TREE-VISIBILITY(P) constructs a tree T such that the reference variable R points to the root of T. This means that any subroutine TREE-VISIBILITY(K, N, INC), recursively called during the execution of the algorithm, either recursively called $l \geqslant 1$ subroutines, where l is the number of connected components of $K - \text{MAX}(K)$, or terminated by creating a leaf of the final tree T since $V(K) \setminus \text{MAX}(K) = \emptyset$. Consider T as a directed tree $\boldsymbol{T}$ with the root specified by R. For any $v \in V$ the corresponding subtree $\boldsymbol{T}_v$ consists of those nodes of T that have a label set containing v.

We show that $(\boldsymbol{T}, \langle \boldsymbol{T}_v,\ v \in V(P)\rangle)$ is a tree-visibility model of P. First we claim that $\boldsymbol{T}_v$ is a subtree of $\boldsymbol{T}$ for any $v \in V(P)$. Suppose not. Then there is a $v \in V(P)$ such that $\boldsymbol{T}_v$ is a disconnected subtree of $\boldsymbol{T}$. On the one hand if v does not belong to the label set of a node N but belongs to the label set of its father then v is not an element of the current order, when the node N is created. Moreover, since v does not belong to the label set of N, v does not belong to the current INC and hence v does not belong to an INC, for any recursive call creating a node, which is a successor of N in T. On the other hand, if v belongs to the label set of a node N and never appears before on a node in the path from the root of $\boldsymbol{T}$ to N, then v is a maximal element in one of the connected components of $\tilde{K} - \mathrm{MAX}(\tilde{K})$ where $\tilde{K}$ is the current order when N' the father node of N has been created. This guarantees that v cannot belong to the current INC when N' has been created, that v is not a maximal element of $\tilde{K}$, and that v is not an element of any of the other order obtained when applying the subroutine to the remaining connected components of $\tilde{K} - \mathrm{MAX}(\tilde{K})$. Thus N is the root of $\boldsymbol{T}_v$.

It remains to show that the final tree indeed creates a tree-visibility model of P. This follows immediately when noting that our algorithm guarantees that when calling the subroutine TREE-VISIBILITY($K;N$; INC) the set INC is actually the set of all elements of P that belong to the label set of a node in the path from the root of the tree to N and that are incomparable to all elements of K. This is ensured by the use of the auxiliary sets L_i and M_i in the For loop.

Let $u \prec_P v$. Consider the first subroutine TREE-VISIBILITY($K;N$; INC) executed during the algorithm for which $u \in \mathrm{MAX}(K)$ holds. Clearly $v \notin$ INC and $v \notin V(K)$. Hence v is not in the label set of node N and $\boldsymbol{T}_u$ and $\boldsymbol{T}_v$ have no node in common. On the other hand, there is a directed path from the root of $\boldsymbol{T}_v$ to the node N, i.e., the root of u, since K is a suborder of $\tilde{K}$, the current order when creating the root of $\boldsymbol{T}_u$.

Finally, consider the execution of TREE-VISIBILITY($K;N$; INC) and suppose $v \in V$ is not an element in the label set of node N but it appears in the label set of the father N' of N. Then v is an element that has all elements of the order K as successor. This is guaranteed by the construction of the current orders for the recursive call.

The recognition algorithm can be implemented such that its running time is $\mathcal{O}(nm)$. The important fact to notice is that the tree T, which is isomorphic to the recursion tree of the algorithm, has at most n vertices since each node N of T has in its label set an element which only appears in the label set of nodes belonging to the subtree of $\boldsymbol{T}$ rooted in N. Indeed, if N' is the father of N then there is an element in the label set of N that belongs to the maximal elements of the connected component of $K - \mathrm{MAX}(K)$ inducing the node N where K is the current order when creating the node N'. Thus this element can appear only in the label set of nodes of the subtree of $\boldsymbol{T}$ rooted in N. Finally the running time of each subroutine TREE-VISIBILITY($K;N$; INC) without counting the recursive calls is $O(m)$ when a linear time algorithm for the computation of the connected components of a graph is applied. □

Remark 4. One can also show that the height of the visibility-tree constructed by our algorithm is minimal among the height of all visibility trees $\boldsymbol{T}$ in any tree-visibility model $(\boldsymbol{T}, \langle \boldsymbol{T}_x,\ x \in V(P)\rangle)$ of the order P.

6. Conclusion

We introduced a new class of orders defined by visibility on subtrees of rooted directed trees. This definition leads to two main types of further investigation related to the efficient coding of orders:

- the characterization of orders defined by visibility on isomorphic subtrees,
- the characterization of orders defined by visibility on subtrees with a bounded number of leaves.

A well known illustration of the former type is given by semiorders and intervals orders. For the latter type recall that a subtree of a rooted directed tree is completely defined by its root and its leaves, and that a rooted directed tree is the transitive reduction of a 2-dimentional order. Thus for orders P, defined by visibility on subtrees with at most k leaves, k a fixed positive integer, one can answer the query '$x \prec_P y$?' in constant time. As noted in [9], in the case $k = 1$ the corresponding class of orders are exactly the duals of generalized interval orders, but the question is still open for any fixed k with $k \geqslant 2$.

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