Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Discrete Mathematics 309 (2009) 987–990
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

A generalization of Fan's results: Distribution of cycle lengths in graphs[☆]

Jun Zhang[a,*], Jinghua Xiang[b]

[a] *Fuzhou No 3 High School, Fuzhou, Fujian 350003, PR China*
[b] *Department of Shu Li, Fujian University of Technology, Fuzhou, Fujian 350014, PR China*

Received 27 April 2006; received in revised form 10 January 2008; accepted 10 January 2008
Available online 4 March 2008

Abstract

Fan [G. Fan, Distribution of cycle lengths in graphs, J. Combin. Theory Ser. B 84 (2002) 187–202] proved that if G is a graph with minimum degree $\delta(G) \geq 3k$ for any positive integer k, then G contains $k+1$ cycles $C_0, C_1, \ldots, C_k$ such that $k+1 < |E(C_0)| < |E(C_1)| < \cdots < |E(C_k)|$, $|E(C_i) - E(C_{i-1})| = 2$, $1 \leq i \leq k-1$, and $1 \leq |E(C_k)| - |E(C_{k-1})| \leq 2$, and furthermore, if $\delta(G) \geq 3k+1$, then $|E(C_k)| - |E(C_{k-1})| = 2$. In this paper, we generalize Fan's result, and show that if we let G be a graph with minimum degree $\delta(G) \geq 3$, for any positive integer k (if $k \geq 2$, then $\delta(G) \geq 4$), if $d_G(u) + d_G(v) \geq 6k - 1$ for every pair of adjacent vertices $u, v \in V(G)$, then G contains $k+1$ cycles $C_0, C_1, \ldots, C_k$ such that $k+1 < |E(C_0)| < |E(C_1)| < \cdots < |E(C_k)|$, $|E(C_i) - E(C_{i-1})| = 2$, $1 \leq i \leq k-1$, and $1 \leq |E(C_k)| - |E(C_{k-1})| \leq 2$, and furthermore, if $d_G(u) + d_G(v) \geq 6k + 1$, then $|E(C_k)| - |E(C_{k-1})| = 2$.

Keywords: Cycle length; 2-connected graph; String of cycles

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. We use [1] for terminology and notation not defined here. A *cycle* is a connected 2-regular graph. Let G be a graph. For $S \subseteq V(G)$, $G - S$ denotes the graph obtained from G by deleting all the vertices of S together with all the edges with at least one end in S. When $S = \{x\}$, we simplify this notation to $G - x$. A connected graph is *nonseparable* if it has no cut vertex. A *block* of G is a maximal nonseparable subgraph of G, and an *end-block* of G is a block that contains at most one cut vertex of G. For $A, B \subseteq V(G)$, $e(A, B)$ is the number of edges with one end in A and the other end in B. When $A = \{a\}$, we simplify the notation to $e(a, B)$. For $x, y \in V(G)$, an (x, y)-path is a path from x to y; and an (x, y)-path is *trivial* if $x = y$, in which the path consists of a single vertex. An edge is *contracted* if it is removed and its two ends are identified. Contraction might create multiple edges. By *removing multiple edges* of a graph, we mean the removal of $m - 1$ edges between every two vertices joined by m edges. Sometimes we identify a graph with its edge set.

[☆] Research supported by the Natural National Science Foundation of China (No. 10431020).
* Corresponding author.
E-mail address: zhangjun6110@126.com (J. Zhang).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2008.01.017

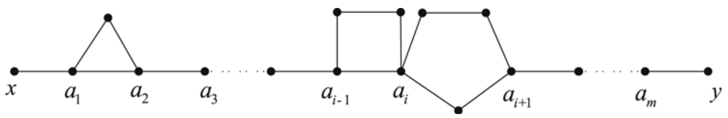

Fig. 1. Strings of cycles.

Erdös [3] asked whether every graph with minimum degree at least three contains two cycles whose lengths differ by 1 or 2. Using nonseparating induced cycles, Bondy and Vince [2] answered the question affirmatively. By a different approach, using strings of cycles, Fan [4] proved the following more general result.

Theorem 1.1 (*[4]*). *If G is a graph with minimum degree $\delta(G) \geq 3k$ for any positive integer k, then G contains $k+1$ cycles $C_0, C_1, \ldots, C_k$ such that $k+1 < |E(C_0)| < |E(C_1)| < \cdots < |E(C_k)|$, $|E(C_i)-E(C_{i-1})| = 2$, $1 \leq i \leq k-1$, and $1 \leq |E(C_k)| - |E(C_{k-1})| \leq 2$, and furthermore, if $\delta(G) \geq 3k+1$, then $|E(C_k)| - |E(C_{k-1})| = 2$.*

In [2], Bondy and Vince asked whether there exists a function $f(k)$ such that every nonbipartite 3-connected graph with minimum degree at least $f(k)$ contains cycles of k consecutive lengths. Fan [3] has proved the following Theorem 1.2, which answered the question in the affirmative.

Theorem 1.2 (*[4]*). *If G is a nonbipartite 3-connected graph with minimum degree at least $3k$ for any positive integer k, then G contains $2k$ cycles of consecutive lengths $m, m+1, m+2, \ldots, m+2k-1$ for some integer $m \geq k+2$.*

Using a method similar to the one which Fan used in [4], we prove the following theorems, which strengthen and expand Fan's results.

Theorem 1.3. *Let G be a graph with minimum degree $\delta(G) \geq 3$. For any positive integer k (if $k \geq 2$, then $\delta(G) \geq 4$), if $d_G(u)+d_G(v) \geq 6k-1$ for every pair of adjacent vertices $u, v \in V(G)$, then G contains $k+1$ cycles $C_0, C_1, \ldots, C_k$ such that $k+1 < |E(C_0)| < |E(C_1)| < \cdots < |E(C_k)|$, $|E(C_i)| - |E(C_{i-1})| = 2$, $1 \leq i \leq k-1$, and $1 \leq |E(C_k)| - |E(C_{k-1})| \leq 2$, and furthermore, if $d_G(u) + d_G(v) \geq 6k+1$, then $|E(C_k)| - |E(C_{k-1})| = 2$.*

We note that in Theorem 1.3, the last two cycles C_{k-1} and C_k have lengths of at least $3k$. The theorem is best possible in the following sense. If G is the complete graph on $3k+1$ or $3k+2$ vertices, then we cannot have more than $k+1$ cycles with the property described. Furthermore, if G is the complete graph on $3k$ vertices, then we cannot have k cycles with the property described.

Theorem 1.4. *Let G be a nonbipartite 3-connected graph. For any positive integer k (if $k \geq 2$, then $\delta(G) \geq 4$), if $d_G(u) + d_G(v) \geq 6k-1$ for every pair of adjacent vertices $u, v \in V(G)$, then G contains $2k$ cycles of consecutive lengths $m, m+1, m+2, \ldots, m+2k-1$ for some integer $m \geq k+2$.*

2. Strings of cycles

For an edge $uv \in E(G)$, by *replacing uv with a cycle*, we mean the operation of deleting the edge uv and adding a new cycle C such that $V(C) \cap V(G) = \{u, v\}$. An (x, y)-string (of k cycles) is the graph obtained from an (x, y)-path by replacing k edges of the path with k cycles, one edge with one cycle. Fig. 1 is an (x, y)-string of three cycles obtained from the path $xa_1a_2 \cdots a_m y$ by replacing a_1a_2, $a_{i-1}a_i$ and a_ia_{i+1} with cycles.

When there is no need to specify the ends, we simply use *strings*, instead of (x, y)-strings. In a string, if C is the cycle replacing uv, then u and v are called the *connection vertices* of C. C is *t-defective* if the two segments of C divided by u and v differ in length by t. A string is *t-defective* if each of its cycles is t-defective. We note that in a string of cycles distinct cycles can intersect only at connection vertices. Fig. 1 is a string of three cycles in which the first and the last cycles are 1-defective and the second one is 2-defective.

An (x, y)-string S of k cycles can be represented by $S = P_0C_1P_1C_2 \cdots C_kP_k$, where C_i is a cycle with connection vertices y_i and x_i, $1 \leq i \leq k$, P_j is a path from x_j to y_{j+1}, $0 \leq j \leq k$, $x_0 = x$ and $y_{k+1} = y$. (P_j may be the trivial path consisting of a single vertex for which $x_j = y_{j+1}$.) For each i, $1 \leq i \leq k$, let C'_i and C''_i be the two segments of C_i divided by its connection vertices such that $|E(C''_i)| \geq |E(C'_i)|$. The length of S is defined by

$$\ell(S) = \sum_{i=1}^{k} |E(C'_i)| + \sum_{i=0}^{k} |E(P_i)|,$$

which is the minimum length of a path from x to y in S. For any s, $1 \le s \le k$, let P be a path from x_s to y in $P_s C_{s+1} \cdots C_k P_k$, for instance, let $P = P_s C'_{s+1} \cdots C'_k P_k$; then $P_0 C_1 P_1 \cdots C_s P$ is an (x, y)-string of s cycles. Fan [4] has given the following observation and proved the following lemmas.

Observation 2.1 (*[4]*). If $P_0 C_1 P_1 \cdots C_k P_k$ is an (x, y)-string of k cycles, then for any s, $1 \le s \le k$, there is a path P such that $\bigcup_{i=s}^{k} E(P_i) \subseteq E(P) \subseteq E(P_s C_{s+1} \cdots C_k P_k)$ and $P_0 C_1 P_1 \cdots C_s P$ is an (x, y)-string of s cycles.

Lemma 2.2 (*[4]*). *Let S be a t-defective (x, y)-string of k cycles. Then S contains (x, y)-paths of lengths $m, m + t, m + 2t, \ldots, m + kt$, where $m = \ell(S)$.*

Lemma 2.3 (*[4]*). *Let S be an (x, y)-string of k cycles in which s cycles are 1-defective and the remaining $k - s$ cycles are 2-defective. If $s \ge 1$, then S contains (x, y)-paths of consecutive lengths $m, m+1, m+2, \ldots, m+2k-s$, where $m = \ell(S)$.*

Definition 2.4. Let $S = P_0 C_1 P_1 \cdots C_k P_k$ be an (x, y)-string of k cycles in a graph G. S is *feasible* (with respect to k and G) if all the following three statements hold.

(1) $\sum_{i=0}^{k} |E(P_i)| \ne 0$.
(2) C_i is 2-defective for every i, $1 \le i \le k$, with at most one exception.
(3) If C_j is the exceptional cycle in (2), then C_j is 1-defective, and moreover, there is $uv \in E(C_j)$ such that $\{u, v\} \cap \{x, y\} = \emptyset$ and $d_G(u) + d_G(v) \in \{6k - 1, 6k\}$.

Theorem 2.5. *Let x and y be two distinct vertices in a 2-connected graph G. For any positive integer k, if every vertex other than x and y has degree at least 3 (if $k \ge 2$, then replace 3 with 4) and $d_G(u) + d_G(v) \ge 6k - 1$ for every pair of adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, then G contains a feasible (x, y)-string of k cycles.*

Proof. Since the proof is almost identical to the that Fan used in [4, Theorem 2.5], we omit it. □

In the proof of Theorem 2.5, if there is no edge $uv \in E(G - \{x, y\})$ such that $d_G(u) + d_G(v) \in \{6k - 1, 6k\}$, then statement (3) in Definition 2.4 cannot occur and so the string is feasible 2-defective. As an immediate consequence of Theorem 2.5, we have that

Theorem 2.6. *Let x and y be two distinct vertices in a 2-connected graph G. For any positive integer k, if every vertex other than x and y has degree at least 3 (if $k \ge 2$, then replace 3 with 4) and $d_G(u) + d_G(v) \ge 6k - 1$ for every pair of adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, and in addition, if there is no edge $uv \in E(G - \{x, y\})$ such that $d_G(u) + d_G(v) \in \{6k - 1, 6k\}$, then G contains a feasible 2-defective (x, y)-string of k cycles.*

3. Proof of Theorem 1.3

In order to prove Theorem 1.3, we present the following Corollaries 3.1–3.4 first.

Corollary 3.1. *Let x and y be two distinct vertices in a 2-connected graph G. For any positive integer k, if $d_G(v) \ge 3$ (if $k \ge 2$, then $d_G(v) \ge 4$) for every $v \in V(G) \setminus \{x, y\}$, and $d_G(u) + d_G(v) \ge 6k - 1$ for every pair of adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, then G contains $k+1$ (x, y)-paths $R_0, R_1, \ldots, R_k$ such that $k < |E(R_0)| < |E(R_1)| < \cdots < |E(R_k)|$, $|E(R_i)| - |E(R_{i-1})| = 2$, $1 \le i \le k - 1$, and $1 \le |E(R_k)| - |E(R_{k-1})| \le 2$.*

Proof. By Theorem 2.5, G contains a feasible (x, y)-string S of k cycles and $\ell(S) = \sum_{i=1}^{k} |E(C'_i)| + \sum_{i=0}^{k} |E(P_i)| \ge k + 1$. If S is 2-defective, then by Lemma 2.2 [4], S contains $k + 1$ (x, y)-paths of lengths $m, m + 2, \ldots, m + 2k$, where $m = \ell(S) > k$, and we are done. Otherwise, one and only one cycle in S is 1-defective, and by Lemma 2.3 [4], S contains (x, y)-paths P_i of lengths $m + i$, $0 \le i \le 2k - 1$, where $m = l(S) > k$. Let $R_i = P_{2i}$, $0 \le i \le k - 1$, and $R_k = P_{2k-1}$. Then $R_0, R_1, \ldots, R_k$ are $k + 1$ (x, y)-paths with the required property. This proves the corollary. □

Corollary 3.2. *Let x and y be two distinct vertices in a 2-connected graph G. For any positive integer k, if $d_G(v) \ge 3$ (if $k \ge 2$, then $d_G(v) \ge 4$) for every $v \in V(G) \setminus \{x, y\}$, and $d_G(u) + d_G(v) \ge 6k + 1$ for every pair of adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, then G contains $k + 1$ (x, y)-paths of consecutive even lengths or consecutive odd lengths $m, m + 2, m + 4, \ldots, m + 2k$ for some integer $m \ge k + 1$.*

Proof. This follows from Theorem 2.6 and Lemma 2.2 [3]. □

In Corollaries 3.1 and 3.2, all the (x, y)-paths have lengths of at least $k + 1 \geq 2$, and hence the edge xy is not contained in any of the (x, y)-paths. So it is clear that Corollaries 3.3 and 3.4 follows from Corollaries 3.1 and 3.2, respectively, by adding xy to those (x, y)-paths.

Corollary 3.3. *Let xy be an edge in a 2-connected graph G. For any positive integer k, if every vertex other than x and y has degree at least 3 (if $k \geq 2$, then replace 3 with 4), and $d_G(u) + d_G(v) \geq 6k - 1$ for every adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, then xy is contained in $k + 1$ cycles $C_0, C_1, \ldots, C_k$ such that $k + 1 < |E(C_0)| < |E(C_1)| < \cdots < |E(C_k)|$, $|E(C_i) - E(C_{i-1})| = 2$, $1 \leq i \leq k - 1$, and $1 \leq |E(C_k)| - |E(C_{k-1})| \leq 2$.*

Corollary 3.4. *Let xy be an edge in a 2-connected graph G. For any positive integer k, if every vertex other than x and y has degree at least 3 (if $k \geq 2$, then replace 3 with 4), and $d_G(u) + d_G(v) \geq 6k + 1$ for every pair of adjacent vertices $u, v \in V(G) \setminus \{x, y\}$, then xy is contained in $k + 1$ cycles of consecutive even lengths or consecutive odd lengths $m, m + 2, m + 4, \ldots, m + 2k$ for some integer $m \geq k + 2$.*

Apply Corollaries 3.3 and 3.4 to a 2-connected component or an end-block of a graph G with minimum degree $\delta(G) \geq 3$ (if $k \geq 2$, then $\delta(G) \geq 4$) and $d_G(u) + d_G(v) \geq 6k - 1$ for every pair of adjacent vertices $u, v \in V(G)$, then Theorem 1.3 follows.

4. Proof of Theorem 1.4

Instead of proving Theorem 1.4 directly, we shall prove Theorem 3.7 in this section. First, we introduce some additional notation. Let C be an odd cycle. The *diameter-graph* of C is the graph with vertex set $V(C)$ in which two vertices u and v are joined by an edge if and only if the two segments of C divided by u and v differ in length by 1. A cycle C is *nonseparating* in G if $G - V(C)$ is connected. In [4], Fan proved the following two lemmas.

Lemma 4.1 ([4]). *The diameter-graph of any odd cycle is connected.*

Lemma 4.2 ([4]). *Let G be a graph with minimum degree at least 4. If G contains a nonseparating induced odd cycle, then G contains a nonseparating induced odd cycle C such that either C is a triangle or $e(v, C) \leq 2$ for every $v \in V(G) \setminus V(C)$ which is not a cut vertex of $G - V(C)$.*

Theorem 4.3. *Let G be a 2-connected graph with minimum degree $\delta(G) \geq 3$. For any positive integer k (if $k \geq 2$, then $\delta(G) \geq 4$), if $d_G(u) + d_G(v) \geq 6k - 1$ for every pair of adjacent vertices $u, v \in V(G)$ and G contains a nonseparating induced odd cycle, then G contains $2k$ cycles of consecutive lengths $m, m + 1, m + 2, \ldots, m + 2k - 1$ for some integer $m \geq k + 2$.*

Proof. Since our proof is almost the same as the proof that Fan used in [4, Theorem 3.5], we omit it.

Proof of Theorem 1.4. By [2, Lemma 2] (or from the proof of [2, Theorem 2]), G contains a nonseparating induced odd cycle, and the theorem follows from Theorem 3.7. (The case of $k = 1$ is a result of Bondy and Vince [2].) □

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, MacMillan, London, 1976.

[2] J.A. Bondy, A. Vince, Cycles in a graph whose lengths differ by one or two, J. Graph Theory 27 (1998) 11–15.

[3] P. Erdös, Some of my recent problems in combinatorial number theory, geometry and combinatorics, in: Y. Alavi, A. Schwend (Eds.), Graph Theory, Combinatorics and Algorithms, in: Proc. Seventh Quadrennial International Conference on the Theory and Applications of Graphs, vol. 1, Wiley, New York, 1995, pp. 335–349.

[4] Genghua Fan, Distribution of cycle lengths in graphs, J. Combin. Theory Ser. B 84 (2002) 187–202.