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A Multidimensional Nonlinear Gronwall Inequality

V. B. HEADLEY

Brock University, St. Catharines, Ontario, Canada

Submitted by Richard Bellman

1. INTRODUCTION

A result of fundamental and far-reaching importance in the study of existence, uniqueness, boundedness, and stability properties of ordinary differential equations is the Gronwall–Bellman inequality [1, 2]. Several authors (see, e.g., [3], [4], [5], [6]) have developed extensions of the inequality to functions of more than one independent variable and exhibited applications to partial differential equations. Rasmussen [7] has recently obtained a nonlinear two-dimensional version of the inequality by using ideas previously applied to functions of one independent variable by Opial [8] and others. In the present note we show that these techniques can be further exploited to obtain nonlinear extensions to any number of independent variables.

Let G be an open connected (possibly unbounded) set contained in N-dimensional Euclidean space R^N. For any two points x and y in G, with $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$, define the set $G(x, y)$ to be the closed rectangular parallelepiped with one diagonal joining the points x and y; that is,

$$G(x, y) = \{t \in R^N \mid t_j = (1 - \lambda_j)\, y_j + \lambda_j x_j\, ,\ 0 \leqslant \lambda_j \leqslant 1, j = 1, 2, \ldots, N\}.$$

We remark that the identity $G(x, y) = G(y, x)$ is an immediate consequence of the definition of $G(x, y)$. This symmetry wil enable us to drop the requirement in [7] that the line joining the points x and y have non-negative (though not necessarily finite) slope.

For fixed ξ in G, define the integral operator K by setting

$$(Kv)\,(x) = \int_{G(x,\xi)} k(t, v(t))\, dt, \tag{1}$$

where v and k are real-valued functions (k being continuous on $G \times R^1$), x is a point of G, the set $G(x, \xi)$ is contained in G, and dt is Lebesgue measure on R^N.

2. Results

In this section we prove a Gronwall-type inequality for nonlinear integral operators on functions of N independent variables. Theorem 1 extends the corresponding result in [7] not only to N dimensions but also to more general linear operators. Consequently, Theorem 2, the N-dimensional analog of the main result of [7], follows readily. It should be noted that Theorem 2 also contains the analogous two-dimensional result of Snow [5] for linear integral operators.

THEOREM 1. *Let ξ and y be points in a (possibly unbounded) domain $G \subset R^N$ such that $G(\xi, y) \subset G$. Let g and k be real-valued functions, with g continuous on G, and with k continuous on $G \times R^1$ and nondecreasing with respect to its last argument. Let (ϵ_n) $(n = 1, 2,...)$ be a strictly decreasing sequence of real numbers with limit zero. Suppose that there exists a family $\{v_n \mid n = 1, 2,...\}$ of functions continuous on $G(y, \xi)$ such that, for $n = 1, 2,...$, and all x in $G(y, \xi)$,*

$$v_n(x) = g(x) + \epsilon_n + (Kv_n)(x). \tag{2}$$

Let U be the maximal solution on $G(y, \xi)$ of the nonlinear Volterra integral equation

$$u(x) = g(x) + (Ku)(x). \tag{3}$$

Then $\lim_{n\to\infty} v_n = U$ *on* $G(y, \xi)$.

Proof. If $\xi_j = y_j$ for some j, then the parallelepiped $G(y, \xi)$ has volume zero and the result is trivially true. We therefore suppose that $\xi_j \neq y_j$ for $j = 1, 2,..., N$. We shall show that the sequence (v_n) is strictly decreasing and satisfies the hypotheses of the Ascoli–Arzelà Theorem [9, p. 112]. Accordingly, we first note that $v_m(\xi) - v_n(\xi) = \epsilon_m - \epsilon_n < 0$ whenever $m > n$. If the sequence (v_n) were not strictly decreasing, then it would follow from the continuity of the functions v_m and v_n that, for some z in $G(y, \xi)$, with $z \neq \xi$, we would have $v_m < v_n$ on the set $G(z, \xi) - \{z\}$, whilst $v_m(z) = v_n(z)$. But then it follows from the definition of v_n and the monotonicity of (ϵ_n) and k that

$$\begin{aligned} v_n(z) &= g(z) + \epsilon_n + (Kv_n)(z) \\ &> g(z) + \epsilon_m + (Kv_m)(z) = v_m(z), \end{aligned}$$

whenever $m > n$. This contradicts the definition of z and shows that $v_m(x) < v_n(x)$ whenever $m > n$ and $x \in G(y, \xi)$. It follows that the sequence (v_n) $(n = 1, 2,...)$ is bounded above by $M_1 = \max\{v_1(x) \mid x \in G(y, \xi)\}$. Let

u be any solution of (3) on $G(y, \xi)$. Then $u(\xi) < v_n(\xi)$ for $n = 1, 2, \ldots$. We now show that $u(x) < v_n(x)$ for $n = 1, 2, \ldots$ and all x in $G(y, \xi)$. If this were not true, then there would exist some function v_m satisfying (2) and some point η in $G(y, \xi)$, with $\eta \neq \xi$, such that $u(x) < v_m(x)$ for all x in $G(\eta, \xi) - \{\eta\}$, whilst $v_m(\eta) = u(\eta)$. But then it follows from (2), (3) and the monotonicity of k that

$$u(\eta) = g(\eta) + Ku(\eta) < g(\eta) + \epsilon_m + (Kv_m)(\eta) = v_m(\eta).$$

This contradicts the definition of η and shows that $u(x) < v_n(x)$ for each positive integer n and all x in $G(y, \xi)$. Consequently, the sequence (v_n) is bounded below by

$$M_2 = \min\{u(x) \mid x \in G(y, \xi)\}.$$

To prove equicontinuity, let ϵ be any positive number, and let y' and y'' be any two points in $G(y, \xi)$, with $y' = (y_1', \ldots, y_N')$ and $y'' = (y_1'', \ldots, y_N'')$. For $n = 1, 2, \ldots$, the first two terms on the right side of the identity

$$v_n(y') - v_n(y'') = (Kv_n)(y') - (Kv_n)(y'') + g(y') - g(y'') \tag{4}$$

may be written in the form

$$(Kv_n)(y') - (Kv_n)(y'') = \int_E K(t, v_n(t))\, dt - \int_F K(t, v_n(t))\, dt, \tag{5}$$

where

$$E = G(y', \xi) - G(y'', \xi) \qquad \text{and} \qquad F = G(y'', \xi) - G(y', \xi),$$

the negative signs denoting relative complementation.

The set E may be decomposed into N disjoint (possibly degenerate) rectangular parallelepipeds $E_1, \ldots, E_N$ as follows

$$E_1 = \{x \in E \mid \min(y_1', y_1'') \leqslant x_1 \leqslant \max(y_1', y_1'')\},$$

$$E_j = \{x \in E - E_{j-1} \mid \min(y_j', y_j'') \leqslant x_j \leqslant \max(y_j', y_j'')\} \qquad (j = 2, 3, \ldots, N).$$

Similarly, the set F may be decomposed into N disjoint (possibly degenerate) rectangular parallelepipeds $F_1, \ldots, F_N$ defined by

$$F_1 = \{x \in F \mid \min(y_1', y_1'') \leqslant x_1 \leqslant \max(y_1', y_1'')\},$$

$$F_j = \{x \in F - F_{j-1} \mid \min(y_j', y_j'') \leqslant y_j \leqslant \max(y_j', y_j'')\} \qquad (j = 2, \ldots, N).$$

For $j = 1, 2, \ldots, N$, consider the parallelepipeds E_j and F_j. Each edge parallel to the x_j-axis has length $| y_j' - y_j'' |$, and each edge parallel to the

x_r-axis $(r \neq j)$ has length not greater than $\max(|y_r' - \xi_r|, |y_r'' - \xi_r|)$, and therefore not greater than $|y_r - \xi_r|$. It follows from (5) that, for $n = 1, 2, \ldots,$

$$|(Kv_n)(y') - (Kv_n)(y'')| \leqslant 2AB \sum_{j=1}^{N} |y_j' - y_j''|,$$

where

$$A = \sup\{|k(x, u)| : x \in G(y, \xi), M_2 \leqslant u \leqslant M_1\}$$

and

$$B = \max\{|y_r - \xi_r| : r = 1, 2, \ldots, N\}.$$

We now introduce the norm

$$\| y' - y'' \| = \sum_{j=1}^{N} |y_j' - y_j''|.$$

Since g is continuous and therefore uniformly continuous on the compact set $G(y, \xi)$, there exists a positive number δ_1 such that $|g(y') - g(y'')| < \epsilon/2$ whenever $y', y'' \in G(y, \xi)$ and $\| y' - y'' \| < \delta_1$. For $A > 0$, choose δ_2 so that $0 < \delta_2 < \epsilon/(4AB)$, and let $\delta = \min(\delta_1, \delta_2)$. It follows from (4) that, for $n = 1, 2, \ldots,$ $|v_n(y') - v_n(y'')| < \epsilon$ whenever $y', y'' \in G(y, \xi)$ and $\| y' - y'' \| < \delta$.

If $A = 0$, this estimate is trivially true for all $\delta < \delta_1$. We have therefore shown that the sequence (v_n) is equicontinuous and uniformly bounded on the compact set $G(y, \xi)$. According to the Ascoli–Arzelà theorem, there exists a subsequence v_{n_i} $(i = 1, 2, \ldots)$ which converges uniformly on $G(y, \xi)$. The sequence (v_n), being decreasing and bounded below, converges on $G(y, \xi)$. It is thus clear that $\lim_{n\to\infty} v_n = \lim_{i\to\infty} v_{n_i}$. If we let $i \to \infty$ in the identity

$$v_{n_i}(x) = g(x) + \epsilon_{n_i} + (Kv_{n_i})(x) \qquad (x \in G(y, \xi)),$$

we see that, in view of the continuity of k and the uniform convergence of the sequence (v_{n_i}), the function $\lim_{i\to\infty} v_{n_i}$ is a solution of (3). Moreover, since each solution u of (3) satisfies the inequality $u(x) < v_n(x)$ for x in $G(y, \xi)$ and $n = 1, 2, \ldots,$ so does the maximal solution U; consequently

$$U(x) \leqslant \lim_{n\to\infty} v_n(x) = \lim_{i\to\infty} v_{n_i}(x) \leqslant U(x)$$

for all x in $G(y, \xi)$. This completes the proof of the theorem.

We remark that the existence of maximal solutions for the integral equations in this section is a consequence of general results proved in Walter's monograph [10, pp. 131, 139].

THEOREM 2. *Let ξ and y be points in a (possibly unbounded) domain $G \subset R^N$ such that $G(\xi, y) \subset G$. Let g, v, and k be real-valued functions, with g and v continuous on G and with k continuous on $G \times R^1$ and nondecreasing with respect to its last argument. Let v be a solution on $G(y, \xi)$ of the nonlinear integral inequality*

$$v(x) \leqslant g(x) + (Kv)(x).$$

Then $v(x) \leqslant U(x)$ for all x in $G(y, \xi)$, where U is the maximal solution on $G(y, \xi)$ of the integral equation

$$u(x) = g(x) + (Ku)(x).$$

Proof. If $\xi_j = y_j$ for some j, then the result is trivially true. We therefore suppose that $\xi_j \neq y_j$ for $j = 1, 2,..., N$. Let (ϵ_n) $(n = 1, 2,...)$ be a strictly decreasing sequence of real numbers with limit zero. For $n = 1, 2,...$, let v_n be a continuous solution on $G(y, \xi)$ of the integral equation

$$v_n(x) = g(x) + \epsilon_n + (Kv_n)(x).$$

We now show that $v(x) < v_n(x)$ for all positive integers n and all x in $G(y, \xi)$. If this were false for some v_m at some point of $G(y, \xi)$, then, in view of the inequalities

$$v(\xi) \leqslant g(\xi) < g(\xi) + \epsilon_m = v_m(\xi),$$

it follows from the continuity of v and v_m that there must exist some point z in $G(y, \xi)$, with $z \neq \xi$, such that $v(x) < v_m(x)$ on the set $G(y, \xi) - \{z\}$, whilst $v(z) = v_m(z)$. But then

$$v(z) \leqslant g(z) + (Kv)(z) < g(z) + \epsilon_m + (Kv_m)(z) = v_m(z).$$

This contradicts the definition of z and shows that $v(x) < v_n(x)$ for $n = 1, 2,...$, and all x in $G(y, \xi)$. It follows from Theorem 1 that, for all x in $G(y, \xi)$,

$$v(x) \leqslant \lim_{n\to\infty} v_n(x) \leqslant U(x).$$

This completes the proof of the theorem.

It should be noted that G need not be connected. It is enough if ξ, y and $G(\xi, y)$ lie in the same component (= maximal connected subset) of G.

In closing, we point out that inequalities similar to that proved here have been successfully applied by several authors to the study of parabolic partial differential equations in N independent space variables. A detailed and comprehensive account of typical applications, together with a full bibliography, may be found in [10].

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