A Solution to the Generalized Cevian Problem
Using Forest Polynomials

Matthew Hudelson

Department of Mathematics, Washington State University, Pullman, Washington 99164-3113

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In this paper, we examine the effect of dissecting an \( n \)-dimensional simplex using cevians (cross-sections passing through \( n-1 \) of the vertices of the simplex). We describe a formula for the number of pieces the simplex is dissected into using a polynomial involving only the number of each type of cevian. The polynomial in question involves terms involving the edges of the simplex, but discarding those terms involving cycles of the underlying graph. Thus, we call such a polynomial a “forest polynomial.” © 1999 Academic Press

A cevian of a triangle \( T \) is a line segment connecting a point on an edge of \( T \) to the vertex opposite this edge. To extend this definition for higher dimensional simplices, we define a cevian of a non-degenerate simplex \( S \) to be a simplex joining a point on an edge of \( S \) to the remaining vertices of \( S \). Alternatively, we may view a cevian as a cross-section of \( S \) which passes through all but two of its vertices. By this definition, a cevian of a tetrahedron, as depicted in Fig. 1, is a triangle sharing two vertices of the tetrahedron and having its third vertex on the edge joining the other two vertices of the tetrahedron.

We wish to examine and later generalize to higher dimensions the answer to the following question:

**Question 1.** Given triangle \( T \) with vertices \( A, B, \) and \( C \), suppose we draw \( a \) cevians from vertex \( A \), \( b \) cevians from \( B \), and \( c \) cevians from \( C \) so that no three cevians intersect in one point in the interior of \( T \). Into how many pieces is \( T \) dissected?

The following solution was given by a fifth-grade student (see [1] for details). Suppose the solution depends only on \( a, b, \) and \( c \), and not on the location of the cevians. Then draw a new picture with the cevians “interfering” as little as possible as depicted in Fig. 2.
Now, it is a simple matter to count the pieces in the regions of $\triangle ABC$:

<table>
<thead>
<tr>
<th>Region</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\triangle BFI$</td>
<td>$ab$</td>
</tr>
<tr>
<td>$\triangle CDG$</td>
<td>$bc$</td>
</tr>
<tr>
<td>$\triangle AEH$</td>
<td>$ac$</td>
</tr>
<tr>
<td>$\square BEHI$</td>
<td>$a$</td>
</tr>
<tr>
<td>$\square CFIG$</td>
<td>$b$</td>
</tr>
<tr>
<td>$\square ADGH$</td>
<td>$c$</td>
</tr>
<tr>
<td>$\triangle GHI$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Summing these, we find that these cevians dissect $\triangle ABC$ into $1 + a + b + c + ab + ac + bc$ pieces.

Of course, this solution lacks a proof of the independence of the location of the cevians. In the two-dimensional case, such independence follows by observing that any cevian drawn from one corner of the triangle is dissected into the same number of segments by the cevians drawn from the other two corners, provided the cevians are in general position. This independence will be proved in general when we consider the cevian problem in higher dimensions:

**Question 2.** Given an $n$ simplex $S$ with vertices $v_0$ through $v_n$, suppose we construct $x_{km}$ cevians each with a vertex on the interior of the edge $(v_k, v_m)$ connecting $v_k$ to $v_m$. Here, no $n + 1$ cevians intersect in one point in the interior of $S$. Into how many pieces is $S$ dissected?

Notice that the answer for triangles is a polynomial in the variables $a$, $b$, $c$ consisting of monomials with monic coefficients. Furthermore, no variable in this polynomial has degree exceeding one. Finally, the
monomial \( abc \) is missing. Such a polynomial is an example of a forest polynomial which we now discuss.

Let \( G \) be an undirected graph with vertices \( \{v_1, v_2, \ldots, v_n\} \) and edges \( \{e_{ij}\} \), where edge \( e_{ij} \) joins vertices \( v_i \) and \( v_j \). Associate with each edge \( e_{ij} \) the variable \( x_{ij} = x_{ji} \).

**Definition 1.** A unomial for \( G \) is a monic monomial in the variables \( x_{ij} \). The unomial will be called cyclic if the underlying set of edges contains a cycle in \( G \). Otherwise the unomial will be called a forest unomial. Any unomial using the same underlying edge twice will be considered cyclic, while the constant 1 is a forest unomial.

**Definition 2.** The forest polynomial for a graph \( G \) is the sum of all of the forest unomials for \( G \).

For instance, if \( H \) is the graph in Fig. 3, then

\[ x_{12} x_{23} x_{31}, \ x_{12} x_{23} x_{31} x_{14}, \ \text{and} \ x_{12}^2 = x_{12} x_{31} \]

are all cyclic unomials for \( G \), while \( x_{12}, \ x_{23}, \ x_{14}, \ \text{and} \ 1 \)

are all forest unomials.

![Diagram of graph H](image.png)

**FIG. 3.** The graph \( H \).
The forest polynomial for $H$ is

$$
1 + x_{12} + x_{13} + x_{14} + x_{23} + x_{12}x_{13} + x_{12}x_{23} \\
+ x_{12}x_{14} + x_{13}x_{23} + x_{13}x_{14} + x_{23}x_{14} \\
+ x_{12}x_{13}x_{14} + x_{12}x_{23}x_{14} + x_{13}x_{23}x_{14}.
$$

Notice that the answer to the cevian question for triangles is provided by the forest polynomial for the complete graph on three vertices. This turns out to be the case in general, namely, that the answer to the cevian question for $n$-dimensional simplices is provided by the forest polynomial for the complete graph on $n+1$ vertices. To establish this connection between cevians and forest polynomials, we describe a cevian of an $n$-simplex more precisely.

**Definition 3.** If $S$ is a non-degenerate $n$-simplex with vertices $v_0$ through $v_n$, then a $(v_k, v_m)$-cevian of $S$ is the convex hull of a set of the form

$$
V(k, m; s) = \{v_i : 0 \leq i \leq n, i \neq m, i \neq k\} \cup u(k, m; s),
$$

where $0 < s < 1$ and $u(k, m; s) = sv_k + (1-s)v_m$.

A $(v_k, v_m)$-cevian will be said to “arise from” the edge joining $v_k$ and $v_m$.

Now, we may state more formally our main result connecting cevians and forest polynomials.

**Theorem 1.** Suppose $S$ is an $n$-simplex with vertices $\{v_0, v_1, \ldots, v_n\}$. Suppose further that for each edge joining two vertices $v_i$ and $v_j$ of $S$, there are $x_{ij}(v_i, v_j)$ cevians, and all cevians are in general position. Then $S$ is dissected into $P_n(x_{01}, \ldots, x_{n-1,n})$ pieces where $P_n$ is the forest polynomial for the complete graph on $n+1$ vertices.

To show this, we make the following critical observations about intersections of cevians. We enumerate “cevians of cevians” and so clearing the way for an induction argument in proving our theorem. The first observation concerns cevians arising from edges which have a vertex in common.

**Lemma 1.** Suppose $A$ is a $(v_a, v_b)$-cevian having endpoint $x = u(a, c; s)$ and $B$ is a $(v_a, v_c)$-cevian having endpoint $y = u(b, c; t)$, $0 < s, t < 1$. Then $A \cap B$ is a $(x, v_b)$-cevian of $A$ and a $(y, v_a)$-cevian of $B$. 

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Proof. Suppose \( A \) is a \((v_a, v_c)\)-cevian and \( B \) is an \((v_b, v_c)\)-cevian of a simplex \( S \). The cevian \( A \) is the convex hull of
\[
\{v_i : 0 \leq i \leq n, i \neq a, i \neq c\} \cup \{u\},
\]
while \( B \) is the convex hull of
\[
\{v_i : 0 \leq i \leq n, i \neq b, i \neq c\} \cup \{w\}.
\]

We claim that \( A \cap B \) equals the set \( C \) defined to be the convex hull of
\[
\{v_i : 0 \leq i \leq n, i \neq a, i \neq b, i \neq c\} \cup \{p\},
\]
where
\[
p = \frac{s(1-t)}{1-st} v_a + \frac{t(1-s)}{1-st} v_b + \frac{(1-s)(1-t)}{1-st} v_c
\]

\[
= \frac{1-t}{1-st} u(a,c,s) + \frac{t(1-s)}{1-st} v_b = \frac{1-s}{1-st} u(b,c,t) + \frac{s(1-t)}{1-st} v_a
\]

\[
= \frac{1-t}{1-st} x + \frac{t(1-s)}{1-st} y = \frac{1-s}{1-st} v_b + \frac{s(1-t)}{1-st} v_a.
\]

It is clear from this that any point in \( C \) is also a point in \( A \) and \( B \), and so \( C \subset A \cap B \).

To show \( A \cap B = C \), choose \( v \in A \cap B \) and write \( v = x_1 v_1 + \cdots + x_n v_n \), where the \( x_i \) are nonnegative and \( \sum x_i = 1 \). Suppose \( x_a + x_b + x_c = u \). Since \( v \in A \), we know that \( x_a = u s/(1-st) \). Likewise, since \( v \in B \), we know that \( x_b = u t/(1-st) \). Therefore,
\[
x_c \left(1 + \frac{s}{1-st} + \frac{t}{1-st}\right) = u,
\]
or, solving for \( x_c \), and substituting for \( x_a \) and \( x_b \), we obtain the equations
\[
x_a = u \frac{s(1-t)}{1-st},
\]
\[
x_b = u \frac{t(1-s)}{1-st},
\]
\[
x_c = u \frac{(1-s)(1-t)}{1-st}.
\]
Therefore,
\[ v = \sum_{i \neq (a, b, c)} x_i v_j + u \left( \frac{s(1-t)}{1-st} v_a + \frac{t(1-s)}{1-st} v_b + \frac{(1-s)(1-t)}{1-st} v_c \right) \]
and so \( v \in C \) as desired.

The second observation concerns cevians arising from edges which have no vertices in common.

**Lemma 2.** Suppose \( a, b, c, \) and \( d \) are all distinct, \( A \) is a \((v_a, v_b)\)-cevian having edgepoint \( x = u(a, b; s) \) and \( B \) is a \((v_c, v_d)\)-cevian having edgepoint \( y = u(c, d; t) \). Then \( A \cap B \) is a \((v_a, v_b)\)-cevian of \( A \) and a \((v_c, v_d)\)-cevian of \( B \).

The proof is similar to that of the previous lemma and is omitted.

Now, we prove the theorem. Suppose \( S \) having vertices \( v_0, \ldots, v_n \) is dissected by a number of cevians in general position. For vertices \( u \) and \( v \) of \( S \), let the variable \( x_{uv} \) denote the number of \((u, v)\)-cevians of \( S \). Notice that \( x_{uv} = x_{vu} \) and so we may identify these variables with one another. For ease of notation, we will write \( x_{ij} \) in place of \( x_{vi vj} \).

Let \( ! \) be the set of cevians in general position used to dissect \( S \) and
\[ f_n(S, \zeta) \]
denote the number of regions into which the \( n \)-simplex \( S \) is dissected.

In the course of the discussion, we will prove that \( f_n \) depends only on the number of each type of cevian rather than their specific locations, provided the cevians are in general position.

Our proof proceeds by induction on \( n \). For \( n = 1 \), a 1-simplex consists of two vertices \( u \) and \( v \), and a single edge \((u, v)\). A cevian will be a single point lying on this edge. If there are \( x_{uv} \) of these cevians in general position, then no two coincide and so the segment is dissected into \( x_{uv} + 1 \) smaller segments. Therefore, \( f_1(S, \zeta) = x_{uv} + 1 = P_1(x_{uv}) \), and \( f_1 \) depends only on the number of cevians rather than their positions.

Now, assume the theorem is true for \( n - 1 \). We must show it is true for \( n \). We proceed by strong induction on the number of cevians of our \( n \)-simplex \( S \). Suppose \( S \) has no cevians. Then \( S \) is still in one piece. Thus, \( f_n(S, \{\}) = 1 = P_n(0, 0, \ldots, 0) \) since the forest polynomial for any graph has constant term 1.

Now, we assume \( k > 0 \) and
\[ f_k(S, \zeta) = P_n(x_{01}, \ldots, x_{ab}, \ldots, x_{n-1,n}) \]
if the number of cevians in \( \zeta \) is less than \( k \). Let \( \zeta \) be a set of \( k \) cevians in general position. If necessary, re-index the vertices so that \( x_{n-1,n} > 0 \).
We examine any \((v_{n-1}, v_n)\)-cevian \(A\) of \(S\) in general position. The number of regions into which this cevian is dissected (by the remaining cevians) equals the number of regions removed from the dissection of \(S\) by \(\zeta\) if cevian \(A\) is deleted. Let \(A\) have edgepoint \(u\) and let \(\sigma\) be the set of cevians of \(A\) induced by the other cevians of \(S\). Let \(x'_{ab}\) represent the number of \((a, b)\)-cevians of \(A\). By the induction hypothesis on \(n\),

\[
f_{n-1}(A, \sigma) = P_{n-1}(x'_{01}, \ldots, x'_{n-2,u})
\]

where \(u\) has replaced vertex \(v_{n-1}\).

By Lemma 1, if \(c < n - 1\), the intersection of \(A\) with any \((v_{n-1}, v_c)\)-cevian or \((v_n, v_c)\)-cevian of \(S\) is a \((u, v_c)\)-cevian of \(A\). By Lemma 2, if \(c\) and \(d\) are both less than \(n - 1\), the intersection of \(A\) with any \((v_c, v_d)\)-cevian of \(S\) is a \((v_c, v_d)\)-cevian of \(A\). Therefore, \(x'_{uv} = x_{v,n-1} + x_{n,u}\) and \(x'_{uv} = x_{uv}\). It follows from these considerations and from the inductive hypothesis that \(f_{n-1}(A, \sigma)\) is independent of the choice of the edgepoint \(u\), and so any \((v_{n-1}, v_n)\)-cevian in general position contributes \(P_{n-1}(x'_{01}, \ldots, x'_{n-2,u})\) regions to the dissection of \(S\). This means that if \(\zeta'\) is the set \(\zeta\) with all \((v_{n-1}, v_n)\)-cevians removed, then

\[
f_n(S, \zeta) = f_u(S, \zeta') + x_{n-1,n} f_{n-1}(A, \sigma).
\]

Since \(\zeta'\) has a smaller cardinality than \(\zeta\), the strong induction hypothesis yields

\[
f_n(S, \zeta') = P_n(x_{01}, \ldots, x_{n-2,n}, 0).
\]

This polynomial consists of all unomials which do not contain cycles and which do not have a factor of \(x_{n-1,n}\). To complete the proof, we must show that \(x_{n-1,n} f_{n-1}(A, \sigma)\) is the polynomial over \(\{x_{01}, \ldots, x_{n-1,n}\}\) of all of whose unomials have a factor of \(x_{n-1,n}\) and do not contain cycles.

Suppose \(M\) is a product of \(x_{ab}\)’s which contains \(x_{n-1,n}\). Suppose also that that \(M\) appears in the expansion of a product of \(x_{n-1,n}\) and \(x_{cd}\)’s arising from a subset of \(\sigma\) which contains a cycle. Let \(\gamma = x_{ab}x_{bc} \cdots x_{zu}\) be a primitive cycle in this product. If none of \(a, b, \ldots, z\) is the vertex \(u\), then \(\gamma = x_{ab}x_{bc} \cdots x_{zu}\) which is a cycle among the \(x_{km}\)’s. The other possibility is that \(u\) is among \(a, b, \ldots, z\). Without loss of generality, suppose \(a = u\). Then

\[
x_{n-1,\gamma} = x_{n-1,n} x_{ab} x_{bc} \cdots x_{zu} x_{zu}'
\]

\[
= x_{n-1,n} (x_{n-1,b} + x_{ab}) x_{bc} \cdots x_{zu} (x_{zu,n-1} + x_{zu})
\]

\[
= x_{n-1,n} x_{n-1,b} x_{bc} \cdots x_{zu,n-1} + x_{n-1,n} x_{n-1,b} x_{bc} \cdots x_{zu}
\]

\[
+ x_{n-1,n} x_{ab} x_{bc} \cdots x_{zu,n-1} + x_{n-1,n} x_{ab} x_{bc} \cdots x_{zu}.
\]
All four of the terms in the last expression contain cycles among the $x_{km}$'s. Now, suppose $M$ contains $x_{n-1,n}$ as well as a cycle in the $x_{ab}$'s. Let the cycle $P = x_{ab}x_{bc} \ldots x_{za}$ be a primitive cycle in $M$. If none of $a, b, \ldots, z$ are $v_{n-1}$ or $v_n$, then $P$ is appears only in products containing $x_{ab}x_{bc} \ldots x_{za}$ which is a cycle among the $x_{ab}$'s. If either $v_{n-1}$ or $v_n$ are among $a, b, \ldots, z$, but not together, then $P$ appears only in products containing $x_{ab}x_{bc} \ldots x_{za}$ which where the vertex $u$ replaces any appearance of $v_{n-1}$ or $v_n$. These products all contain cycles of the $x_{ab}$'s. If $x_{n-1,n}$ is in $P$, say as $a = v_{n-1}$ and $b = v_n$, then $P$ appears only in products containing $x_{n-1,n}x_{bc} \ldots x_{za}$ which contains a cycle of $x_{ab}$'s.

We have shown that $M$ contains cycles among the $x_{ab}$'s if and only if $M$ appears in the expansion of a cyclic product of $x_{ab}$'s. This implies that $M$ is in the expansion of $x_{n-1,n}/n-1(A, \sigma)$ if and only if $M$ has a factor of $x_{n-1,n}$ and $M$ contains no cycles, which completes the proof of the theorem.

One interesting consequence of this theorem is that if an $n$-simplex $S$ dissected by one cevian per edge in general position, then $S$ is dissected into the number of terms in the forest polynomial $P_n(x_{01}, \ldots, x_{n-1,n})$. This follows by substituting $x_{ij} = 1$ for all $i$ and $j$. In turn, the number of terms of the forest polynomial $P_n$ is the number of labelled forests on $n+1$ vertices. The number of labelled trees on $n+1$ vertices is known to be $(n+1)^{n+1}$. Furthermore, using standard graph enumeration techniques (see Wilf [2] for a nice discussion), if

$$f(x) = x + 2x^2 + 7x^3 + 38x^4 + 291x^5 + \frac{2932x^6}{6!} + \cdots$$

is the exponential generating function for the number of labelled trees on $n$ vertices, then $g(x) = e^{f(x)} - 1$ is the generating function for the number of labelled forests on $n$ vertices.

Calculating the first few coefficients of this function, we find

$$g(x) = x + 2x^2 + \frac{7x^3}{3!} + \frac{38x^4}{4!} + \frac{291x^5}{5!} + \frac{2932x^6}{6!} + \cdots$$

Thus, $P_1$ has two terms, $P_2$ has seven terms, $P_3$ has 38, $P_4$ has 291, and $P_5$ has 2932 terms. We note that if we choose one cevian per edge passing through the center of the simplex, then $S$ is dissected into $(n+1)!$ pieces. This is shown easily by induction. Therefore, we note that $P_n$ has more than $(n+1)!$ terms. It is a challenge to visualize how one cevian per edge (in general position) of the regular tetrahedron dissects it into 38 pieces.
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REFERENCES