Finite Length and Pure-Injective Modules over a Ring of Differential Operators

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Let $k$ be an algebraically closed field of characteristic zero, $\mathcal{C}_n = k[[x_1, \ldots, x_n]]$ the ring of formal power series over $k$, and $\mathcal{D}_n$ the ring of differential operators over $\mathcal{C}_n$. Suppose that $\rho$ is a prime ideal of $\mathcal{C}_n$ of height $n - 1$; i.e., $A = \mathcal{C}_n/\rho$ is a curve. We prove that every indecomposable finite length module over $\mathcal{D}_n$ with support on $\rho$ is uniserial with isomorphic or alternating composition factors. For the ring $\mathcal{D}(A)$ of differential operators over $A$ we also classify indecomposable pure-injective modules and show that the Cantor–Bendixson rank of the Ziegler spectrum over $\mathcal{D}(A)$ is equal to 2.

1. INTRODUCTION

Let $k$ be an algebraically closed field of characteristic zero, let $k[[x_1, \ldots, x_n]] = \mathcal{C}_n$ be the ring of formal power series over $k$, and let $\mathcal{D}_n = \mathcal{C}_n[\partial/\partial x_1, \ldots, \partial/\partial x_n]$ be the ring of differential operators over $\mathcal{C}_n$.

The problem of classification of simple $\mathcal{D}_n$-modules has a long story. It was considered first by Manin [6] for the case of $\mathcal{D}_1$-modules with regular singularity and without $\mathcal{C}_1$-torsion. Here "regular singularity" means that the differential operator $Y = x \partial$ has an eigenvector. Further investigations dealt mostly with the case of simple modules with regular singularity. Exhaustive historical remarks can be found in [2].

Remarkable progress has been made by van den Essen and Levelt [3], who classified simple $\mathcal{D}_1$-modules including the case of irregular singularity.

In this paper we investigate and completely classify the finite length (equivalently holonomic) $\mathcal{D}_1$-modules. In particular we prove that every such module is uniserial and either homogeneous or with alternating
composition factors. By using the Morita duality we show that the same is true for $D_n$-modules supported on a curve.

Our description relies upon the classification of finite length modules over the ring $D(K)$ of differential operators over the Laurent series field $K = k(x)$ in Puninski [9], which in turn used very heavily the results of Zimmermann [13]. Notice that the crucial role in this classification is played by the trivial $D(K)$-module $K$. In fact, over $D(K)$, finite length modules are organized into infinitely many homogeneous tubes; hence every such module is uniserial homogeneous. The situation changes slightly when we pass to $D_1$—all tubes but one are preserved. But the trivial module produces a new tube with two simples on the mouth.

Bearing this picture in mind we produce a classification of indecomposable pure-injective modules over $D_1$ similarly to that over integers, separating them into “prüfer,” “adic,” and “rationals.” For instance, the Cantor–Bendixson rank of the Ziegler spectrum over $D_1$ is equal to 2 and there is no superdecomposable pure-injective module. Another similarity we show to the abelian groups is that the category of finite length modules over $D_1$ admits almost split sequences with at most two terms in the middle. We also prove that there exists a duality between categories of right and left finite length $D_1$-modules.

From the results of Stafford and Smith [12] we derive that similar conclusions are true for $D_n$-modules supported on a curve.

2. BASIC NOTIONS AND FACTS

All the notions from ring and module theory used in the paper are quite standard. For rings and modules of differential operators we follow [2]. For instance, for modules $M, N$ over a commutative ring $A$, $D_A(M, N)$ will denote a collection of differential operators from $M$ to $N$. Then $D(M) = D_A(M, M)$, $D(N) = D_A(N, N)$ are rings, and $D_A(M, N)$ is a $(D_A(N), D_A(M))$-bimodule.

For the basic notions in the model theory of modules the reader is referred to [7]. In particular a monomorphism of right modules $M \to N$ is called pure, if for every left module $K$, the induced morphism $M \otimes K \to N \otimes K$ is mono. A module is called pure-injective if it is injective with respect to pure monomorphisms. A positive–primitive formula ($pp$-formula) $φ(x)$ in one free variable for modules over a ring $R$ is an existentially quantified formula $∃(y_1, \ldots, y_n)$ such that $(y_1, \ldots, y_n)A = x(b_1, \ldots, b_m)$, where $A$ is an $n \times m$ matrix over $R$ and $b_j \in R$. In particular, for $a, b \in R$ we obtain a divisibility formula $a|x$ and an annihilator formula $xb = 0$. Every $pp$-formula $φ(x)$ defines in a module $M$ a $pp$-subgroup
\( \varphi(M) = \{ m \in M | M \models \varphi(m) \}. \) In particular, \((a|x)(M) = Ma \) and \((xb = 0)(M) = \{ m \in M | mb = 0 \} = \text{ann}_M(b) \).

We write \( \psi \rightarrow \varphi \) for pp-formulae \( \psi, \varphi \) if \( \psi(M) \subseteq \varphi(M) \) for every module \( M \). For such pp-formulae we say that \( (\psi, \varphi) \) is a pair of pp-formulae. The Ziegler spectrum \( Zg_R \) over a ring \( R \) is a topological space whose underlying set consists of (isomorphism types of) indecomposable pure-injective \( R \)-modules. A basis of open sets for \( Zg_R \) is given by \( (\psi, \varphi) = \{ M \in Zg_R | \psi(M) \subsetneq \varphi(M) \} \).

We will use freely some model theoretic dualities. One of these (see [7, Chap. 8]) defines an antismetomorphism between the lattices of left and right pp-formulae, such that \( \varphi \rightarrow \psi \) iff \( D\psi \rightarrow D\varphi \) for any pp-formulae \( \varphi, \psi \). Another (see Herzog [4]) acts on the level of theories of modules. For instance, over a right noetherian ring \( R \) the theory \( T \text{inj} \) of injective right \( R \)-modules is dual to the theory \( T \text{flat} \) of flat left \( R \)-modules. \( \text{PE}(\mathcal{M}) \) will denote the pure-injective envelope of a module \( M \) and \( \mathcal{E}(M) \) its injective envelope. For an element \( m \) of a module \( M \), \( pp_m(m) \) is the pp-type of \( m \), i.e., the set of all pp-formulae which are satisfied on \( M \). We say that a pair of pp-formulae \( (\psi, \varphi) \) is minimal if \( \psi < \varphi \) and there is no pp-formula strongly between \( \psi \) and \( \varphi \).

**Fact 2.1** [7, Corollary 9.3]. For every minimal pair of pp-formulae \( (\psi, \varphi) \), there exists a unique indecomposable pure-injective module \( M \) such that \( \psi(M) \subsetneq \varphi(M) \).

A nonsplit short exact sequence of finitely generated modules \( 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \) is called an AR-sequence if (1) every morphism \( A \rightarrow M \) which is not a split monomorphism factors through \( f \) and (2) every morphism \( N \rightarrow C \) which is not a split epimorphism factors through \( g \). The basic properties of almost split sequences can be found in [1].

Recall that \( k \) is an algebraically closed field of characteristic zero, \( \mathcal{O} = k[[x]] \) is the ring of formal power series, and \( \mathcal{D} = \mathcal{O}[\partial] \) is the ring of differential operators acting from the right. It is well known (see [12]) that \( \mathcal{D} \) is a simple hereditary noetherian domain with Krull dimension 1.

The quotient ring of \( \mathcal{O} \) is the Laurent series field \( K = k((x)) \), and \( \mathcal{D}(K) \) denotes the ring of differential operators over \( K \). If \( Y \) is the differential operator \( x \partial \) and \( f = f(x) \in K \), then

\[
(f)(xY - Yx) = (fx^2)' - (fx)'x = f'x^2 + f \cdot 2x - f'x^2 - fx = (f)x;
\]

hence \( xY = Yx + x \) in \( \mathcal{D}(K) \) and similarly \( g(x)Y = Yg(x) + d(g) \), where \( d \) is the derivation of \( K \) given by \( x' = x \).

Thus \( \mathcal{D}(K) \) is the differential polynomial ring \( R = K[Y, d] \); hence \( R \) is a left and right principal ideal domain. So, every simple module over \( R \) is of the form \( R/bR \) for an irreducible polynomial \( b(Y) \in R \). In fact, by [3],
there is such a $b$ of arbitrarily large degree, and it is possible to choose $b$
 of a very nice form. For $b = Y$ we obtain the trivial module $K = R/YR$,
 where an $R$-module structure on $K$ is given by

$$g(x)Y = Yg(x) + dg = d(g) \quad \text{for } g(x) \in K.$$  

The following is a rephrasing of the classification of simple $\mathcal{D}$-modules
from [3].

**Fact 2.2.** Let $M$ be a simple module over $R$. Then either

1. $M$ is isomorphic to the trivial $R$-module $K$, and then $M$ is a
   uniserial $\mathcal{D}$-module with the composition series $0 \subset \mathcal{O} \subset K$,
or
2. $M$ is not isomorphic to the trivial $R$-module $K$, and then $M$ is a
   simple $\mathcal{D}$-module.

Moreover, if $M$ is a simple $\mathcal{D}$-module, then $M \cong \mathcal{O}$, or $M \cong K/\mathcal{O}$,
or $M$ admits the canonical structure of a nontrivial simple $R$-module.

Note that $\mathcal{D}/\partial \mathcal{D} \cong \mathcal{O}$ with respect to the map $1 \to 1$ and
$\mathcal{D}/x \mathcal{D} \cong K/\mathcal{O}$, where $1 \to 1/x$. Similarly, for $\alpha \notin \mathbb{Z}$,
the map $1 \to x^\alpha$ generates the isomorphism
$\mathcal{D}/(x \partial - \alpha - 1) \mathcal{D} \cong x^\alpha K$ and the latter is a simple $\mathcal{D}$-mod-
ule (see [3]). Moreover, for $\alpha, \beta \notin \mathbb{Z}$, $x^\alpha K \cong x^\beta K$ iff $\alpha - \beta \in \mathbb{Z}$.

Since $K = d(K) \oplus 1 \cdot k$, the classification of finite length $R$-modules
from Puninski [9] holds true. Let us recall it. For every irreducible $a \in R$,
$M_k(a)$ denotes the $R$-module with generators $x_0, \ldots, x_k$, and relations

$$\sum_{i=0}^{s} \frac{a^{s-i}}{(s-i)!} x_i = 0, \quad 0 \leq s \leq k,$$

where $a'$ is the usual derivative by $Y$, and we put $a^{(0)} = a, 0! = 1$. For
instance $M_0(a) = R/aR$ is simple and $M_1(a)$ is the module $(x_0, x_1|_{x_0 a = 0},
 x_0 a' + x_1 a = 0)$.

**Fact 2.3.** (1) For any irreducible $a \in R$, $M_k(a)$ is a homogeneous
uniserial $R$-module of length $k + 1$ whose composition factors are isomor-
phic to $R/aR$;

2. every indecomposable finite length module over $R$ is isomorphic to
   $M_k(a)$ for some $k$ and an irreducible $a \in R$.

3. $\text{Hom}(M_k(a), M_k(b)) = \text{Ext}(M_k(a), M_k(b)) = 0$ if $R/aR \n \cong R/bR$;
   the $k$-dimension of $\text{Hom}(M_k(a), M_k(a))$ and $\text{Ext}(M_k(a), M_k(a))$
   is $\min(k, l) + 1$;
(4) there is an AR-sequence starting from $M_k(a)$,

$$0 \rightarrow M_k(a) \xrightarrow{f} M_{k-1}(a) \oplus M_{k+1}(a) \xrightarrow{g} M_k(a) \rightarrow 0,$$

where $f(x_i) = (x_{i-1} + x_i)/2$ for $x_i \in M_k(a)$, $g(x_i) = x_i$ for $x_i \in M_{k-1}(a)$, and $g(x_i) = -x_{i-1}$ for $x_i \in M_{k+1}(a)$ (we put $x_{-1} = 0$).

3. FINITE LENGTH MODULES

**Lemma 3.1.** Suppose that a simple $R$-module $R/aR$ for $0 \neq a \in \mathcal{D}$ is nontrivial. Then $M_k(a)$ is a uniserial homogeneous $\mathcal{D}$-module of length $k + 1$.

**Proof.** Let us prove that $\text{Soc}_{\mathcal{D}}(M_k(a)) = M_0(a)$. Since by [9, Corollary 3.5] we have $M_k(a)/M_0(a) \cong M_{k-1}(a)$, the result will follow by induction. By [9, Lemma 4.9], $\text{Hom}_R(R/bR, M_k(a)) = 0$ for an irreducible $b \in R$ if $R/bR \not\cong R/aR$, and $\text{Hom}_R(R/aR, M_k(a)) = x_0 \cdot k$.

Since $M_k(a)$ is an $R$-module, $\text{Hom}_R(R/aR, M_k(a)) = x_0 \cdot k$. If a simple $\mathcal{D}$-module $M$ is isomorphic to a nontrivial simple $R$-module $R/bR$, then similarly $\text{Hom}_R(M, M_k(a)) = 0$.

So it remains to prove that $\text{Hom}(\mathcal{D}, M_k(a)) = \text{Hom}(K/\mathcal{D}, M_k(a)) = 0$. If the former $\text{Hom}$ is nonzero, then (since $\mathcal{D} \equiv \mathcal{D}/\partial \mathcal{D}$) there is $0 \neq m \in M_k(a)$ such that $m \partial = 0$. Since $M_0(a)$ is a $K$-module, there exists $n \in M_k(a)$ with $nx = m$. Then $nY = nx \partial = m \partial = 0$; hence $m = 0$, since $R/aR$ is nontrivial (see Fact 2.3).

Similarly, letting the latter $\text{Hom}$ be nonzero, we find $0 \neq m \in M_k(a)$ with $mx = 0$. But $x$ is invertible in $K$; hence $m = 0$, a contradiction. 

**Lemma 3.2.** Every finitely presented module over $\mathcal{D}$ is a direct summand of a direct sum of modules $\mathcal{D}/a\mathcal{D}$, $a \in \mathcal{D}$. In particular, every indecomposable finite length module over $\mathcal{D}$ is a direct summand of a module $\mathcal{D}/a\mathcal{D}$, $0 \neq a \in \mathcal{D}$.

**Proof.** Since $\mathcal{D}$ is a hereditary simple noetherian domain, $\mathcal{D}$ is a Dedekind prime ring. Now it suffices to apply [10, Corollaries 2.11, 2.14].

Let $M_k$ be the $R$-module $M_k(Y)$. So $M_k$ is generated by $x_0, \ldots, x_k$ with relations $x_0Y = 0$, $x_0 + x_1Y = 0$, \ldots, $x_{k-1} + x_kY = 0$. Thus, $M_k$ is generated by $x_k$ with the relation $x_kY^{k+1} = 0$.

**Lemma 3.3.** $M_k$ is a uniserial $\mathcal{D}$-module of length $2k + 2$ with the composition series $0 \subset x_0 \mathcal{D} \subset x_0 \mathcal{D} \subset \cdots \subset x_k \mathcal{D} \subset x_k \mathcal{D}$, whose composition factors alternate: $x_0 \mathcal{D}/x_0 \mathcal{D} \cong K/\mathcal{D}$ and $x_0 \mathcal{D}/x_0 \mathcal{D} \cong \mathcal{D}$.

**Proof.** We prove that $\text{Soc}(M_k) = x_0 \mathcal{D}$. Indeed, $\text{Hom}_R(R/aR, M_k) = 0$; hence $\text{Hom}_R(R/aR, M_k) = 0$ for every nontrivial simple $R$-module $R/aR$. If $\text{Hom}(K/\mathcal{D}, M_k) \neq 0$, then there is $0 \neq m \in M_k$ with $mx = 0$. 


But $M_k$ is a $K$-module; hence $m = 0$, a contradiction. Now every $f \in \text{Hom}(\mathcal{O}, M_k)$ is determined by $m = f(1) \in M_k$ such that $m \vartheta = 0$. Let us choose $n \in M_k$ with $nx = m$. Then the map $1 \to n$ defines an $R$-homomorphism $g$ from $K = R/\mathcal{O}R$ to $M_k$ which extends $f$. Therefore $n \in x_0 \cdot k$ by [9, Lemma 4.9]; hence $m = nx \in x_0 \cdot k$.

Let us prove that $\text{Soc}_\mathcal{O}(M_k) = x_0 \mathcal{O}/x_0x \mathcal{O}$. For a nontrivial simple module $R/aR, a \in \mathcal{O}$, every morphism $f: R/aR \to M_k/x_0x \mathcal{O} = \mathcal{O}/Y^kx \mathcal{O}$ is given by left multiplication by $r \in \mathcal{O}$; hence $ra = Y^kxs$ for $s \in \mathcal{O}$. By the result just proved, the composition of $f$ with the projection $\mathcal{O}/Y^kx \mathcal{O} \to \mathcal{O}/Y^k \mathcal{O}$ should be zero; hence $r = Y^kt$ for $t \in \mathcal{O}$. Then $Y^kta = Y^kxs$ yields $ta = xs$. Thus, left multiplication by $t$ defines a homomorphism $R/aR \to \mathcal{O}/x \mathcal{O}$ which is zero. So, $t = xt$ for $h \in \mathcal{O}$; hence $r = Y^kxh$ is zero in $\mathcal{O}/Y^k \mathcal{O}$ and $f$ is zero.

Suppose that there is a homomorphism $g: \mathcal{O} \to \mathcal{O}/Y^kx \mathcal{O}$ given by left multiplication by $r$. Since $\mathcal{O} = \mathcal{O}/x \mathcal{O}, r \vartheta = Y^kxs$ for some $s \in \mathcal{O}$. When $g$ is combined with the projection $\mathcal{O}/Y^kx \mathcal{O} \to \mathcal{O}/Y^k \mathcal{O}$, by the result proved above, $r = Y^{k-1}x \alpha + Y^kt$ for $\alpha \in k, t \in \mathcal{O}$. Then $(Y^{k-1}x \alpha + Y^kt) \vartheta = Y^kxs$ implies $(x \vartheta = Y^kt) \alpha + t \vartheta = xs$. If $\alpha \neq 0$, then $1 \in \mathcal{O} \vartheta + x \mathcal{O}$, which is not the case. Otherwise $\alpha = 0$ and $t \vartheta = xs$ defines a homomorphism $\mathcal{O}/x \mathcal{O} \to \mathcal{O}/x \mathcal{O}$ which should be zero. Thus $t = xt, h \in \mathcal{O}$, therefore $r = Y^kxh$ and $g$ is zero.

Now we consider a homomorphism $K/\mathcal{O} \to \mathcal{O}/Y^kx \mathcal{O}$ given by left multiplication by $r$; hence $nx = Y^kxs$ for $s \in \mathcal{O}$. Taking into consideration the projection $\mathcal{O}/Y^kx \mathcal{O} \to \mathcal{O}/Y^k \mathcal{O}$ by proved above $r = Y^kt$ for $t \in \mathcal{O}$. Then $Y^ktx = Y^kxs$ implies $tx = xs$; hence $t \in \text{End}(K/\mathcal{O}) = k$. So $r \in Y^k \cdot k$; therefore $\text{Soc}_\mathcal{O}(M_k) = x_0 \mathcal{O}/x_0x \mathcal{O}$.

Now the result follows by easy induction.

The following proposition describes the Ext-graph for simples over $\mathcal{O}$.

**Proposition 3.4.** Let $R/aR, a \in \mathcal{O}$, be a simple nontrivial $R$-module. Then $\text{Ext}_\mathcal{O}(R/aR, M) = \text{Ext}_\mathcal{O}(M, R/aR) = 0$ for every simple $\mathcal{O}$-module $M$ with $M \ncong R/aR$, and $\text{Ext}_\mathcal{O}(R/aR, R/aR) = k$. Moreover $\text{Ext}(\mathcal{O}, \mathcal{O}) = \text{Ext}(K/\mathcal{O}, K/\mathcal{O}) = 0$ and $\text{Ext}(K/\mathcal{O}, \mathcal{O}), \text{Ext}(\mathcal{O}, K/\mathcal{O})$ are of $k$-dimension 1. So the Ext-graph for simples over $\mathcal{O}$ is drawn in Fig. 1.

![FIGURE 1](image-url)
Proof. For a simple nontrivial $R$-module $M$ one can apply [9, Lemma 4.5]. Since $R/aR$ is a $K$-module, right multiplication by $x$ acts as an isomorphism of $k$-space $R/aR$; hence $\text{Ext}(K/\mathcal{S}, R/aR) = \text{Ext}(\mathcal{D}/x\mathcal{D}, R/aR) = 0$. By [9, Lemma 4.5] again the action on $R/aR$ by right multiplication by $Y$ is an isomorphism of a $k$-space $R/aR$. Since $Y = x\partial$ and $R/aR$ is a $K$-module, the same is true for the action on $R/aR$ by right multiplication by $\partial$. Thus $\text{Ext}(\mathcal{S}, R/aR) = \text{Ext}(\mathcal{D}/\partial\mathcal{D}, R/aR) = 0$.

Now $\text{Ext}(R/aR, \mathcal{D}/x\mathcal{D}) = 0$ and $\text{Ext}(R/aR, \mathcal{D}/x\mathcal{D} = 0)$ are the same as $\text{Ext}(\mathcal{D}/x\mathcal{D}, R/Ra) = 0$ and $\text{Ext}(\mathcal{D}/\partial\mathcal{D}, R/Ra) = 0$, respectively, which follows by symmetry.

For the remainder, every element $m \in \mathcal{S}$ has a canonical form $m = \sum_{i=0}^{n} \alpha_i x^i$, $\alpha_i \in k$. Thus the action by right multiplication by $\partial$ on $\mathcal{S}$ is onto; hence $\text{Ext}(\partial, \mathcal{S}) = 0$. The image of the action on $\mathcal{S}$ by right multiplication by $x$ is of $k$-codimension one (1 is not in this image); hence $\text{Ext}(K/\mathcal{S}, \mathcal{S}) \cong k$.

Similarly $\text{Ext}(K/\mathcal{S}, K/\mathcal{S}) = 0$ and $\text{Ext}(\partial, K/\mathcal{S})$ is of $k$-dimension one.

In the rest of the paper “word” means a finite word consisting of alternating letters $x$ and $\partial$. For instance $xx$ is not a word.

**Lemma 3.5.** Let $w$ be a word. Then

1. $\text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/xw\mathcal{D}) \cong k$ and $\text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/xw\mathcal{D}) = 0$;
2. $\text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/xw\mathcal{D}) \cong k$ and $\text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/xw\mathcal{D}) = 0$ as soon as $xw$ or $xw$ is defined.

**Proof.** We prove only (1). Applying $\text{Hom}(\mathcal{D}/x\mathcal{D}, -)$ to the short exact sequence

$$0 \rightarrow \mathcal{D}/w\mathcal{D} \rightarrow \mathcal{D}/\partial w\mathcal{D} \rightarrow \mathcal{D}/\partial\mathcal{D} \rightarrow 0$$

we get the exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/w\mathcal{D}) \rightarrow \text{Hom}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/\partial w\mathcal{D}) \rightarrow \text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/w\mathcal{D}) \rightarrow \text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/\partial w\mathcal{D}) \rightarrow 0.$$

Since $w = xw'$, $\text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/w\mathcal{D}) = 0$ by the induction assumption. Thus $\text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/xw\mathcal{D}) \cong \text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/\partial w\mathcal{D})$, which is of $k$-dimension one by Proposition 3.4. Similarly we obtain the exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/w\mathcal{D}) \rightarrow \text{Hom}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/xw\mathcal{D}) \rightarrow \text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/w\mathcal{D}) \rightarrow \text{Ext}(\mathcal{D}/x\mathcal{D}, \mathcal{D}/xw\mathcal{D}) \rightarrow 0.$$
Since \( w = \partial w' \), one may assume that \( \text{Ext}(\mathcal{D}/x, \mathcal{D}/w) \cong k \). Clearly \( \text{Hom}(\mathcal{D}/x, \mathcal{D}/x) \cong k \) and \( \text{Ext}(\mathcal{D}/x, \mathcal{D}/x) = 0 \) by Proposition 3.4. Moreover \( \text{Hom}(\mathcal{D}/x, \mathcal{D}/w) \cong \text{Hom}(\mathcal{D}/x, \mathcal{D}/xw) \), since \( \mathcal{D}/xw \) is a uniserial module. So the results follow.

For a word \( w \) we put \( M(w) = \mathcal{D}/w \). For instance if \( w = \partial x \), then \( M(w) \) is the module \( \mathcal{D}/\partial x \) whose socle and top are isomorphic to \( \mathcal{D}/\partial \mathcal{D} \).

**Theorem 3.6.** Let \( M \) be an indecomposable finite length module over \( \mathcal{D} \). Then \( M \) is isomorphic either to \( M_\alpha(a) \) for a simple nontrivial \( R \)-module \( R/a \) or to \( M(w) \) for a word \( w \) from alternating letters \( x \) and \( \partial \).

**Proof.** Let \( \mathcal{A} \) be the class of finite length \( \mathcal{D} \)-modules all whose composition factors are isomorphic to either \( \mathcal{D}/x \) or \( \mathcal{D}/\partial \). Clearly this class is closed with respect to submodules, factor modules, and extensions. So every finite length module \( M \) over \( \mathcal{D} \) contains the largest submodule \( T(M) \) from \( \mathcal{A} \).

**Step 1.** Let \( N \) be a finite length module over \( \mathcal{D} \) such that \( T(N) = 0 \).

By induction on length we prove that \( N \) is a direct sum of \( R \)-modules \( M_\alpha(a) \) for nontrivial simple \( R \)-modules \( R/a \). For simple \( N \) the conclusion is evident. Suppose that \( N \) is not simple and choose a maximal submodule \( 0 \neq K \subset N \). One may assume that \( K = M_\alpha(a) \oplus \cdots \oplus M_\beta(b) \) for nontrivial simple \( R \)-modules \( R/a, \ldots, R/b \). If the simple module \( N/K \) is isomorphic to \( M_\gamma(c) \) for some simple nontrivial \( R \)-module \( R/c \), then \( N \) admits a natural structure as an \( R \)-module; hence the desired result follows by [9, Proposition 5.2].

Otherwise \( N/K \cong \mathcal{D}/x \) or \( N/K \cong \mathcal{D}/\partial \). But by Proposition 3.4 using the heredity of \( \mathcal{D} \), we obtain \( \text{Ext}(\mathcal{D}/x, K) = 0 \) and \( \text{Ext}(\mathcal{D}/\partial, K) = 0 \). Therefore \( N = K \oplus N/K \); hence \( T(N) \neq 0 \), a contradiction.

**Step 2.** \( T(M) \) is a direct summand of every finite length module \( M \) over \( \mathcal{D} \).

Indeed by what we just proved \( K = M/T(M) \) is a module of the form \( M_\alpha(a) \oplus \cdots \oplus M_\beta(b) \) for nontrivial simple \( R \)-modules \( R/a, \ldots, R/b \). As above, using heredity, we obtain \( \text{Ext}(K, T(M)) = 0 \), which is the desired result.

**Step 3.** So it suffices to consider only the case of indecomposable \( M \) with \( M = T(M) \). We apply induction on the length of \( M \), where the case of simple \( M \) is clear. Let us consider a maximal submodule \( N \subset M \), where one may assume that \( N \) has the desired structure. Thus \( N = N_1 \oplus \cdots \oplus N_m \), where \( N_i = M(w_i) \). For \( m = 1 \) the result follows by Lemma 3.5. So \( m > 1 \) and by symmetry it suffices to consider the case \( M/N \equiv \mathcal{D}/x \).
Choose \( n \in M \setminus N \) with \( nx = (n_1, \ldots, n_m) \in N \). So \( M \) is generated by \( N \cup n \) with the relation \( nx = (n_1, \ldots, n_m) \).

If \( n_i = 0 \) for some \( i \), then \( M \) is decomposable and the result follows. Similarly if \( n_i \in N, x \), say \( n_i = kx \), then replacing \( n \) by \( n - k \), we obtain zero on the \( i \)th position, which yields the desired result. Otherwise \( n_i \notin N, x \) for every \( i \); hence, by Lemma 3.5, \( w_i = \partial w_i \). Since \( \text{Ext}(D/xD, N_i) \equiv k \) by the same lemma, \( \alpha_i = n_i - k_i x \) for some \( k_i \in N, i \neq 0 \neq \alpha_i \in k \). Replacing \( n \) by \( n' = n - k_1 - \cdots - k_m \), we get \( n' x = (\alpha_1, \ldots, \alpha_m) \). W.l.o.g. one may assume that the length of \( N_i \) is the largest, i.e., \( w_1 = w_i h_i \) for every \( i \geq 2 \). Then we choose a new basis \( \alpha_1 + \cdots + \alpha_m, \alpha_2, \ldots, \alpha_m \) for \( N \). Indeed clearly this set generates \( N \). Let \( k = (\alpha_1 + \cdots + \alpha_m)s = \alpha_2 s_2 + \cdots + \alpha_m s_m \) for \( s, s_i \in D \). Then \( \alpha_i s = 0 \) implies \( s \in w_1 D \); hence \( s \in w_i D \) for every \( i \geq 2 \). Therefore \( \alpha_i s = 0 \) for every \( i \geq 2 \); hence \( k = 0 \). So \( M \) is decomposable, a contradiction.

**COROLLARY 3.7.** The category of finite length modules over \( D \) is a direct sum of infinitely many uniserial categories of global dimension one.

Now the following can be easily proved using induction on the length similarly to [9, Corollary 4.15].

**COROLLARY 3.8.** Let \( M \) be a finite length \( D \)-module and \( \varphi \) a pp-formula. Then \( \varphi(M) \) is either finite dimensional or cofinite dimensional over \( k \).

**PROPOSITION 3.9.** The category of finite length modules over \( D \) admits almost split sequences. Precisely, let \( M \) be an indecomposable finite length module over \( D \). Then either \( \text{Soc}(M) \) is a nontrivial \( R \)-module and \( M \) is included in the AR-sequence (4) from Fact 2.3, or \( M = M(w) \) for \( w = w'b \), which is included in the AR-sequence

\[
0 \to M(w) \xrightarrow{f} M(w') \oplus M(aw) \xrightarrow{g} M(aw') \to 0,
\]

where \( f(m) = (\pi(m), am) \) and \( g(m, n) = (am - \pi(n)) \). For instance,

\[
0 \to D/\partial xD \xrightarrow{f} D/\partial D \oplus D/\partial xD \xrightarrow{g} D/\partial xD \to 0
\]

and

\[
0 \to D/\partial xD \xrightarrow{f} D/\partial xD \xrightarrow{g} D/\partial \to 0
\]

are AR-sequences.

**Proof.** Let \( M \) be an indecomposable finite length \( D \)-module.

If \( \text{Soc}(M) \) is a nontrivial simple \( R \)-module, then the conclusion follows from [9, Proposition 3.8]. Otherwise \( M = M(w) \) and let \( h: M \to N \) be a nonsplit morphism, where we may assume that \( N \) is indecomposable;
hence $N = M(w_1)$ for some word $w_1$. If $h$ is not mono then $h$ clearly factors through the projection $\pi: M(w) \to M(w')$, hence through $f$. So we may assume that $h$ is mono, hence $h(1) = n \neq 0$. Let the first letter of $w$ be $\partial$. If $n \in Nx$, then choose $m \in N$ such that $mx = n$ and put $u(1) = m$. Clearly, $u$ defines a homomorphism from $M(xw)$ to $N$ and $h$ factors through the embedding $M(w) \to M(aw)$.

Otherwise $n \notin Nx$, hence $w_1 = \partial w_1'$. We prove that $n \in \text{ann}_0(w') + Nx$, which is clearly enough. For $K = \mathcal{D}/w_1' \mathcal{D} \subset \mathcal{D}/w_1 \mathcal{D}$, by Lemma 3.5 we have $K = Kx$. Since $nw = 0$, $nw = 0$ in the factor $N/K = \mathcal{D}/\partial \mathcal{D}$. Considering the canonical form $(\Sigma \alpha_i x^i)$ for elements of $\mathcal{D}/\partial \mathcal{D} = \mathcal{E}$, we get $n = \alpha_0$ in $N/K$. Since $1 \cdot w' = 0$ in $N/K$, the result follows.

**Proposition 3.10.** There exists a duality between categories of right and left finite length modules over $\mathcal{D}$.

**Proof.** Since (see [13]) there exists a duality between categories of finite length modules over $\mathcal{D}(K)$, it suffices to consider the case of modules $M(w)$ for $w$ being a word consisting of letters $x$ and $\partial$.

If $w = cw'$, then put $DM(w) = \mathcal{D}/\partial w'.b$. For instance, $D(\mathcal{D}/x \partial \mathcal{D}) = \mathcal{D}/\partial bx$. Note that this definition corresponds to Auslander’s formula: the right-hand term of the AR-sequence with source $M$ should be $D(\text{Tr} M)$.

Let $i(w)$ denote the monomorphism $M(w) \to M(aw)$ given by left multiplication by $a$ and let $p(w)$ be an epimorphism $M(w) \to M(w')$, where $w = w'.b$. Similarly to [11, pp. 117–118] one can prove that the category of morphisms between $M(w)$'s is $k$-linear; i.e., every morphism is written uniquely as a linear combination of products of some $p(w)$'s followed by $i(w)$’s. Now the duality can be defined so that the image of $p(w)$ is a left monomorphism $i(w')$ for a suitable word $w'$ and similarly for $i(w)$.

**4. Pure-Injective Modules**

In this section we classify indecomposable pure-injective modules over a ring $\mathcal{D} = \mathcal{E}[\partial]$ of differential operators over $\mathcal{E}$.

For a simple $R$-module $R/aR$, $M_1(a)$ will denote the direct limit of the chain $M_1(a) \subset M_1(a) \subset \cdots$. From [9, Lemma 5.5], $M_1(a)$ is an injective uniserial $R$-module. We define $M_\infty(x)$ as the union of the chain $\mathcal{D}/x \mathcal{D} \subset \mathcal{D}/\partial x \mathcal{D} \subset \cdots$, and $M_\infty(\partial)$ similarly.

**Lemma 4.1.** $M_\infty(t)$ is a uniserial injective $\mathcal{D}$-module.

**Proof.** Let us consider the case $M_\infty(a)$. Since $M_\infty(a)$ is injective as an $R$-module the same is true over $\mathcal{D}$. If $R/aR$ is nontrivial, then $M_\infty(a)$ is
uniserial homogeneous by Lemma 3.1. Otherwise \( M_t(\partial) \cong M_t(Y) \), which is homogeneous with alternating factors over \( \mathcal{D} \) by Lemma 3.3.

Clearly \( M_t(\partial) = M_t(Y) \); hence the result follows. Moreover \( M_t(x) \cong M_t(\partial)/x_0\mathcal{D} \), where \( x_0\mathcal{D} = \text{Soc}(M_t(\partial)) \).

**Theorem 4.2.** Let \( M \) be an indecomposable pure-injective module over \( \mathcal{D} = \mathcal{D}[\partial] \). Then exactly one of the following holds:

1. \( M \cong \text{PE}(N) \) for an indecomposable finite length module \( N \) over \( \mathcal{D} \);
2. \( M \cong \text{E}(N) \) for a simple module \( N \) over \( \mathcal{D} \); hence \( M \) is an indecomposable direct summand of \( \text{PE}(\mathcal{D}_0) \); hence \( M \) is a flat (= torsion-free) \( \mathcal{D} \)-module;
3. \( M \) is an indecomposable direct summand of \( \text{PE}(\mathcal{D}_0) \); hence \( M \) is a flat (= torsion-free) \( \mathcal{D} \)-module;
4. \( M \cong \text{E}(\mathcal{D}_0) \).

Moreover, the isolated points in the Ziegler spectrum \( Z_{\mathcal{D}_0} \) are exactly the modules (1). The points of CB-rank 1 are exactly the modules (2) and (3). Finally, the unique point of CB-rank 2 is \( \text{E}(\mathcal{D}_0) \). Thus, the Cantor–Bendixson rank of the Ziegler spectrum over \( \mathcal{D} \) is 2.

**Proof.** Let \( M \) be an indecomposable pure-injective module over \( \mathcal{D} \) and assume that \( M \) is not torsion-free. Then \( \text{Soc}(M) \) is nonzero and hence contains either a simple nontrivial \( R \)-module \( R/aR \) or one of the modules \( \mathcal{D}/\partial \mathcal{D}, \mathcal{D}/a\mathcal{D} \). Choose \( 0 \neq m \in M \) such that \( mt = 0 \), where \( t = a \) or \( t = x \), \( t = \partial \), respectively. Let \( p = \text{pp}_M(m) \) be the pp-type of \( m \) in \( M \).

From the description of the almost split sequences over \( \mathcal{D} \), there exists the descending chain of pp-formulae \( \varphi_0 = \langle xt = 0 \rangle > \varphi_1 > \cdots \) such that \( (\varphi_{t+1}, \varphi_t) \) is a minimal pair over \( \mathcal{D} \). Indeed for the case \( t = a \) the precise form of \( \varphi_t \) was given in [9, Sect. 5]. For \( t = x \) we define \( \varphi_0(z) = xz = 0 \), \( \varphi_t(z) = x\partial z \wedge zx = 0 \), and so on. The definition for \( t = \partial \) is similar.

Therefore, if \( \varphi_k \in p \) and \( \varphi_{k+1} \notin p \), then \( M \) is isolated by a minimal pair. Since \( M_t(\partial) \) is indecomposable, it has a local endomorphism ring; hence \( \text{PE}(M_t(\partial)) \) is an indecomposable module by [7, Sect. 11.3]. Since \( M_t(\partial) \) realizes this pair, \( M \cong \text{PE}(M_t(\partial)) \) by Fact 2.1.

Otherwise, \( \varphi_k \in p \) for every \( k \); hence there is an embedding \( f_k: M_k(t) \to M \) such that \( f_k(x_0) = m \), thus an embedding \( f: M_t(\partial) \to M \). Since \( M_t(\partial) \) is injective by Lemma 4.1, and \( M \) is indecomposable, \( M \cong M_t(\partial) \).

Suppose that \( M \) is torsion-free; in particular, \( \mathcal{D}_2 \subseteq M \). If \( M \) is injective, then \( M \cong \text{E}(\mathcal{D}_2) \). Otherwise clearly there exist \( t \) which is \( a, x, \) or \( \partial \) and \( m \in M \), such that \( m \in M \setminus Mt \). By symmetry, the pair of left pp-formulae \( (z = 0; tz = 0) \) is minimal in the theory \( T_{\text{inj}} \) of injective left modules. By duality the pair of right pp-formulae \( (t|z; z = z) \) is minimal in the theory
$T_{\text{flat}}$ of flat right modules. Since $M$ and $\mathcal{D}$ are both flat and realize this pair, $M$ is isomorphic to a direct summand of $\text{PE}(\mathcal{D})$.

The counting of the Cantor–Bendixson rank is similar to [9]. Indeed, all points $\text{PE}(N)$ for the indecomposable finite length module $N$ are isolated in $Z_{\mathcal{D}}$. Since every simple module over $\mathcal{D}$ is not injective, every isolated point in $Z_{\mathcal{D}}$ is of this form by [8, Lemma 3.6].

Clearly the pair $(z = 0; zt = 0)$ isolates $M_t(t)$ in $Z_{\mathcal{D}}$ on the level one. Then indecomposable direct summands of $\text{PE}(\mathcal{D})$ are of Cantor–Bendixson rank one by duality. It remains to prove that $E$ is torsion submodule of $\text{PE}$. Since every simple module over $k$ is isomorphic to a direct summand of $\text{PE}$. Moreover $N$ is not injective, every isolated point in $Z_{\mathcal{D}}$ is of this form by [8, Lemma 3.6].

**Corollary 4.3.** Every pure-injective module over $\mathcal{D}$ is a pure-injective envelope of a direct sum of indecomposable pure-injective modules. So there exists no superdecomposable pure-injective module over $\mathcal{D}$.

**Proof.** In the proof of Theorem 4.2 we have been able to find an indecomposable direct summand in any pure-injective module over $\mathcal{D}$.  

**Lemma 4.4.** Let $N$ be an indecomposable finite length module over $\mathcal{D}$ and let module $\mathcal{D}/a\mathcal{D}$ be simple. Then $\text{Hom}_\mathcal{D}(\mathcal{D}/a\mathcal{D}, N) = \text{Hom}_\mathcal{D}(\mathcal{D}/a\mathcal{D}, \text{PE}(N))$, which is either zero- or one-dimensional over $k$.

**Proof.** If $\mathcal{D}/a\mathcal{D} \neq \text{Soc}_\mathcal{D}(N)$ then $\text{Hom}(\mathcal{D}/a\mathcal{D}, N) = 0$ since $N$ is uniserial. Hence $\text{Hom}(\mathcal{D}/a\mathcal{D}, \text{PE}(N)) = 0$ since modules $N$ and $\text{PE}(N)$ are elementarily equivalent.

Otherwise by Fact 2.3 and Lemma 3.5, $\text{Hom}_\mathcal{D}(\mathcal{D}/a\mathcal{D}, N)$ is one-dimensional over $k$. The following arguments are due to Nick Granger. Suppose that there exists $m \in \text{PE}(N) \setminus N$ such that $ma = 0$. Since $N$ is pp-essential in $\text{PE}(N)$, there are $n \in N$ and a pp-formula $\varphi(x, y)$ such that $\text{PE}(N) \models \varphi(m, n) \land \neg \varphi(m, 0)$. Since $N$ is an elementary substructure of $\text{PE}(N)$, there is $m' \in N$ such that $N \models \varphi(m', n) \land \neg \varphi(m', 0)$. Then $\varphi(N, 0)$ is a proper nonzero $k$-subspace of $\text{ann}_N(a)$, a contradiction.  

**Remark 4.5.** Let $N$ be an indecomposable finite length module over $\mathcal{D}$. Then $N \subset \text{PE}(N)$ and $\text{PE}(N)/N \cong E(\mathcal{D})^{(a)}$ for some $a$. Moreover $N$ is a fully invariant submodule of $\text{PE}(N)$ and $\text{End}(N) = \text{End}(\text{PE}(N))$.

**Proof.** Similarly to [13, after Theorem 7] one can prove that $N \subset \text{PE}(N)$; i.e., $N$ is not pure-injective. Let us prove that $N$ coincides with the torsion submodule of $\text{PE}(N)$. Otherwise $mt = n \in N$ for some $m \in \text{PE}(N) \setminus N, t \in \mathcal{D}$, where either $t$ is irreducible in $R$ or $t = x, t \in \mathcal{D}$. Since
N is an elementary substructure of PE(N), there exists \( m' \in N \) such that \( m' = n \). By Lemma 4.4, \( m - m' \in N \); hence \( m \in N \), a contradiction.

We prove that \( K = PE(N)/N \) is an injective module (cf. [13, Corollary 19]). It suffices to prove that \( Kr = K \), where the module \( D/tD \) is simple. If \( \text{soc}(N) \neq D/tD \), then \( Nt = N \) by Fact 2.3 and Lemma 3.5. Hence \( PE(N)t = PE(N) \) since \( N \) and \( PE(N) \) are elementarily equivalent. Otherwise by the same reference \( N/Nt \) is of \( k \)-dimension one and we choose \( n \in N \setminus Nt \).

Then \( ns = 0 \) for some \( 0 \neq s \in \mathcal{D} \); hence \( N = Nt + \text{ann}_N(s) \), and therefore the same decomposition takes place for \( PE(N) \). Since \( N \) is the torsion part of \( PE(N) \), the result follows. Thus \( K \) is an injective torsion-free module.

Every morphism \( f \in \text{End}(PE(N)) \) preserves its torsion part; hence \( N \) is a fully invariant submodule and \( f \) induces a homomorphism from \( \text{End}(PE(N)) \) to \( \text{End}(N) \). If this map is not an embedding then there is \( 0 \neq f \in \text{End}(PE(N)) \) such that \( f|_N = 0 \). Therefore \( f \) induces a nonzero homomorphism from \( K \) which is injective to \( PE(N) \) and the image of which is a direct summand, a contradiction.

Note that in [9] the ring \( \text{End}(N) \) was described explicitly, which in particular yields the following corollary.

**Corollary 4.6.** Let \( N \) be an indecomposable module over \( \mathcal{D} \) of length \( n \). Then \( S = \text{End}(PE(N)) \) is an artinian commutative valuation ring of length \( n \) such that \( S/\text{Jac}(S) = k \).

### 5. GENERALIZATIONS

Let \( \mathcal{O}_n \) be the ring of formal power series over \( k \) and let \( \mathcal{D}_n = \mathcal{O}_n[x_1, \ldots, x_n] \) be the ring of differential operators over \( \mathcal{O}_n \). Let us consider a prime ideal \( \rho \) of \( \mathcal{O}_n \) of height \( n - 1 \) and the corresponding curve \( A = \mathcal{O}_n/\rho \). The following fact is contained in [2].

**Fact 5.1.** The following categories of modules are Morita equivalent:

1. the category of \( \mathcal{D}_n \)-modules supported on \( \rho \) (module \( M \) over \( \mathcal{D}_n \) is called supported on \( \rho \) if \( \text{Supp}(M) \subseteq V(\rho) \));
2. the category of \( \mathcal{D}(A) \)-modules;
3. the category of \( \mathcal{D} \)-modules.

From the above we immediately obtain the following.

**Corollary 5.2.** Every indecomposable finite length \( \mathcal{D}_n \)-module supported on a curve is uniserial and either homogeneous or with alternating composition factors. Moreover this category admits almost split sequences with at most two terms in the middle.
PROPOSITION 5.3. Let $A = \mathcal{O}/\rho$ be a curve and let $M$ be an indecomposable pure-injective module over $\mathcal{D}(A)$. Then exactly one of the following holds true:

1. $M$ is isolated in $Z_{\mathcal{D}(A)}$; equivalently $M \cong \text{PE}(N)$ for an indecomposable finite length $A$-module $N$ (hence $N$ is uniserial);

2. $M$ is of Cantor–Bendixson rank 1 in $Z_{\mathcal{D}(A)}$; equivalently either $M \cong E(S)$ for a simple $A$-module $S$ or $M$ is an indecomposable direct summand of a module $\text{PE}(\mathcal{D}(A, \mathcal{O}))$ (hence $M$ is flat);

3. $M$ is of Cantor–Bendixson rank 2 in $Z_{\mathcal{D}(A)}$, which is equivalent to $M \cong E(\mathcal{D}(A, \mathcal{O}))$.

In particular the Cantor–Bendixson rank of $Z_{\mathcal{D}(A)}$ is equal to 2; hence there exists no superdecomposable pure-injective module over $\mathcal{D}(A)$.

Proof. We may assume that $x_i \notin \rho$; hence $A \subseteq \mathcal{O} = \mathcal{O}_1$. In view of [2] the categories of right $\mathcal{D}$-modules and right $\mathcal{D}(A)$-modules are equivalent where the functor $F: \mathcal{D}\text{-Mod} \to \mathcal{D}(A)\text{-Mod}$ is given by $M \mapsto M \otimes_{\mathcal{O}} \mathcal{D}(A, \mathcal{O})$. Since the image of $\mathcal{D}$ is $\mathcal{D}(A, \mathcal{O})$, the result follows from Theorem 4.2.

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