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Maximal Subgroups of the Harada–Norton Group

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1. INTRODUCTION

In this paper we find the maximal subgroups of the simple group HN (also known as F and F_{5+}) of order 273, 030, 912, 000, 000 = $2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$, and of its automorphism group HN:2. Much of this work was originally done by the first author several years ago (only partly recorded in [3]), but the methods of the second author have led to several simplifications.

THEOREM 1. HN *has* 14 *conjugacy classes of maximal subgroups*:

A_{12}	$2 \cdot HS{:}2$	$U_3(8){:}3$
$2_+^{1+8} \cdot (A_5 \times A_5){:}2$	$(D_{10} \times U_3(5)) \cdot 2$	$5_+^{1+4}{:}2_-^{1+4}.5.4$
$2^6 \cdot U_4(2)$	$(A_6 \times A_6) \cdot D_8$	$2^3.2^2.2^6.(3 \times L_3(2))$
$5^2.5_+^{1+2}.4A_5$	$M_{12}{:}2$ (2 *classes*)	
$3^4{:}2 \cdot (A_4 \times A_4).4$	$3_+^{1+4}{:}4 \cdot A_5$	

THEOREM 2. HN:2 *has* 13 *conjugacy classes of maximal subgroups*:

HN	S_{12}	$4 \cdot HS{:}2$
$U_3(8){:}6$	$2_+^{1+8} \cdot (A_5 \times A_5).2^2$	$5{:}4 \times U_3(5){:}2$
$5_+^{1+4}.4 \cdot 2^4.5.4$	$2^6 \cdot U_4(2){:}2$	$(S_6 \times S_6) \cdot 2^2$
$2^3.2^2.2^6.(S_3 \times L_3(2))$	$5^2.5_+^{1+2}.4S_5$	$3^4{:}2(S_4 \times S_4).2$
$3_+^{1+4}{:}4S_5$		

Note. Our notation for groups, conjugacy classes, characters and so on follows the ATLAS [1].

The arguments we use refer explicitly to subgroups of HN, but apply equally to HN:2, showing that its maximal subgroups (except HN itself) contain those of HN to index 2.

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2. Preliminaries

2.1. *The Graph*

The original construction of the group HN is described in [3], using the work of Harada [2] in characterizing the group, and the computational work of Smith [5]. Many properties of the group HN are given in [2] and [3], though the notation there may differ from that used here, which follows [1]. We sometimes quote such properties without explicit reference.

There is a graph of valence 462 on 1 140 000 nodes on which HN acts. The nodes correspond to the subgroups A_{12}, and are joined if and only if the corresponding A_{12}'s intersect in $A_6 \times A_6$. The suborbit lengths are 1, 462, 2520, 2520, 10 395, 16 632, 30 800, 69 300, 166 320, 166 320, 311 850, and 362 880. The corresponding 2-node stabilizers are A_{12}, $\frac{1}{2}(S_6 wr 2)$, M_{12}, M_{12}, $2^5 S_6$, $\frac{1}{2}(PGL_2(5) wr 2)$, $\frac{1}{2}(S_3 wr A_4)$, $\frac{1}{3}(A_4 wr S_3)$, $2 \times S_6$, $2 \times S_6$, $2^5 S_4$, and $L_2(11)$. In HN:2 suborbits of equal length are fused.

We specify the elements of a given A_{12} in terms of their action on the 12 *letters* $a, b, c, d, e, f, g, h, i, j, k, l$.

2.2. *How to Find Maximal Subgroups*

If M is a maximal subgroup of a simple group G, and K is a minimal normal subgroup of M, then K is a characteristically simple group (i.e., a direct product of isomorphic simple groups) and $M = N_G(K)$. The local case (when K is Abelian) is considered in Section 3, and the nonlocal case (when K is non-Abelian) is treated in Sections 4 and 5.

Now for any triple (X, Y, Z) of conjugacy classes of G, we write $\xi(X, Y, Z)$ for the expression

$$\frac{|G|}{|C(x)|\,|C(y)|\,|C(z)|} \sum \frac{\chi(x).\chi(y).\chi(z)}{\chi(1)},$$

where $x \in X$, $y \in Y$, $z \in Z$, and the sum is taken over all irreducible characters χ of G. This is called the (symmetrized) structure constant, and it is a standard result that

$$\xi(X, Y, Z) = \sum \frac{1}{|C(x, y, z)|},$$

where the sum is taken over all conjugacy classes of ordered triples (x, y, z) with $x \in X$, $y \in Y$, $z \in Z$, and $xyz = 1$. Clearly if $\xi(X, Y, Z) < 1$ and H can be generated by elements $x \in X$, $y \in Y$, $z \in Z$ with $xyz = 1$, then H must have nontrivial centralizer. The following obvious lemma will then deal with most cases:

LEMMA. *If $H < G$ and $C_G(H)$ contains a nontrivial elementary Abelian* *characteristic subgroup K (in particular, if $C_G(H)$ is a nontrivial soluble group) then $N_G(H)$ is contained in the local subgroup $N_G(K)$.*

Some minor alterations are required when G is not simple but G' is. The reader is referred to [6] for a detailed discussion of this case.

3. Local Subgroups

3.1. *The 2-Local Subgroups*

There are just two classes of elements of order 2 in HN, with normalizers $N(2A) \cong 2 \cdot HS{:}2$ and $N(2B) \cong 2_+^{1+8} \cdot (A_5 \times A_5){:}2$. The fusion map from these groups to HN can be seen from [2]. They both have subgroups of index 2, and in each case any four-group containing the centre lies in this subgroup if and only if it contains an even number of $2A$-elements. It follows from this that any elementary Abelian 2-group T in HN supports a bilinear form, with values in $\mathbf{F}_2$, whose value on a pair of elements of T is 1 just when they generate a four-group with an odd number of $2A$-elements. This is associated with a quadratic form whose values are 1 on $2A$-elements and 0 on $1A$- and $2B$-elements. Using this we can define a subgroup T' of T to consist of all elements of T on which the bilinear form is trivial. Then clearly $N(T)$ is contained in $N(T')$, so that it is sufficient to consider the cases where T is one of the following:

(a) a $2A$-pure group, which must necessarily have rank at most 2.

(b) a group on which the bilinear form is nondegenerate. This necessarily has even rank, which must be at least 4 as otherwise the situation reduces to case (a).

(c) a $2B$-pure group.

We deal with these three cases in turn. There are just two classes of $2A$-pure subgroups, with normalizers $N(2A) \cong 2 \cdot HS{:}2$ and $N(2A^2) \cong (A_4 \times A_8){:}2 < A_{12}$. If T is of type (b), it contains a $2A$-pure four-group, and hence lies in the A_{12} containing the normalizer of this four-group. Note that involutions in A_{12} are of class $2B$ just when their cycle-shape is 2^4. In terms of our notation for A_{12} our four-group will have shape $\langle (ab)(cd), (ac)(bd) \rangle$, and T will have one of the following forms:

(1) $\langle (ab)(cd), (ac)(bd), (ef)(gh), (eg)(fh), (ij)(kl), (ik)(jl) \rangle$.

In this case $N(T) \cong 2^6 \cdot U_4(2)$, because it must act transitively on $2A$-pure four-groups. Here the nonsplitness follows from Section 4, where we show that $U_4(2)$ is not contained in HN.

(2) $\langle (ab)(cd), (ac)(bd), (ef)(gh), (eg)(fh) \rangle$.

(3) $\langle (ab)(cd), (ac)(bd), (ef)(gh), (eg)(fh)(ij)(kl) \rangle$.

In these two cases $O_2(C(T))$ will be the above 2^6-group, so that $N(T)$ will lie in the normalizer thereof.

To deal with the case of $2B$-pure groups, we need to know more about the structure of $C(2B)$. We take a 4-dimensional vector space over $\mathbf{F}_4 = \{0, 1, \omega, \bar{\omega}\}$, supporting a quadratic form of maximal Witt index. This form is fixed by a group $O_4^+(4) \cong A_5 \times A_5$ acting on the space. We take the split extension of translation group 2^8 by this group, and further adjoin the field automorphism. Then $C(2B)$ is a double cover of this group in which translation by a nonzero vector lifts to an element of class $2B, 2A, 4A$ or $4A$ according as its norm is $0, 1, \omega$, or $\bar{\omega}$, and in which diagonal involutions of the $A_5 \times A_5$ lift to involutions (of class $2B$), but nondiagonal involutions lift to elements of order 4.

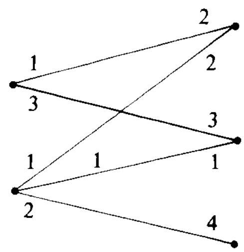
We may then check, using the structure constant $\xi(2B, 2B, 2B)$, that there are just two classes of $2B$-pure four-group in HN. The first type is generated by the centre and any other $2B$-element of the extraspecial group 2^{1+8} in $C(2B)$, while the second type is generated by the centre and a diagonal involution of $A_5 \times A_5$. Two noncentral commuting $2B$-elements in 2^{1+8} generate a four-group of the first type if and only if they correspond to a 1-space over $\mathbf{F}_4$.

Consideration of centralizer orders shows that given a four-group of the second type, there is a unique 2^{1+8} containing it, in which the generators correspond to vectors generating an isotropic 2-space. Any element extending this four-group to a larger $2B$-pure group must also lie in this extraspecial group, as neither diagonal involutions of $A_5 \times A_5$ nor field automorphisms can fix a pair of vectors generating an isotropic 2-space. So any $2B$-pure group T extending our four-group lies in a unique 2^{1+8}, and hence $N(T)$ lies in $C(2B) \cong 2_+^{1+8} \cdot (A_5 \times A_5){:}2$.

It therefore only remains to consider $2B$-pure groups in which every four-group is of the first type. The first two generators of any such group we may take as the centre of a $C(2B)$ and the translation by a given isotropic vector. As pairs of vectors generating an isotropic 2-space correspond to four-groups of the second type, we may only extend this by the translations by other vectors in the same 1-space. The resulting group is in fact exemplified in A_{12} by the 2^3-group fixing four letters and acting regularly on the other eight. The normalizer of this group is $N(2B^3) \cong 2^3.2^2.2^6.(3 \times L_3(2))$. Furthermore, as a four-group of the first type extends uniquely to such an 2^3-group, its normalizer is contained in the above group.

3.2. *The 3-Local Subgroups*

There are just two classes of elements of order 3 in HN, with normalizers $N(3A) \cong (3 \times A_9){:}2$, contained in A_{12}, and $N(3B) \cong 3_+^{1+4}{:}4 \cdot A_5$. Since the latter group contains a Sylow 3-subgroup, it contains a conjugate of every elementary Abelian 3-group. First, we consider the elementary Abelian subgroups of 3^{1+4}_{+}. These, whether or not they contain the centre, correspond to the isotropic 1- and 2-dimensional subspaces ("points" and "lines," respectively) of a 4-dimensional symplectic space over $\mathbf{F}_3$ (the symplectic form being given by the commutator map on 3^{1+4}_{+}). These fall into the following orbits under the action of $4A_5$:

Points			Lines	
Type	Number	Incidence	Number	Type
			10	$3A_2B_2$
$3A$	20			
			20	$3A_3B_1$
$3B$	20			
			10	$3B_4$

In the above table, the "type" refers to the class distribution of the corresponding elementary Abelian subgroup of 3^{1+4}_{+} not containing the centre. In the table below, we give the normalizers of these groups in the left-hand column, and the normalizers of the groups obtained by adjoining the centre, in the right-hand column. An explanation follows the table.

Type	Normalizer	Type	Normalizer
$3A$	$(3\times A_9){:}2 < A_{12}$	$3A_3B_1$	$< N(3B)$
$3B$	$3^{1+4}_{+}{:}4A_5$	$3B_4$	$3^2.(3\times 3^2).2S_4 < N(3^4)$ (c)
$3A_2B_2$	$< N(3^4)$ (c)	$3B_1A_6B_6$	$< N(3B)$ and $N(3^4)$ (a), (c)
$3A_3B_1$	$< N(3B)$	$3B_4A_9$	$< N(3^4)$ (c)
$3B_4$	$< N(3B)$ (b)	$3B_1B_{12}$	$3^3.3^2.2S_4 < N(3B)$ (a)

Notes. (a) Since 7 does not divide the order of $GL_3(3)$, and 13 does not divide the order of HN, it follows that the orbits on $3B$-elements in the groups of type $(3B_1A_6B_6)$ and $(3B_1B_{12})$ cannot fuse, and thus their normalizers are contained in $N(3B)$.

(b) See below.

(c) The 3^3-group of type $(3B_1A_6B_6)$ has centralizer of order 3^4, which can be seen from the structure of $C(3A)$ to be elementary Abelian of type $(3A_{24}B_{16})$. Since this is the only conjugacy class of elementary Abelian

3^4 (see below), it follows that its normalizer is transitive on the $3A$-elements it contains, and therefore has order $2^7 \cdot 3^6$. In fact the normalizer has the shape $3^4{:}2\cdot(A_4\times A_4).4 \cong 3^4{:}GO_4(3)\cdot 2$, and there is a quadratic form on the 3^4-group which is preserved up to sign. Under this quadratic form the $3A$-elements are nonisotropic, and the $3B$-elements are isotropic. We can now see the normalizers of some of the above elementary Abelian 3-groups inside this group. Indeed, if X is a $3B$-pure subgroup of this 3^4-group, then it has order at most 9, and we have just one new case, $N(3B^2) \cong 3^2.(3\times 3^2).2S_4 < N(3^4)$. Otherwise, X contains $3A$-elements, and we may assume that either X is $3A$-pure, or it is generated by $3B$-elements. In either case the restriction q to X of the quadratic form on 3^4 is non-singular, since the radical is equal to the intersection of all the maximal $3B$-pure subgroups. We may also assume that X has order at least 9, so the following cases arise:

Dim(X)	Type(X)	Sign(q)	$N(X)$
2	$3A_2B_2$	+	$< N(3^4)$ (see above)
2	$3A_4$	−	$(3^2{:}4\times A_6).2.2 < (A_6\times A_6)\cdot D_8$
3	$3B_4A_3A_6$		$< N(3^4)$ since the centralizer $= 3^4$
4	$3A_{24}B_{16}$	+	$3^4{:}2\cdot(A_4\times A_4).4$

We must check that $N(X)$ fixes q up to sign. This is obvious except when X has type $3A_4$, in which case each $3A$-element x is orthogonal to a unique other $3A$-element, which is the one in the A_9 in $C(x)\cong 3\times A_9$.

Now consider the case when our elementary Abelian 3-group is not in a group 3^{1+4}. We must first determine the conjugacy classes of 3-elements in $3^{1+4}2A_5\backslash 3^{1+4}$. Now the centralizer in 3^{1+4} of any such 3-element is contained in a "line" L of type $(3B_1A_6B_6)$. Factoring $S=3^{1+4}{:}3$ by this we obtain an elementary Abelian 3^3-group $\bar{S}$ (this is clear by looking in $N(3^4)$). The 9 complements to $3^{1+4}/3^3 \cong 3^2$ in $\bar{S}$ fall into orbits of sizes 1, 4, and 4 under the action of $N(L)\cong 3^{1+4}{:}(2\times 2\cdot S_3)$. The orbit of size 1 gives rise to the elementary Abelian 3^4-group described above. The complements in one of the orbits of size 4 lift only to elements of order 9. Those in the other orbit of size 4 act nontrivially on L, and hence give rise to a single class of self-centralizing elementary Abelian 3^3-groups, each of which intersects L in a "point" P of type $(3B_4)$. Furthermore, since we have already accounted for all the $3A$-elements in $N(3B)$, it follows that this 3^3-group is $3B$-pure. But now $C(P)$ has the shape $3^{2+3}\cong 3^2.(3\times 3^2)\cong 3^4{:}3$, and our 3^3-group is obtained from P by adjoining a "diagonal" element of the quotient $C(P)/P\cong 3\times 3^2$. But if we adjoin to this also the left hand factor then we obtain a group 3^{1+4}. Hence our 3^3-group is conjugate to the "line" of type $(3B_1B_{12})$. The following statements follow immediately:

(i) there are just two classes of elementary Abelian 3-group of type $(3B_4)$

(ii) they are both conjugate to subgroups of the "line" of type $(3B_1B_{12})$

(iii) one has normalizer $3^2.(3\times 3^2).2S_4 < N(3^4)$, as shown above

(iv) the other has centralizer equal to the "line" of type $(3B_1B_{12})$, so has normalizer contained in $N(3B)$. This is case (b) in the above table of subgroups of 3^{1+4}.

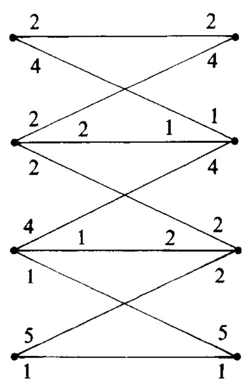
3.3. *The 5-Local Subgroups*

There are 5 classes of 5-elements in HN, with normalizers:

$$N(5A)\cong(D_{10}\times U_3(5))\cdot 2,$$
$$N(5B)\cong 5_+^{1+4}{:}2_-^{1+4}.5.4,$$
$$N(5CD)\cong 5^3{:}4\cdot A_5 < 5^2.5_+^{1+2}{:}4A_5 = N(5B^2),$$
$$N(5E)\cong(5\times 5_+^{1+2}{:}2^2).4 < N(5B).$$

Since $N(5B)$ contains a Sylow 5-subgroup of HN, it contains a conjugate of every elementary Abelian 5-group. We consider first the elementary Abelian subgroups of the group 5_+^{1+4}. Just as in the 3-local case, these correspond to isotropic 1- and 2-dimensional subspaces ("points" and "lines," respectively) of a 4-dimensional symplectic space over $\mathbf{F}_5$. Calculations in $S_4(5)$ show that $2_-^{1+4}.5.4$ has 8 orbits on non-trivial isotropic subspaces, as follows:

Points		Incidence	Lines	
Type	Number		Number	Type
$5A$	20		20	$5A_2E_4$
$5E$	40		80	$5A_1(CD)_4E_1$
$5CD$	80		40	$5B_2(CD)_2E_2$
$5B$	16		16	$5B_1(CD)_5$



We have already considered the normalizers of groups of order 5. A group of order 25 corresponding to a point has type $(5B_1?_5)$, so has normalizer contained in $N(5B)$ unless it has type $(5B_6)$. In the latter case, we shall show later that there is a unique class of $5B$-pure 5^2-group with centralizer of order 5^5, so its normalizer is transitive on the $5B$-elements it contains, and is therefore $5^2.5^{1+2}.4A_5$. (Note: the middle factor is an extraspecial group 5_+^{1+2} rather than an elementary Abelian group 5^3, since otherwise there would be an elementary Abelian subgroup of order 5^4.)

Consider next the elementary Abelian groups of order 25 corresponding to lines. In the first three cases, the centralizer contains a unique Sylow 5-subgroup, which is just the 5^3-group obtained by adjoining the centre of 5_+^{1+4}, so the normalizer is contained in the normalizer of the appropriate 5^3-group. Now these have types $5B_1A_{10}E_{20}$, $5B_1A_5(CD)_{20}E_5$, and $5B_1B_{10}(CD)_{10}E_{10}$, respectively, and since 11 does not divide the order of $GL_3(5)$, it follows that in every case the normalizer is contained in $N(5B)$. In the case of the 16-orbit of lines, the 5^2-group has type $5B_1(CD)_5$, so its normalizer is contained in $N(5B)$, while the corresponding 5^3-group has type $5B_6(CD)_{25}$, so its normalizer is contained in $N(5B^2) \cong 5^2.5^{1+2}.4A_5$.

Finally we must consider the case of an elementary Abelian 5-group X not in 5_+^{1+4}. The existence of self-centralizing elements of order 25 implies that any 5-element in $5^{1+4}.2^{1+4}.5.4 \backslash 5^{1+4}$ centralizes a subgroup of order at most 25 in 5^{1+4}. This is elementary Abelian, of pure $5B$-type, so X is contained in $C(5B^2) \cong 5^2.5^{1+2}$. Hence it can be conjugated into 5^{1+4} by a suitable element of $N(5B^2) \cong 5^2.5^{1+2}.4A_5$. (Note that we have shown in particular that if there is any $5B^2$-group (containing the centre) in $5^{1+4}{:}5$ that is not in 5^{1+4}, then it has centralizer of order 5^3, thus justifying the assertion made above that there is a unique class of $5B^2$-group with centralizer of order 5^5. We proved this before we used the structure of $N(5B^2)$, so the argument is not circular!)

3.4. *Other Local Subgroups*

The normalizers of the remaining elements of prime order are:

$$N(7A) \cong (7{:}3 \times A_5){:}2 < A_{12},$$

$$N(11A) \cong 2 \times 11{:}10 < 2 \cdot HS{:}2,$$

$$N(19AB) \cong 19{:}9 < U_3(8){:}3.$$

4. Candidates for Nonlocal Subgroups

The following is a complete list of the non-Abelian simple groups whose order divides the order of HN:

$A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}$	
$L_2(7)$, $L_2(8)$, $L_2(11)$	$L_2(19)$
$U_3(5)$, $U_3(8)$	$U_3(3)$, $L_3(4)$, $U_4(2)$, $U_4(3)$, $U_5(2)$
	$S_6(2)$, $O_8^+(2)$
M_{11}, M_{12}	M_{22}, HS, M^cL, J_1, J_2.

Of these, all the groups in the left-hand column are in the known subgroups A_{12}, $U_3(5)$, or $U_3(8)$ (see [3]). The groups in the right-hand column are not in HN: to prove this it suffices to prove it for the cases $L_2(19)$, $U_3(3)$, $L_3(4)$, $U_4(2)$, and J_1.

For $U_3(3)$, the 3-central 3-elements must be of class $3B$, since they have centralizer $3^{1+2}{:}4$. Hence the 12-elements are of class $12C$, and so the elements of $U_3(3)$-class $4AB$ are of HN-class $4C$. But $4C$-elements have no square roots, so there is no class restriction to $U_3(3)$.

There is no subgroup J_1, since the latter group contains D_{38}, whereas HN does not.

Now any $L_2(19)$ contains $9A$-elements, so $3B$-elements, so (see Sect. 5.1) A_5's of type $(2B, 3B, 5E)$. Hence it can be generated by two such A_5's intersecting in A_4. But an A_5 of type $(2B, 3B, 5E)$ has normalizer S_5, and can be embedded in A_{12} with orbits $1+5+6$ (2 classes) or $1+1+10$ on the 12 letters. In each case the full S_5 is in A_{12}, and it follows that an A_5 of this type fixes just three nodes in the 1 140 000-node graph. Similarly, the A_4 fixes just five nodes, so any $L_2(19)$ would have to fix a node, i.e. be contained in A_{12}. This contradiction proves that there is no $L_2(19)$ in HN. This argument simplifies that used in [3].

Any $L_3(4)$ contains a group $2^4{:}A_5$ in which all the involutions are conjugate, so of class $2B$. Such a group must be contained in $2^{1+8}\cdot(A_5 \times A_5)$, and the 2^4-group corresponds to a totally isotropic 2-space over $\mathbf{F}_4$. Now there is a single orbit of $A_5 \times A_5$ on such spaces, and the stabilizer is $A_4 \times A_5$. So on factoring out by the group 2^{1+8}, our group 2^4A_5 must map onto one of the factors of $A_5 \times A_5$. But the involutions in such an A_5 do not lift to involutions in $2^{1+8}(A_5 \times A_5)$, so this is impossible.

Now suppose there is a subgroup $U_4(2)$. If all the involutions are of class $2B$, then the same argument produces a contradiction. Otherwise, it follows from the 2-local analysis that there is a unique class of $2^4{:}A_5$ in which the normal subgroup 2^4 contains $2A$-elements and the A_5 acts on it as $O_4^-(2)$. Furthermore this may be embedded in A_{12}, fixing two letters and acting imprimitively on the rest. It contains a unique class of A_5, whose normalizer in A_{12} is $2 \times S_5$. Now $U_4(2)$ can be constructed by taking a group $2^4{:}A_5$ and extending a subgroup A_5 to S_5. But the normalizer of our A_5 in HN is $(2^2 \times A_5){:}2$ (see Sect. 5.1), so it extends to exactly two groups S_5, both of which may be seen in our A_{12}. So since A_{12} does not contain $U_4(2)$, there is no group $U_4(2)$ in HN.

5. The Individual Cases

5.1. A_5 *and* $A_5 \times A_5$

The nonzero (2, 3, 5)-structure constants are

$$\xi(2A, 3A, 5A) = 1/2520, \qquad \xi(2A, 3A, 5E) = 1/10,$$

$$\xi(2B, 3A, 5E) = 1/4, \qquad \xi(2B, 3B, 5E) = 1.$$

Now $C(5A) \cong 5 \times U_3(5)$ and $C(3A) \cong 3 \times A_9$, and the largest intersection of these is A_7. It follows that there is a unique class of A_5 of type $(2A, 3A, 5A)$, and it has normalizer $(A_5 \times A_7){:}2$, contained in A_{12}. Now $C(5E) \cong 5 \times 5^{1+2}{:}2^2$, so any other A_5 has centralizer a subgroup of $5^{1+2}{:}2^2$. Also the only 5-elements which centralize an A_5 have class $5A$, and normalizer $(D_{10} \times U_3(5)) \cdot 2$. This contains two classes of A_5, one containing $5A$-elements, the other with normalizer $D_{10} \times S_5$, contained in $(D_{10} \times U_3(5)) \cdot 2$. Hence the latter is the unique class of A_5's of type $(2A, 3A, 5E)$.

Thus the centralizer of any other A_5 is a subgroup of $\mathrm{Syl}_2(C(5E))$, which is a four-group of type $(2A, 2B, 2B)$. So there is a unique class of A_5 of type $(2B, 3A, 5E)$, and it has normalizer $(2^2 \times A_5){:}2$, contained in $2 \cdot HS{:}2$. Finally, an A_5 of type $(2B, 3B, 5E)$ can have centralizer of order at most 2 (type $2B$) since $3B$-elements do not centralize $2A$-elements. Now the only involutions in $2^{1+8} \cdot (A_5 \times A_5){:}2$ are either in 2^{1+8} or in the outer half, or correspond to diagonal involutions of $A_5 \times A_5$. But as the diagonal 3-elements therein are of class $3A$, there is no A_5 of type $(2B, 3B, 5E)$ in $C(2B)$. Hence there is a unique class of such A_5 in HN, and its normalizer is S_5, contained in A_{12}.

There are two classes of $A_5 \times A_5$, and their normalizers are $(S_5 \times S_5){:}2$, contained in A_{12}, and $(A_5 \times A_5){:}4$, contained in $(A_6 \times A_6) \cdot D_8$.

5.2. A_8 *to* A_{12}

In any of these groups, the 5-point A_5 centralizes a 3-element, so must be of type $(2A, 3A, 5A)$. The group A_n, and indeed its full normalizer, can be obtained by extending the normalizer of a four-group. But the entire normalizers of the A_5 and the four-group are contained in A_{12}, so the normalizer of any such A_n is contained in A_{12}.

5.3. A_6 *and* $A_6 \times A_6$

If A_6 contains an A_5 of type $(2A, 3A, 5A)$, then by the same argument it is contained in A_{12}. But there is a unique class of such A_6 in A_{12}, and its normalizer in HN is $(A_6 \times A_6) \cdot 2^2$, a subgroup of index 2 in $N(A_6 \times A_6) \cong (A_6 \times A_6) \cdot D_8$.

Any other A_6 contains $5E$-elements, and using the fact that the 4-elements square to the involutions, the only relevant nonzero $(2, 4, 5)$-structure constants are:

$$\xi(2A, 4B, 5E) = 4/5, \qquad \xi(2B, 4A, 5E) = 19/4, \qquad \xi(2B, 4C, 5E) = 6.$$

Thus we need only consider A_6's containing $2B$-elements. If the A_6 contains any $3B$-elements, we use the fact that it is generated by two A_5's intersecting in an A_4. Now as we have seen in Section 4 above, these A_5's fix three nodes each, whereas the A_4 fixes five nodes. Thus any such A_6 fixes at least one node, i.e., it is in A_{12}. Its orbits on the 12 letters are either $1+1+10$ or $6+6$, and in either case its normalizer is $A_6 \cdot 2^2$, and is contained in A_{12} or $(A_6 \times A_6) \cdot D_8$.

Since $\xi(2B, 3A, 4C) = 0$, the only case left is $(2B, 3A, 3A, 4A, 5E)$. We construct such an A_6 by taking an A_5 and extending a subgroup A_4 to S_4. Now there is a unique class of A_5 of type $(2B, 3A, 5E)$, represented by the A_5 with orbits $5+5+1+1$ in A_{12}, in which the orbits of an A_4 are $4+4+1+1+1+1$. Such an A_4 extends to four S_4's within A_{12}, one with orbits $8+1+1+1+1$ and three with orbits $4+4+2+2$. But the 2-local analysis shows that all A_4's of type $(2B, 3A)$ are conjugate, and the structure constants $\xi(2B, 3A, 3A) = 1/96$ and $\xi(2B, 3A, 4A) = 1/24$ show that such an A_4 extends to exactly four S_4's of type $(2B, 3A, 4A)$. It now follows that any A_6 of this type is in A_{12}. Furthermore its orbit structure must be $6+6$, and the involution that interchanges the two orbits is the unique $2A$-element that centralizes the A_6. (Indeed, we saw above that there is a unique $2A$-element centralizing a subgroup A_5.) Thus the normalizer of such an A_6 lies in $N(2A) \cong 2 \cdot HS{:}2$.

There is a unique class of $A_6 \times A_6$, it is contained in A_{12}, and has normalizer $(A_6 \times A_6) \cdot D_8$. The action of the D_8 is as follows: the central element extends each A_6 to S_6, another element extends one A_6 to $PGL_2(9)$ and the other to M_{10}, and a further element interchanges the two A_6's.

5.4. A_7

If A_7 contains an A_5 of type $(2A, 3A, 5A)$, then by the same argument as used above for A_8 to A_{12}, its normalizer is contained in A_{12}. Any other A_7 containing $2A$-elements can be constructed by taking an S_5 of type $(2A, 3A, 5E)$ and extending S_4 to $(A_4 \times 3){:}2$. Now there are two classes of A_4 of type $(2A, 3A)$, with normalizers $(A_4 \times A_8){:}2$ and $(A_4 \times 3){:}2 \times A_5$, respectively, both contained in A_{12}. The former extends only to A_5's of type $(2A, 3A, 5A)$, as the entire $3A$-normalizer is contained in A_{12}. Hence our S_5 contains the latter class of A_4. Now the only $2A$-elements in this A_4-normalizer are either in the $(A_4 \times 3){:}2$ or in the A_5, and so there is a unique way of making the required extension. Hence there is a unique class of

A_7 of type $(2A, 5E)$, and it has normalizer $D_{10} \times A_7$, contained in $(D_{10} \times U_3(5)) \cdot 2$.

Remark. A similar argument shows that there is a unique class of A_6 of type $(2A, 5E)$, with normalizer $D_{10} \times M_{10}$, also contained in $(D_{10} \times U_3(5)) \cdot 2$.

5.5. $L_2(7)$ *and* $L_2(8)$

The nonzero (2, 3, 7)-structure constants are

$$\xi(2A, 3A, 7A) = 2/15, \qquad \xi(2B, 3A, 7A) = 1/12, \qquad \xi(2B, 3B, 7A) = 19/3.$$

Hence we need only consider $L_2(7)$'s of type $(2B, 3B, 4A/C, 7A)$ and $L_2(8)$'s of type $(2B, 3B, 7A, 9A)$.

To deal with $L_2(8)$ we consider the subgroup $2^3{:}7$. All involutions and four-groups are conjugate here, so that as $2B$-pure groups the latter must all be of one type (see Sect. 3.1). This cannot be the second type, as the normalizer of any $2B$-pure group containing such a subgroup lies in $C(2B)$ and hence has no element of order 7. So, by the results of our 2-local analysis, we may represent the 2^3 in A_{12} as fixing four letters and acting regularly on the rest. We note, by comparing centralizers, that this group fixes 64 nodes of our graph, so that $2^3{:}7$ fixes at least one. It must therefore fix the unique node fixed by an element of order 7, which is also fixed by any element normalizing it. Hence this node is fixed by any group generated in this way by $2^3{:}7$ and D_{14}, in particular by any $L_2(8)$. Therefore $L_2(8)$ and its normalizer are contained in A_{12}. This normalizer is in fact $3 \times L_2(8){:}3$.

Some detailed knowledge of the group HN, as given in [3], is required to deal with $L_2(7)$. We use the following general property of permutation groups. If a group G acts transitively on a set S, then the number of elements of a conjugacy class of G taking a point of S into a given suborbit O_j relative to that point is given as follows. If the permutation character is $\chi_1 + \chi_2 + \cdots + \chi_n$ and the eigenvalue of χ_i on the suborbit O_j is a_{ij}, then the number of elements conjugate to g that take a given point into the corresponding O_j-suborbit is $\sum_{i=1}^{n} a_{ij}\chi_i(g)$. For each class of HN, and each suborbit of the 1140000-node graph, these sums are given in Table 3 of [3].

Now if we have a group $L_2(7)$ of type $(2B, 3B, 7A)$, then each of the eight subgroups of order 21 fixes a unique node. We ask which of the suborbits corresponding to one of these nodes the other seven lie in. But there is an element of class $3B$ stabilizing any pair of these eight nodes, so that the corresponding suborbit stabilizer must include a $3B$-element. This means that its orbit length must be 1, 462, 30 800, 69 300, or 2520 (there are two orbits with this last length). But it cannot be 1, as the entire $L_2(7)$

would lie in an A_{12}; however, A_{12} contains no such subgroup with 3-elements of class $3B$ (i.e. cycle shape $3^3 1^3$). Now in our putative $L_2(7)$, there are $2B$- and $3B$-elements taking one of the 8 nodes to any other. Hence the orbit length cannot be 462 or 2520, since no $3B$-element takes the fixed node into one of these orbits, and similarly it cannot be 69300, since no $2B$-element takes the fixed node into this orbit (see Table 3 of [3]). So the 30800-suborbit is the only possibility. The stabilizer in HN:2 of the corresponding suborbit is $S_3 \, wr \, A_4$, in which just the even permutations are in HN. Nodes in this suborbit may be described by a decomposition of the 12 letters permuted by our A_{12} into four triples. (Strictly speaking, there are two nodes corresponding to a given decomposition, which are interchanged by odd permutations of the triples.)

If, with the usual notation for the 12 letters, we take the intersection of our putative $L_2(7)$ and the A_{12} to be $\langle (abcdefg), (bce)(dgf)(hij) \rangle$, then one of the other seven nodes permuted by the $L_2(7)$ will be fixed by $(bce)(dgf)(hij)$. The corresponding decomposition into triples may then be taken, without loss of generality, to be one of $\{bce, dgf, akl, hij\}$, $\{bgh, cfi, dej, akl\}$ or $\{bdh, cgi, efj, akl\}$.

In the first of these cases, the subgroup of HN fixing all eight nodes would be non-trivial (it is actually generated by (hij)), so that our $L_2(7)$ would have to lie in its normalizer. But this normalizer is contained in our A_{12}, and we have already seen that this contains no $L_2(7)$ with 3-elements of class $3B$.

To deal with the other cases, it is sufficient to show that the eight nodes do not lie in the 30 800-suborbits corresponding to one another. If they did, then the inner product of the corresponding vectors in the 133-dimensional representation would be 3 (on the scale, used in [3], where the vectors have norm 21). We can show by computation that this does not happen. Alternatively, we calculate using [3] that the six vectors fixed by $\langle (abc), (def), (ghi), (jkl) \rangle$ (or the corresponding group for any other splitting of the 12 letters into triples) sum to zero in the 133-space, as the norm of their sum is zero. We note that one of these six vectors lies in the 1-suborbit, three in the 462-suborbit, and two in the 30 800-suborbit. If we try to determine the inner product table between the vectors fixed by $\langle (bgh), (cfi), (dej), (akl) \rangle$ and those fixed by $\langle (cah), (dgi), (efj), (bkl) \rangle$ we quickly reach a contradiction. The same arguments hold in the third case, and we therefore conclude that HN contains no $L_2(7)$ of type $(2B, 3B, 7A)$.

5.6. $L_2(11)$ *and* M_{11}

Since $\xi(2A, 3A, 11A) = 1/2$ it follows that there is a unique class of $L_2(11)$ of type $(2A, 3A)$, with normalizer $2 \times L_2(11){:}2$, contained in $2 \cdot HS{:}2$. Any other $L_2(11)$ contains an A_5 of type $(2B, 3A/B, 5E)$, so has type $(2B, 3A, 5E, 6B, 11A)$ or $(2B, 3B, 5E, 6C, 11A)$. In the former case the subgroup A_5 contains $2B$-pure four-groups of the first type (see Sect. 3.1) while the subgroup D_{12} contains $2B$-pure four-groups of the second type, and so there is no such $L_2(11)$. In the latter case, the group may be generated by two A_5's intersecting in an A_4, and by the argument used above for A_6 and $L_2(19)$, it follows that such an $L_2(11)$ is contained in A_{12}. Hence there is a unique conjugacy class in HN:2, with normalizer $L_2(11){:}2$, contained in $M_{12}{:}2 < \text{HN}$.

Now M_{11} can be constructed from $L_2(11)$ by extending A_5 to S_5. (But note that $M_{12}{:}2$ can also be constructed in this way, using the $L_2(11)$ which is maximal in M_{12}.) If we start with the $L_2(11)$ of type ($2A$, $3A$, $5E$, $6A$, $11A$), then the corresponding A_5 has normalizer $D_{10} \times S_5$, so there are 6 ways of extending A_5 to S_5. Two of these are centralized by the involution centralizing $L_2(11)$, so the normalizer of the group so generated is contained in $2 \cdot HS{:}2$. The other four fall into two orbits of size 2 under the centralizing involution, and give rise to the two classes of $M_{12}{:}2$.

If we start with the $L_2(11)$ of type ($2B$, $3B$, $5E$, $6C$, $11A$), then the A_5 has normalizer S_5. Thus there is a unique class of M_{11} of this type, and it is self-normalizing and contained in A_{12}.

5.7. M_{12}

The elements of order 5 in M_{12} are of HN-class $5E$, since they normalize elements of order 11. Then the 10-elements are rational and square to these, so are of class $10F$, and so the elements of M_{12}-class $2A$ are of HN-class $2A$. Hence, any M_{12} contains an A_5 of type ($2A$, $3A$, $5E$), and can be constructed from this A_5 by extending A_4 to $A_4 \times S_3$. Now we have seen above (see section 5.4) that the normalizer of this A_4 in HN is $(A_4 \times 3){:}2 \times A_5$. Thus there are 10 ways of making the required extension, and these fall into two orbits of size 5 under the D_{10} centralizing our A_5. Hence there are just two classes of M_{12}, and in each case the normalizer is $M_{12}{:}2$, since extending A_5 to S_5 normalizes both groups.

Remark. The 133-dimensional character restricts to M_{12} as $1aa + 16ab + 45a + 54a$, and so any M_{12} has type ($2A$, $2B$, $3B$, $3A$, $4A$, $4A$, $5E$, $6A$, $6C$, $8B$, $8B$, $10F$, $11A$). Then the two classes of $M_{12}{:}2$ account fully for the structure constant $\xi_{\text{HN}}(2A, 3B, 11A) = 2$, thus furnishing an alternative proof that there is no other M_{12}.

5.8. $U_3(5)$

The 133-dimensional character restricts to $U_3(5)$ as $21a + 28bbcc$, so any $U_3(5)$ has type ($2A$, $3A$, $4B$, $5B$, $5A$, $5E$, $5E$, $6A$, $7A$, $8A$, $10A$). Hence any $U_3(5)$ can be constructed by taking an A_5 of type ($2A$, $3A$, $5E$) and extending a subgroup 2^2 to an A_5 of type ($2A$, $3A$, $5A$). Now the 2^2-group extends to just 8 groups A_5 of type $(2A, 3A, 5A)$ (since its normalizer is contained in A_{12}). But the centralizer of our first A_5 has order 10, so every $U_3(5)$ has nontrivial centralizer. As $U_3(5)$ is contained uniquely in $2 \cdot HS{:}2$, it follows that there is a unique class of $U_3(5)$ in HN, and it has normalizer $(D_{10} \times U_3(5)) \cdot 2 = N(5A)$.

5.9. $U_3(8)$

The group $U_3(8)$ may be constructed by taking a group $3 \times L_2(8)$, and extending the subgroup $3 \times 2^3{:}7$ to $2^{3+6}{:}21$. Now there is a unique class of $3 \times L_2(8)$, and they have normalizer $3 \times L_2(8){:}3$. Furthermore, the normalizer of the relevant 2^3-group in HN is $2^3.2^2.2^6.(3 \times L_3(2))$, in which there is a unique way of making the required extension. Hence there is a unique class of $U_3(8)$ in HN, with normalizer $U_3(8){:}3$.

Remark. The existence of $U_3(8){:}3$ in HN was originally proved in [3] by a related construction. An alternative proof, using the existence of the Fischer–Griess Monster and a result of Thompson, can be obtained by the method used in [4].

6. Conclusion

Collecting together the results of Sections 3, 4, and 5, we see that any proper subgroup of HN (resp. HN:2) is contained in one of the groups listed in Theorem 1 (resp. Theorem 2). Conversely, it is easy to see that none of these groups is contained in any other, thus concluding the proof of the theorems.

References

1. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "An ATLAS of Finite Groups," Oxford Univ. Press, London, 1985.
2. K. Harada, On the simple group F of order $2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$, *in* "Proceedings of the Conference on Finite Groups, Utah 1975", (W. R. Scott and F. Gross, Eds.), pp. 119–276, Academic Press, New York, 1976.
3. S. P. Norton, "*F* and Other Simple Groups," Ph. D. thesis, Univ. of Cambridge, England, 1975.
4. S. P. Norton, The Uniqueness of the Fischer–Griess Monster, *in* "Finite Groups—Coming of Age" (J. McKay, Ed.), pp. 271–285, Contemporary Mathematics Series, Vol. 45, 1985.
5. P. E. Smith, "On Certain Finite Simple Groups," Ph. D. thesis, Univ. of Cambridge, England, 1975.
6. R. A. Wilson, Maximal subgroups of automorphism groups of simple groups, *J. London Math. Soc.* (2) **32** (1985), 460–466.