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Nilpotent maximal subgroups of $GL_n(D)$

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Abstract

In [S. Akbari, J. Algebra 217 (1999) 422–433] it has been conjectured that if D is a noncommutative division ring, then D^* contains no nilpotent maximal subgroup. In connection with this conjecture we show that if $GL_n(D)$ contains a nilpotent maximal subgroup, say M, then M is abelian, provided D is infinite. This extends one of the main results appeared in [S. Akbari, J. Algebra 259 (2003) 201–225, Theorem 4].

Keywords: Division rings; Nilpotent groups; Maximal subgroups

1. Introduction

The structure of matrix groups over division rings is completely different from that of linear groups. Linear groups are now very well understood. But when we deal with skew linear groups everything changes. There are a lot of skew linear groups which are very different from linear groups. In this paper we investigate some properties of maximal subgroups of the general skew linear group. The structure of such groups have been studied in various papers (e.g., see [1–4]). An interesting question which has not been answered yet is whether the multiplicative group of every noncommutative division ring has a maximal subgroup. In [4] it is conjectured that the multiplicative group of a division ring contains no nilpotent maximal subgroup. In connection with this conjecture, in [1] it is proved that if D is a division ring with center F and M is a nilpotent maximal subgroup of D^* such that $F[M]\backslash F$ contains an algebraic element over F, then M is abelian. Here using crossed products we omit the above condition and prove a more general statement, namely for any

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natural number n and any infinite division ring D, the group $GL_n(D)$ contains no non-abelian nilpotent maximal subgroup.

2. Notations and conventions

For a group G and a subset S of G we denote by $Z(G)$, $Z_r(G)$, G', $C_G(S)$, and $N_G(S)$ the center, rth center, derived subgroup, centralizer of S in G, and normalizer of S in G, respectively.

Let R be a ring and X a subset of R. The set of all non-zero elements of X is denoted by X^*. The group of units of R is denoted by $U(R)$. Let R be a ring and S a subring of R. Suppose that G is a subgroup of $U(R)$ normalizing S. If $R = S[G]$ (i.e., the ring generated by S and G) and if $N = G \cap S$ is a normal subgroup of G with $R = \bigoplus_{t \in T} tS$ for some transversal T of N to G, we say $(R, S, G, G/N)$ is a *crossed product*. Let D be a division ring with center F and n be a natural number. We denote by $M_n(D)$ the ring of $n \times n$ matrices over D and denote by $GL_n(D)$ its group of units. Also denote by $SL_n(D)$ the derived subgroup of $GL_n(D)$. Suppose that G is a subgroup of $GL_n(D)$. Obviously we can regard D^n as a D–G bimodule. We say that G is *irreducible*, *reducible*, or *completely reducible*, whenever D^n has the corresponding property as D–G bimodule. Also G is called *absolutely irreducible* if $F[G] = M_n(D)$.

3. Results

The structure of maximal nilpotent subgroups of general linear group was extensively studied by Suprunenko; the main results can be found in [11]. Here we study the structure of nilpotent maximal subgroups of $GL_n(D)$ for a natural number n and a division ring D. First we state the following useful lemma.

Lemma 1. *Let D be a division ring with center F and M be a maximal subgroup of D^* such that $Z_2(M) \neq Z(M)$. Then $F(M')^* \subseteq M$.*

Proof. On the contrary suppose $F(M')^* \nsubseteq M$. By [1, Lemma 2], M contains either D' or F^*. If M contains D', then it is a normal subgroup of D^*. Therefore $Z_2(M)$ is a nilpotent normal subgroup of D^*, so it is central, which contradicts the fact that $Z_2(M) \neq Z(M)$. Therefore $F^* \subseteq M$. Suppose that $x \in Z_2(M) \backslash Z(M)$. By considering the homomorphism $\theta : M \to Z(M)$, taken by the rule $\theta(y) = xyx^{-1}y^{-1}$ we conclude that $M/C_M(x)$ is an abelian group, so $M' \subseteq C_M(x)$. Obviously we have that $M \leqslant N_{D^*}(F(M')^*)$. Noticing maximality of M we conclude that either $F(M')^* \lhd D^*$ or $N_{D^*}(F(M')^*) = M$. The first case can not occur, for if $F(M')^* \lhd D^*$, then by Cartan–Brauer–Hua Theorem [5, p. 222] we obtain that either $F(M') = D$ or $M' \subseteq F$. If $F(M') = D$, then we should have $x \in F$, which is a contradiction and if $M' \subseteq F^*$, then $F(M')^*$ as a subgroup of F^* is contained in M. Therefore $N_{D^*}(F(M')^*) = M$, hence $F(M')^* \leqslant M$. This completes the proof of lemma. $\square$

To prove one of our main theorems we need the following interesting theorem.

Theorem A [12, Corollary 1.5]. *If G is a locally solvable absolutely irreducible skew linear group, then G is abelian-by-(locally finite).*

Remark 2. Let D be a division ring, such that D^* is locally solvable. Using the above theorem and the fact that every abelian normal subgroup of D^* is contained in the center we conclude that D^* is center-by-(locally finite). Now Kaplansky's Theorem [5, p. 259] implies that D is a field.

Theorem 3. *Let D be an infinite dimensional division ring and M a locally solvable maximal subgroup of D^*, then $Z_2(M) = Z(M)$.*

Proof. Suppose that $Z_2(M) \neq Z(M)$. By Lemma 1 we have that $F(M')^* \leqslant M$. Therefore $F(M')$ is a division ring such that its multiplicative group is locally solvable. Hence by Remark 2 we conclude that $F(M')^*$ is abelian. Now using the fact that $M' \lhd M$ and Zorn's Lemma we can find a maximal normal abelian subgroup L of M containing M'. If $L \subseteq Z(M)$, then by choosing an element $a \in Z_2(M)\backslash Z(M)$ we can find an abelian normal subgroup $\langle L, a\rangle$ of M which properly contains L, which is a contradiction. Therefore $L \nsubseteq Z(M)$. We claim that $K = L \cup \{0\}$ is a maximal subfield of D. We have that $M \leqslant N_{D^*}(C_D(L)^*)$. Thus by maximality of M we conclude that either $C_D(L)^*$ is a normal subgroup of D^* or $M = N_{D^*}(C_D(L)^*)$. By the fact that $L \nsubseteq F$ and Cartan–Brauer–Hua Theorem we conclude that the first case is impossible. Therefore $M = N_{D^*}(C_D(L)^*)$. Hence $C_D(L)^* \lhd M$. On the other hand $C_D(L)$ is a division ring such that its multiplicative group is locally solvable. Hence Remark 2 implies that $C_D(L)$ is a field. Now by the choice of L we obtain that $C_D(L)^* = L$. Thus K is a maximal subfield of D and K^* is a subgroup of M containing M'. Let N be a subgroup of M which properly contains K^*. Obviously we have that $N \lhd M$; therefore $M \subseteq N_{D^*}(F(N)^*)$. Thus maximality of M implies that either $N_{D^*}(F(N)^*)$ equals M or $F(N)^*$ is a normal subgroup of D^*. If the second case occurs by Cartan–Brauer–Hua Theorem we conclude that $F(N)$ is either central in D or is D itself. But N contains K^*; hence the first case cannot happen. Therefore $F(N) = D$. Now assume that $N_{D^*}(F(N)^*) = M$; so $F(N)^*$ is locally solvable which by Remark 2 we obtain that it is a field. Thus $N \subseteq C_{D^*}(K^*) = K^*$ which is a contradiction. Therefore we proved that if N is a subgroup of M properly containing K^*, then $F(N) = D$.

Now we claim that $M\backslash K$ contains no element which is algebraic over K. Suppose that $x \in M\backslash K$ is algebraic over K. Assume that x satisfies an equation of the form $\sum_{i=0}^{n} k_i x^i = 0$, where $k_i \in K$ for any $0 \leqslant i \leqslant n$ and $k_n = 1$. Using the fact that x normalizes K and the above equality one can easily show that $R = \sum_{i=0}^{n} K x^i$ is a ring that is of finite dimension as a left vector space over K. Therefore it is a division ring. If we set $N = K^*\langle x\rangle$, by the fact that $x \notin K$ we conclude that N is a subgroup of D^* properly containing K^*; hence by what we proved before we obtain that $F(N) = D$. On the other hand obviously we have that $R = F(N)$. Therefore $[D : K]_\ell < \infty$. Thus by [1, Lemma 6] we conclude that D is a finite dimensional division ring, which is a contradiction. Therefore every element of $M\backslash K$ is transcendental over K.

Now let $a \in M\backslash K$ and set $T = K^*\langle a^2\rangle$. Using the fact that a is transcendental over K one can assume the ring $F[T] = \bigoplus_{i\in\mathbb{Z}} Ka^{2i}$ and conclude that $(F[T], K, T, T/K^*)$ is a crossed product. On the other hand $T/K^* \simeq \langle a^2\rangle$ is the infinite cyclic group. Hence by [10, Theorem 1.4.3] we conclude that $F[T]$ is an Ore domain. On the other hand by what we proved before we conclude that $F(T) = D$. Hence the division ring generated by $F[T]$, which is exactly its classical ring of quotients, coincides with D. Therefore every element of D can be written in the form $z_1z_2^{-1}$, where $z_1, z_2 \in F[T]$ and $z_2 \neq 0$. Thus there exist two elements $s_1, s_2 \in F[T]^*$ such that $a = s_1s_2^{-1}$. But every element of $F[T]$ is a polynomial of a^2 with coefficients from K, thus $s_1 = \sum_{i=l}^m k_ia^{2i}$ and $s_2 = \sum_{i=l}^m k_i'a^{2i}$, where $k_i, k_i' \in K$, for any $l \leqslant i \leqslant m$. Hence $\sum_{i=l}^m ak_i'a^{2i} = \sum_{i=l}^m k_ia^{2i}$. If we set $l_i = ak_i'a^{-1}$, for any $1 \leqslant i \leqslant m$, then l_i's are elements of K and we have $\sum_{i=l}^m l_ia^{2i+1} = \sum_{i=l}^m k_ia^{2i}$ which shows that a is algebraic over K, which is a contradiction. This contradiction shows that $Z_2(M) = Z(M)$ which completes the proof. □

Remark 4. The above theorem is not valid if one omits infinite dimensionality of D. For example, in [1, Theorem 1] it was proved that $M = \mathbb{C}^* \cup \mathbb{C}^*j$ is a solvable maximal subgroup of the division ring of real quaternions. But one can easily show that $i \in Z_2(M)\backslash Z(M)$.

Using Theorem 3 and [1, Theorem 4] we conclude the following corollary. This corollary extends Theorem 4 of [1].

Corollary 5. *Let D be a division ring and M a nilpotent maximal subgroup of D^*. Then M is abelian.*

To prove our main result we need the following theorem. A proof of this theorem can be found in a series of papers [6–8].

Theorem B. *If R is a prime ring such that $U(R)$ satisfies a group identity and generates R as a ring, then either R is a domain or R is isomorphic to the algebra of $n \times n$ matrices over a finite field.*

Now we are in a position to prove our main theorem as follows.

Theorem 6. *Let D be an infinite division ring and n a natural number. If M is a nilpotent maximal subgroup of $GL_n(D)$, then M is abelian.*

Proof. The case $n = 1$ was done in Corollary 5, so we can assume that $n \geqslant 2$. Observing Theorems 12 and 13 of [1] we can assume that M is not absolutely irreducible and D is infinite dimensional over its center F. First we show that M is irreducible. If it is not the case, then there exists a natural number $m < n$ such that M is conjugate to the group

$$\left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \,\middle|\, A \in GL_m(D),\ B \in M_{m\times(n-m)}(D),\ C \in GL_{n-m}(D) \right\}.$$

Since M is nilpotent D^* should be nilpotent, so D is a field, a contradiction. Now by a theorem of [10, p. 9], $F[M]$ is a prime ring. But Theorem B shows that $F[M]$ is either a domain or finite. By [2, Theorem 6] the latter case cannot happen, so $F[M]$ is a domain. On the other hand M is nilpotent, so it is locally polycyclic, therefore the group ring FM is locally Noetherian (cf. [9, p. 425]). So $F[M]$ is locally Noetherian. Therefore Theorem 1.4.2 of [10, p. 25] shows that $F[M]$ is an Ore domain. Now by Theorem 5.7 of [10, p. 213], there exists a subdivision ring D_1 of $M_n(D)$ containing $F[M]$. Combining this with maximality of M we conclude that M is abelian, as required. □

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