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Convolution properties for certain classes of multivalent functions

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Abstract

Recently N.E. Cho, O.S. Kwon and H.M. Srivastava [Nak Eun Cho, Oh Sang Kwon, H.M. Srivastava, Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl. 292 (2004) 470–483] have introduced the class $\mathcal{S}^{\lambda}_{a,c}(\eta;\, p;\, h)$ of multivalent analytic functions and have given a number of results. This class has been defined by means of a special linear operator associated with the Gaussian hypergeometric function. In this paper we have extended some of the previous results and have given other properties of this class. We have made use of differential subordinations and properties of convolution in geometric function theory.

Keywords: Carlson–Shaffer operator; Cho–Kwon–Srivastava operator; Convex functions; Starlike functions; Multivalent functions; Hadamard product; Subordination; Differential subordination

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad \bigl(p \in N := \{1, 2, 3, \ldots\}\bigr), \tag{1}$$

which are analytic and p-valent in the open unit disc $U = \{z\colon\ |z| < 1\}$ on the complex plane C. Let $\mathcal{S}^*$, $\mathcal{K}$ denote the subclasses of $\mathcal{A}_1$ consisting of starlike and convex functions, respectively. If f and g are analytic in U, we say that f is subordinate to g in U, written $f \prec g$, if there exists the Schwarz function ω, analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ in U such that $f(z) = g(\omega(z))$ $(z \in U)$. If g is univalent and $g(0) = f(0)$, then $f(U) \subset g(U)$ follows $f \prec g$.

For $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ the Hadamard product (or convolution) is defined by $(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$. For $a \in R$, $c \in R \setminus Z_0^-$, where $Z_0^- := \{\ldots, -2, -1, 0\}$ H. Saitoh introduced in [6] a linear operator

$$\mathcal{L}_p(a, c) : \mathcal{A}_p \to \mathcal{A}_p$$

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defined by

$$\mathcal{L}_p(a,c)f(z) := \phi_p(a,c;z) * f(z) \quad (z \in U;\ f \in \mathcal{A}_p) \tag{2}$$

where

$$\phi_p(a,c;z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} \quad (z \in U), \tag{3}$$

and $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0, \\ x(x+1)\cdots(x+k-1) & \text{for } k \in N = \{1,2,3,\ldots\}. \end{cases}$$

The operator $\mathcal{L}_p(a,c)$ is an extension of the Carlson–Shaffer operator (see [1]). In [2] Cho, Kwon and Srivastava introduced the following family of linear operators $\mathcal{I}_p^\lambda(a,c)$ analogous to $\mathcal{L}_p(a,c)$:

$$\mathcal{I}_p^\lambda(a,c) : \mathcal{A}_p \to \mathcal{A}_p,$$
$$\mathcal{I}_p^\lambda(a,c)f(z) := \phi_p^\dagger(a,c;z) * f(z) \quad \big(a, c \in R \setminus Z_0^-,\ \lambda > -p,\ z \in U;\ f \in \mathcal{A}_p\big), \tag{4}$$

where $\phi_p^\dagger(a,c;z)$ is the function defined in terms of the Hadamard product (or convolution) by the following condition

$$\phi_p(a,c;z) * \phi_p^\dagger(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}}, \tag{5}$$

where ϕ_p is given by (3). Now we find the explicit form of the function $\phi_p^\dagger(a,c;z)$. It is well known that for $\lambda + p > 0$

$$\frac{z}{(1-z)^{\lambda+p}} = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k}{k!} z^{k+1} \quad (z \in U).$$

Thus

$$\frac{z^p}{(1-z)^{\lambda+p}} = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k}{k!} z^{k+p}. \tag{6}$$

Putting (3) and (6) in (5) we get

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} * \phi_p^\dagger(a,c;z) = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k}{k!} z^{k+p}.$$

Therefore the function $\phi_p^\dagger(a,c;z)$ has the following form

$$\phi_p^\dagger(a,c;z) = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k (c)_k}{k!(a)_k} z^{k+p} \quad (z \in U). \tag{7}$$

The authors of [2] have obtained the following properties of the operator $\mathcal{I}_p^\lambda(a,c)$:

$$\mathcal{I}_p^1(p+1,1)f(z) = f(z) \quad \text{and} \quad \mathcal{I}_p^1(p,1)f(z) = \frac{zf'(z)}{p}, \tag{8}$$
$$z\big(\mathcal{I}_p^\lambda(a+1,c)f(z)\big)' = a\mathcal{I}_p^\lambda(a,c)f(z) - (a-p)\mathcal{I}_p^\lambda(a+1,c)f(z), \tag{9}$$

and

$$z\big(\mathcal{I}_p^\lambda(a,c)f(z)\big)' = (\lambda+p)\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \lambda\mathcal{I}_p^\lambda(a,c)f(z). \tag{10}$$

Let $\mathcal{N}$ be the class of functions h with the normalization $h(0) = 1$, which are convex and univalent in U and satisfy the condition $\mathrm{Re}[h(z)] > 0$ for $z \in U$. In [2], by using the operator $\mathcal{I}_p^\lambda(a,c)$ for $0 \leqslant \eta < p$, $p \in N$, $h \in \mathcal{N}$, the following subclasses of $\mathcal{A}_p$ have been defined:

$$\mathcal{S}_{a,c}^{\lambda}(\eta; p; h) = \left\{ f \in \mathcal{A}_p\colon \frac{1}{p-\eta}\left(\frac{z(\mathcal{I}_p^{\lambda}(a,c)f(z))'}{\mathcal{I}_p^{\lambda}(a,c)f(z)} - \eta\right) \prec h(z),\ z \in U \right\},$$

$$\mathcal{S}_{a,c}^{\lambda}(\eta; p; A, B) := \mathcal{S}_{a,c}^{\lambda}\left(\eta; p; \frac{1+Az}{1+Bz}\right) \quad (-1 \leqslant B < A \leqslant 1)$$

and by applying the properties (8)–(10) many of interesting results have been proved. In particular the several inclusion properties of the classes $\mathcal{S}_{a,c}^{\lambda}(\eta; p; h)$ were investigated. In [2] the authors presented a long list of papers connected with the operators (2) and (4) and classes of functions defined by means of those operators. Thus we refer the reader to [2]. In this paper we continue and extend the consideration of the paper [2]. We recall here the fact that, Dziok and Srivastava [3] have introduced and considered more general the Dziok–Srivastava operator

$$\mathcal{H}_p(a_1, a_2, \ldots, a_q; c_1, c_2, \ldots, c_s; z) : \mathcal{A}_p \to \mathcal{A}_p$$

such that

$$\mathcal{I}_p^{\lambda}(a,c)f(z) = \mathcal{H}_p(\lambda + p, c, a; z)f(z).$$

In [3] Dziok and Srivastava, by using the operator $\mathcal{H}_p$, have introduced and deeply examined a class of p-valent functions with negative coefficients. This class and the class $\mathcal{S}_{a,c}^{\lambda}(\eta; p; A, B)$ are equal for suitable chosen parameters.

2. Inclusion properties

The following lemmas will be used in our investigation.

Lemma 1. *Let $a, a_1, a_2 \in R \setminus Z_0^-$, $c, c_1, c_2 \in R \setminus Z_0^-$. Then for $z \in U$*

$$\phi_p^{\dagger}(a, c_1; z) = \phi_p^{\dagger}(a, c_2; z) * \phi_p(c_1, c_2; z) \tag{11}$$

and

$$\phi_p^{\dagger}(a_2, c; z) = \phi_p^{\dagger}(a_1, c; z) * \phi_p(a_1, a_2; z). \tag{12}$$

Proof. From (7) we have

$$\phi_p^{\dagger}(a, c_1; z) = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k (c_1)_k}{k!(a)_k} z^{k+p} = \sum_{k=0}^{\infty} \frac{(\lambda+p)_k (c_2)_k}{k!(a)_k} \frac{(c_1)_k}{(c_2)_k} z^{k+p} = \phi_p^{\dagger}(a, c_2; z) * \phi_p(c_1, c_2; z)$$

and the condition (11) is proved. The proof of (12) is similar to that of (11) and the details involved may be omitted. The proof of Lemma 1 is thus completed. □

Lemma 2. *(See [5, p. 54].) If $f \in \mathcal{K}$, $g \in \mathcal{S}^*$, then for each analytic function h*

$$\frac{(f * hg)(U)}{(f * g)(U)} \subseteq \overline{\text{co}}\, h(U),$$

where $\overline{\text{co}}\, h(U)$ denotes the closed convex hull of $h(U)$.

Lemma 3. *(See [5].) Let $0 < \alpha \leqslant \beta$. If $\beta \geqslant 2$ or $\alpha + \beta \geqslant 3$, then the function*

$$\phi_1(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} z^{k+1} \quad (z \in U)$$

belongs to the class $\mathcal{K}$ of convex functions.

Remark 1. Lemma 3 is a special case of Theorems 2.12 or 2.13 contained in [5].

From (4) and (7) we directly obtain the following useful conclusion.

Corollary 1. *If $f(z)=z^p+a_{p+1}z^{p+1}+\cdots$, then the Cho–Kwon–Srivastava operator has the form*

$$\mathcal{I}_p^\lambda(a,c)f(z)=\sum_{k=0}^{\infty}\frac{(\lambda+p)_k(c)_k}{k!(a)_k}a_{p+k}z^{p+k}\quad (z\in U)\ \big(a\in R\setminus Z_0^-,\ c\in R\setminus Z_0^-,\ \lambda>-p\big).$$

Let $\alpha,\beta\in C$, $\gamma\in C\setminus Z_0^-$. Then the function

$${}_2F_1(\alpha,\beta,\gamma;z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}z^k\quad (z\in U),$$

satisfies the hypergeometric differential equation

$$z(1-z)w''(z)+\big[\gamma-(\alpha+\beta+1)z\big]w'(z)-\alpha\beta w(z)=0$$

and is called hypergeometric function. Thus we can rewrite Corollary 1 in the following form.

Corollary 2. *If $f(z)=z^p+a_{p+1}z^{p+1}+\cdots$, then*

$$\mathcal{I}_p^\lambda(a,c)f(z)=\big[z^p\cdot{}_2F_1(\lambda+p,c,a;z)\big]*f(z)\quad (z\in U)\ \big(a\in R\setminus Z_0^-,\ c\in R\setminus Z_0^-,\ \lambda>-p\big).$$

Theorem 1. *Let $0<a_1\leqslant a_2$, $c\in R\setminus Z_0^-$, $h\in\mathcal{N}$ and*

$$\operatorname{Re}\big[h(z)\big]>1-\frac{1}{p-\eta}\quad (z\in U). \tag{13}$$

If $a_2\geqslant 2$ or $a_1+a_2\geqslant 3$, then

$$\mathcal{S}_{a_1,c}^\lambda(\eta;p;h)\subset\mathcal{S}_{a_2,c}^\lambda(\eta;p;h).$$

Proof. Let $f\in\mathcal{S}_{a_1,c}^\lambda(\eta;p;h)$. Then from the definition of the class $\mathcal{S}_{a,c}^\lambda(\eta;p;h)$ we have

$$\frac{1}{p-\eta}\left(\frac{z(\mathcal{I}_p^\lambda(a_1,c)f(z))'}{\mathcal{I}_p^\lambda(a_1,c)f(z)}-\eta\right)=h\big(\omega(z)\big)$$

where h is convex univalent in U with $\operatorname{Re}h(z)>0$ and $|\omega(z)|<1$ in U with $\omega(0)=0=h(0)-1$. Therefore

$$\frac{z(\mathcal{I}_p^\lambda(a_1,c)f(z))'}{\mathcal{I}_p^\lambda(a_1,c)f(z)}=(p-\eta)h\big(\omega(z)\big)+\eta, \tag{14}$$

and

$$\frac{z[z^{1-p}(\mathcal{I}_p^\lambda(a_1,c)f(z))]'}{z^{1-p}\mathcal{I}_p^\lambda(a_1,c)f(z)}=(p-\eta)h\big(\omega(z)\big)+\eta-p+1\prec\frac{1+z}{1-z}. \tag{15}$$

Applying the definition of $\mathcal{I}_p^\lambda(a,c)$ and (12) and the properties of convolution we obtain

$$\begin{aligned}\frac{z(\mathcal{I}_p^\lambda(a_2,c)f(z))'}{\mathcal{I}_p^\lambda(a_2,c)f(z)}&=\frac{z(\phi_p^\dagger(a_2,c;z)*f(z))'}{\phi_p^\dagger(a_2,c;z)*f(z)}=\frac{z(\phi_p^\dagger(a_1,c;z)*\phi_p(a_1,a_2;z)*f(z))'}{\phi_p^\dagger(a_1,c;z)*\phi_p(a_1,a_2;z)*f(z)}\\&=\frac{\phi_p(a_1,a_2;z)*z(\phi_p^\dagger(a_1,c;z)*f(z))'}{\phi_p(a_1,a_2;z)*\phi_p^\dagger(a_1,c;z)*f(z)}=\frac{\phi_p(a_1,a_2;z)*z(\mathcal{I}_p^\lambda(a_1,c)f(z))'}{\phi_p(a_1,a_2;z)*\mathcal{I}_p^\lambda(a_1,c)f(z)}.\end{aligned}$$

Therefore by using (14) we obtain

$$\begin{aligned}\frac{1}{p-\eta}\left(\frac{z(\mathcal{I}_p^\lambda(a_2,c)f(z))'}{\mathcal{I}_p^\lambda(a_2,c)f(z)}-\eta\right)&=\frac{1}{p-\eta}\left(\frac{\phi_p(a_1,a_2;z)*z(\mathcal{I}_p^\lambda(a_1,c)f(z))'}{\phi_p(a_1,a_2;z)*\mathcal{I}_p^\lambda(a_1,c)f(z)}-\eta\right)\\&=\frac{1}{p-\eta}\left(\frac{\phi_p(a_1,a_2;z)*[(p-\eta)h(\omega(z))+\eta]\mathcal{I}_p^\lambda(a_1,c)f(z)}{\phi_p(a_1,a_2;z)*\mathcal{I}_p^\lambda(a_1,c)f(z)}-\eta\right).\end{aligned} \tag{16}$$

It follows from Lemma 3 that $z^{1-p}\phi_p(a_1,a_2;z)\in\mathcal{K}$ and it follows from (15) that $z^{1-p}\mathcal{I}_p^\lambda(a_1,c)f(z)\in\mathcal{S}^*$. Let us put $s(\omega(z)):=(p-\eta)h(\omega(z))+\eta$. Then applying Lemma 2 we get

$$\frac{\{[z^{1-p}\phi_p(a_1,a_2;z)]*s(\omega)z^{1-p}\mathcal{I}_p^\lambda(a_1,c)f\}(U)}{\{[z^{1-p}\phi_p(a_1,a_2;z)]*z^{1-p}\mathcal{I}_p^\lambda(a_1,c)f\}(U)}\subseteq\overline{\text{co}}\,s\big(\omega(U)\big)$$

because s is convex univalent function. Therefore we conclude that

$$\frac{1}{p-\eta}\left(\frac{\{\phi_p(a_1,a_2;z)*s(\omega)\mathcal{I}_p^\lambda(a_1,c)f\}(U)}{\{\phi_p(a_1,a_2;z)*\mathcal{I}_p^\lambda(a_1,c)f\}(U)}-\eta\right)\subseteq h(U),$$

and hence that (16) is subordinate to the convex univalent function h, and finally that $f\in\mathcal{S}_{a_2,c}^\lambda(\eta;p;h)$. The proof of Theorem 1 is completed. □

Theorem 1 is a generalization of the result [2] of the form: If $a\geqslant p$ and $\lambda>0$, then $\mathcal{S}_{a,c}^\lambda(\eta;p;h)\subset\mathcal{S}_{a+1,c}^\lambda(\eta;p;h)$ $(h\in\mathcal{N})$.

Theorem 2. *Let $a\in R$, $0<c_1\leqslant c_2$, $h\in\mathcal{N}$ and let h satisfies* (13). *If $c_2\geqslant 2$ or $c_1+c_2\geqslant 3$, then*

$$\mathcal{S}_{a,c_2}^\lambda(\eta;p;h)\subset\mathcal{S}_{a,c_1}^\lambda(\eta;p;h).$$

Proof. Let $f\in\mathcal{S}_{a,c_2}^\lambda(\eta;p;h)$. In the same way as we have obtained (15) we get

$$\frac{z[z^{1-p}(\mathcal{I}_p^\lambda(a,c_2)f(z))]'}{z^{1-p}\mathcal{I}_p^\lambda(a,c_2)f(z)}=(p-\eta)h\big(\omega(z)\big)+\eta-p+1\prec\frac{1+z}{1-z}. \tag{17}$$

Using (11) and the same arguments as in the proof of Theorem 1 we obtain

$$\begin{aligned}\frac{z(\mathcal{I}_p^\lambda(a,c_1)f(z))'}{\mathcal{I}_p^\lambda(a,c_1)f(z)}&=\frac{z(\phi_p^\dagger(a,c_1;z)*f(z))'}{\phi_p^\dagger(a,c_1;z)*f(z)}=\frac{z(\phi_p^\dagger(a,c_2;z)*\phi_p(c_1,c_2;z)*f(z))'}{\phi_p^\dagger(a,c_2;z)*\phi_p(c_1,c_2;z)*f(z)}\\&=\frac{\phi_p(c_1,c_2;z)*z(\phi_p^\dagger(a,c_2;z)*f(z))'}{\phi_p(c_1,c_2;z)*\phi_p^\dagger(a,c_2;z)*f(z)}=\frac{\phi_p(c_1,c_2;z)*z(\mathcal{I}_p^\lambda(a,c_2)f(z))'}{\phi_p(c_1,c_2;z)*\mathcal{I}_p^\lambda(a,c_2)f(z)}.\end{aligned}$$

Therefore we get

$$\begin{aligned}\frac{1}{p-\eta}\left(\frac{z(\mathcal{I}_p^\lambda(a,c_1)f(z))'}{\mathcal{I}_p^\lambda(a,c_1)f(z)}-\eta\right)&=\frac{1}{p-\eta}\left(\frac{\phi_p(c_1,c_2;z)*z(\mathcal{I}_p^\lambda(a,c_2)f(z))'}{\phi_p(c_1,c_2;z)*\mathcal{I}_p^\lambda(a,c_2)f(z)}-\eta\right)\\&=\frac{1}{p-\eta}\left(\frac{\phi_p(c_1,c_2;z)*[(p-\eta)h(\omega(z))+\eta]\mathcal{I}_p^\lambda(a,c_2)f(z)}{\phi_p(c_1,c_2;z)*\mathcal{I}_p^\lambda(a,c_2)f(z)}-\eta\right).\end{aligned} \tag{18}$$

By Lemma 3 we have $z^{1-p}\phi_p(c_1,c_2;z)\in\mathcal{K}$ and by (17) we have $z^{1-p}\mathcal{I}_p^\lambda(a,c_2)f(z)\in\mathcal{S}^*$. Hence, by virtue of Lemma 2, we conclude that (18) is subordinate to h and consequently $f\in\mathcal{S}_{a,c_1}^\lambda(\eta;p;h)$. We thus complete the proof of Theorem 2. □

Remark 2. In [2] there are no results concerning inclusion relationships between the classes $\mathcal{S}_{a,c}^\lambda(\eta;p;h)$ with respect to the parameter c. The above theorem is thus the essential supplement of the results of [2].

Corollary 3. *Let $0<a_1\leqslant a_2$ and $a_2\geqslant\min\{2,3-a_1\}$ and let $0<c_1\leqslant c_2$ and $c_2\geqslant\min\{2,3-c_1\}$. Then for $\frac{1-A}{1-B}>1-\frac{1}{p-\eta}$ we have*

$$\mathcal{S}_{a_1,c_2}^\lambda(\eta;p;A,B)\subset\mathcal{S}_{a_1,c_1}^\lambda(\eta;p;A,B)\subset\mathcal{S}_{a_2,c_1}^\lambda(\eta;p;A,B).$$

Proof. Take $h(z)=\frac{1+Az}{1+Bz}$, $-1\leqslant B<A\leqslant 1$. Then we have $h\in\mathcal{N}$ and $\mathrm{Re}[h(z)]>1-\frac{1}{p-\eta}$, $z\in U$. Thus applying Theorems 1 and 2 we obtain the desired result. □

3. Coefficients estimates

Theorem 3. *Let* $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$. *If* $f \in \mathcal{S}^{\lambda}_{a,c}(\eta; p; h)$ *and the function* h *satisfy* (13), *then*

$$|a_{p+k}| \leqslant \frac{(k+1)!(a)_k}{(\lambda+p)_k(c)_k}, \quad k = 0, 1, 2, \ldots.$$

Proof. If we put $a_1 = a$ in the formula (15), then we have

$$z^{1-p}\mathcal{I}^{\lambda}_p(a,c)f(z) \in \mathcal{S}^*$$

because $f \in \mathcal{S}^{\lambda}_{a,c}(\eta; p; h)$ and $\operatorname{Re}[h(z)] > 1 - \frac{1}{p-\eta}$. Therefore by Corollary 1

$$\sum_{k=0}^{\infty} \frac{(\lambda+p)_k(c)_k}{k!(a)_k} a_{p+k} z^{k+1} \in \mathcal{S}^*,$$

and by using the estimation of $(k+1)$th coefficient of starlike function we obtain

$$\left| \frac{(\lambda+p)_k(c)_k}{k!(a)_k} a_{p+k} \right| \leqslant k+1, \quad k = 0, 1, \ldots,$$

which ends the proof. □

Remark 3. If $h(z) = \frac{1}{p-\eta}[\frac{2z}{1-z} + p - \eta]$, then the above estimates of coefficients become sharp. The extremal function is

$$f_0(z) = \sum_{k=0}^{\infty} \frac{(k+1)!(a)_k}{(\lambda+p)_k(c)_k} z^{k+p}.$$

Then we have

$$\frac{1}{p-\eta}\left[\frac{z(\mathcal{I}^{\lambda}_p(a,c)f_0(z))'}{\mathcal{I}^{\lambda}_p(a,c)f_0(z)} - \eta\right] = \frac{1}{p-\eta}\left[\frac{z[\sum_{k=0}^{\infty}(k+1)z^{p+k}]'}{\sum_{k=0}^{\infty}(k+1)z^{p+k}} - \eta\right] = \frac{1}{p-\eta}\left[\frac{2z}{1-z} + p - \eta\right].$$

4. Structural formula

Theorem 4. *A function* f *belongs to the class* $\mathcal{S}^{\lambda}_{a,c}(\eta; p; h)$ *if and only if there exists a Schwarz function* $\omega(z)$ *such that*

$$f(z) = \left[\sum_{k=0}^{\infty} \frac{k!(a)_k}{(\lambda+p)_k(c)_k} z^{k+p}\right] * \left[z^p \exp \int_0^z \frac{(p-\eta)[h(\omega(t)) - 1]}{t}\, dt\right].$$

Proof. Let $f \in \mathcal{S}^{\lambda}_{a,c}(\eta; p; h)$. Then from the definition of the class $\mathcal{S}^{\lambda}_{a,c}(\eta; p; h)$ we have

$$\frac{1}{p-\eta}\left(\frac{z(\mathcal{I}^{\lambda}_p(a,c)f(z))'}{\mathcal{I}^{\lambda}_p(a,c)f(z)} - \eta\right) = h\big(\omega(z)\big),$$

where $h \in \mathcal{N}$ and $|\omega(z)| < 1$ in U with $\omega(0) = 0 = h(0) - 1$. Therefore

$$\frac{(\mathcal{I}^{\lambda}_p(a,c)f(z))'}{\mathcal{I}^{\lambda}_p(a,c)f(z)} - \frac{p}{z} = \frac{(p-\eta)h(\omega(z)) - (p-\eta)}{z}.$$

Thus

$$\log \frac{\mathcal{I}^{\lambda}_p(a,c)f(z)}{z^p} = \int_0^z \frac{(p-\eta)[h(\omega(t)) - 1]}{t}\, dt.$$

Therefore from (4) and (7) we obtain

$$f(z) * \sum_{k=0}^{\infty} \frac{(\lambda+p)_k (c)_k}{k!(a)_k} z^{p+k} = z^p \exp \int_0^z \frac{(p-\eta)[h(\omega(t))-1]}{t}\, dt$$

and our assertion follows immediately. □

Corollary 4. *A function f belongs to the class $\mathcal{S}^{\lambda}_{a,c}(\eta; p; A, B)$ if and only if there exists a Schwarz function $\omega(z)$ such that*

$$f(z) = \left[\sum_{k=0}^{\infty} \frac{k!(a)_k}{(\lambda+p)_k (c)_k} z^{k+p}\right] * \left[z^p \exp \int_0^z \frac{(p-\eta)(A-B)\omega(t)}{t(1+B\omega(t))}\, dt\right].$$

Remark 4. If we apply the previous theorem to the functions

$$h(z) = \frac{1+[2(p-\eta)^{-1}-1]z}{1-z}, \quad \omega(z) = z,$$

then from the structural formula we obtain the function f_0 (compare this with Remark 3).

Lemma 4. *Let H be starlike in U, with $H(0)=0$ and $a \neq 0$. If $P(z) = a + a_n z^n + \cdots$ is analytic in U and satisfies*

$$\frac{zP'(z)}{P(z)} \prec H(z) \quad (z \in U),$$

then

$$P(z) \prec q(z) = a \exp\left[n^{-1} \int_0^z H(t)t^{-1}\, dt\right] \quad (z \in U),$$

and q is the best dominant in the sense that if $P(z) \prec q_1(z)$, then $q(z) \prec q_1(z)$.

Miller and Mocanu proved Lemma 4 in [4, p. 76] taking advantage of a more general result due to Suffridge [7].

Theorem 5. *If $f \in \mathcal{S}^{\lambda}_{a,c}(\eta; p; h)$, then*

$$\frac{\mathcal{I}^{\lambda}_p(a,c)f(z)}{z^p} \prec \exp \int_0^z \frac{(p-\eta)(h(t)-1)}{t}\, dt \quad (z \in U).$$

Proof. Let $P(z) = z^{-p}\mathcal{I}^{\lambda}_p(a,c)f(z) = 1 + a_1 z + \cdots$. Then from the definition of the class $\mathcal{S}^{\lambda}_{a,c}(\eta; p; h)$ we obtain

$$\frac{zP'(z)}{P(z)} \prec H(z) = (p-\eta)\big(h(z)-1\big) \quad (z \in U).$$

The function $H(z) = (p-\eta)(h(z)-1)$ is starlike because $h(z)$ is convex univalent function. Applying Lemma 4 we conclude that

$$P(z) \prec \exp \int_0^z \frac{H(t)}{t}\, dt. \qquad \square$$

Corollary 5. *If $f \in \mathcal{S}^{\lambda}_{a,c}(\eta; p; h)$ and $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, then*

$$\sum_{k=0}^{\infty} \frac{(\lambda+p)_k (c)_k}{k!(a)_k} a_{p+k} z^k \prec \exp \int_0^z \frac{(p-\eta)(h(t)-1)}{t}\, dt \quad (z \in U).$$

Proof. It follows immediately from Corollary 1 and Theorem 5. □

Corollary 6. *If* $f \in \mathcal{S}_{a,c}^{\lambda}(\eta; p; A, B)$ *and* $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, *then*

$$\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^p} \prec (1+Bz)^{\frac{(A-B)(p-\eta)}{B}} \quad (z \in U).$$

5. Convolutions with convex functions

Theorem 6. *Let* $\lambda \geqslant 0$, $a \geqslant p$, $\phi \in \mathcal{K}$, $h \in \mathcal{N}$ *and let h satisfies* (13). *Then*

$$f \in \mathcal{S}_{a,c}^{\lambda}(\eta; p; h) \quad \Rightarrow \quad \left[z^{p-1}\phi\right] * f \in \mathcal{S}_{a,c}^{\lambda}(\eta; p; h).$$

Proof. Let $f \in \mathcal{S}_{a,c}^{\lambda}(\eta; p; h)$ and let $\phi \in \mathcal{K}$. By applying the properties of the convolution and (14) we have

$$\begin{aligned}\frac{z(\mathcal{I}_p^{\lambda}(a,c)((z^{p-1}\phi) * f)(z))'}{\mathcal{I}_p^{\lambda}(a,c)((z^{p-1}\phi) * f)(z)} &= \frac{z(\phi_p^{\dagger}(a,c;z) * (z^{p-1}\phi(z)) * f(z))'}{\phi_p^{\dagger}(a,c;z) * (z^{p-1}\phi(z)) * f(z)} \\ &= \frac{(z^{p-1}\phi(z)) * z(\phi_p^{\dagger}(a,c;z) * f(z))'}{(z^{p-1}\phi(z)) * \phi_p^{\dagger}(a,c;z) * f(z)} \\ &= \frac{(z^{p-1}\phi(z)) * z(\mathcal{I}_p^{\lambda}(a,c)f(z))'}{(z^{p-1}\phi(z)) * z\mathcal{I}_p^{\lambda}(a,c)f(z)} \\ &= \frac{(z^{p-1}\phi(z)) * z[(p-\eta)h(\omega(z)) + \eta]\mathcal{I}_p^{\lambda}(a,c)f(z)}{(z^{p-1}\phi(z)) * z\mathcal{I}_p^{\lambda}(a,c)f(z)}. \end{aligned} \tag{19}$$

Let us put

$$F(z) := \frac{1}{p-\eta}\left(\frac{z(\mathcal{I}_p^{\lambda}(a,c)((z^{p-1}\phi) * f)(z))'}{\mathcal{I}_p^{\lambda}(a,c)((z^{p-1}\phi) * f)(z)} - \eta\right).$$

Then, by using (19), we obtain

$$F(z) = \frac{1}{p-\eta}\left(\frac{\phi(z) * [(p-\eta)h(\omega(z)) + \eta]z^{1-p}\mathcal{I}_p^{\lambda}(a,c)f(z)}{\phi(z) * z^{1-p}\mathcal{I}_p^{\lambda}(a,c)f(z)} - \eta\right).$$

From (15) it follows that $z^{1-p}\mathcal{I}_p^{\lambda}(a,c)f(z) \in \mathcal{S}^*$, hence, by applying the arguments similar to those used in the proof of Theorem 1, we conclude that $F \prec h$ and $\phi * f \in \mathcal{S}_{a,c}^{\lambda}(\eta; p; h)$. This completes the proof. □

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