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Mapping properties for oscillatory integrals in d-dimensions

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Abstract

For $a_j, b_j \geqslant 1$, $j = 1, 2, \ldots, d$, we prove that the operator $Kf(x) = \int_{\mathbb{R}^d_+} k(x, y) f(y)\, dy$ maps $L^p(\mathbb{R}^d_+)$ into itself for $p = 1 + \frac{1}{r}$, where $r = \frac{a_1}{b_1} = \cdots = \frac{a_d}{b_d}$, and $k(x, y) = \varphi(x, y)e^{ig(x,y)}$, $\varphi(x, y)$ satisfies (1.2) (e.g. $\varphi(x, y) = |x - y|^{i\tau}$, τ real) and the phase $g(x, y) = x^a \cdot y^b$. We study operators with more general phases and for these operators we require that $a_j, b_j > 1$, $j = 1, 2, \ldots, d$, or $a_l = b_l \geqslant 1$ for some $l \in \{1, 2, \ldots, d\}$.

Keywords: Oscillatory integrals; L^p mappings

1. Introduction

We consider oscillatory integral operators with kernels defined for $x, y \in \mathbb{R}^d_+ = [0, \infty) \times \cdots \times [0, \infty)$, d-times, and the kernels take the form

$$k(x, y) = \varphi(x, y) \exp\bigl(ig(x, y)\bigr), \tag{1.1}$$

g a real-valued phase function and where $\varphi(x, y)$ for $|x - y| > 0$ satisfies

$$\bigl|\partial_x^\alpha \partial_y^\beta \varphi(x, y)\bigr| \leqslant C_{\alpha\beta}|x - y|^{-|\alpha|-|\beta|}, \quad \forall\alpha, \beta \in \mathbb{N}^d, \mathbb{N} = \{0, 1, \ldots\}. \tag{1.2}$$

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A typical $g(x, y)$ is $x^a \cdot y^b = x_1^{a_1} y_1^{b_1} + \cdots + x_d^{a_d} y_d^{b_d}$ for $a_j, b_j \geqslant 1$, $j = 1, 2, \ldots, d$, and $a, b \in \mathbb{R}^d$. For $s \in \mathbb{R}$ we set $\bar{s} = (s, s, \ldots, s) \in \mathbb{R}^d$ and write $a \geqslant b$ if $a_j \geqslant b_j$ for all $j = 1, 2, \ldots, d$. And so in this paper $a, b \geqslant \bar{1}$.

The operator we study is given by

$$Kf(x) = \int\limits_{\mathbb{R}_+^d} k(x, y) f(y)\, dy, \quad x \in \mathbb{R}_+^d. \tag{1.3}$$

Define for $\mu_0(t) \in C^\infty(\mathbb{R}_+)$, $\mu_0(t) = 1$ for $0 \leqslant t \leqslant 1$, $\mu_0(t) = 0$ for $t \geqslant 2$, $\mu_0(t) + \mu_1(t) = 1$ and $0 \leqslant \mu_0(t), \mu_1(t)$ for all $t \geqslant 0$.

And we define the phase functions as follows

$$\Phi_*\big(x^a, y^b\big) = g(x, y) = x^a \cdot y^b + \mu_{\bar{1}}(x)\mu_{\bar{1}}(y)\Phi\big(x^a, y^b\big), \tag{1.4}$$

with $\mu_{\bar{1}}(x) = \mu_1(x_1) \cdots \mu_1(x_d)$ and where $\Phi(x, y)$ satisfies (2.1) below. Examples of Φ's are $\log(\sum_{j=1}^d x_j + y_j)$ or $(\sum_{j=1}^d x_j + y_j)^l$, $0 \leqslant l \leqslant 1$. We first suppose that $a \geqslant b \geqslant \bar{1}$ and then by duality we settle the cases where $b \geqslant a \geqslant \bar{1}$. These results appear in Proposition 4.2, Corollaries 4.3 and 4.4.

The referee points out that the mixed Hessian, namely $\det(\frac{\partial^2}{\partial_{x_i} \partial_{y_j}} g(x, y))$ for the phase $g(x, y) = x^a \cdot y^b$ ($\Phi = 0$) is simply a monomial in the variables x_i, y_j. However, in Proposition 5.1 of [7] (in 2-d), we showed in case $r = \frac{a_1}{b_1} = \frac{a_2}{b_2}$, $a, b \geqslant \bar{1}$, $g(x, y) = x^a \cdot y^b$ and $|\varphi(x, y)| \geqslant C$, then these operators map L^p into itself if and only if $p = 1 + \frac{1}{r}$. While in the paper [6], with again the phase $x^a \cdot y^b$, but this time no restriction on r, we ended up with weighted estimates. And the proofs in [6] for the most part, were quite different than the ones in [7] (or even here).

The size of the kernels here is essentially $\varphi(x, y) = |x - y|^{i\tau}$, τ-real. In order to have any success in employing Hörmander type estimates (say in 2-d) then $|\varphi(x, y)| \leqslant \frac{C}{|x-y|^2}$ and $|\varphi(x, y) - \varphi(x', y)|$ and $|\varphi(x, y) - \varphi(x, y')|$ must satisfy Hörmander type conditions. See for example Theorem 2.2 in [7].

In this paper, we prove an L^p mapping problem for a class of oscillatory integrals. The Fourier transform is included among this class of operators. Here we take $r = \frac{a_1}{b_1} = \cdots = \frac{a_d}{b_d}$ and $p = 1 + \frac{1}{r}$, and φ satisfies (1.2). And the case of the Fourier transform is when $r = a_l = b_l = 1$, $l \in \{1, 2, \ldots, d\}$. In Corollary 4.3, we show that these more general operators map $L^p(\mathbb{R}_+^d)$ into itself for $p = 1 + \frac{1}{r}$, $r \geqslant 1$ where r is defined above and either $a_j, b_j > 1$ for all $j \in \{1, 2, \ldots, d\}$ or $a_l = b_l \geqslant 1$ for some $l \in \{1, 2, \ldots, d\}$. In Proposition 4.2 we obtain the (p, p) result for $p = 1 + \frac{1}{r}$ with $g(x, y) = x^a \cdot y^b$ and $a, b \geqslant \bar{1}$. This result appears in [3] for $d = 1$ and in [7] for $d = 2$.

The authors [4] have settled the cases where $a = b = \bar{1}$. In [3] and [7] we obtained results for φ's satisfying weaker estimates than (1.2) in case $a, b > \bar{1}$ and for $\varphi(x, y) = |x - y|^{i\tau}$, $\tau \in \mathbb{R}$, in case a or $b = \bar{1}$ in dimensions $d = 1$, $d = 2$, respectively. In [1] the L^p result for φ satisfying (1.2) and $\Phi(x, y) \equiv 0$ was settled in dimension $d = 1$ in case a or $b = 1$ (i.e. $a, b \geqslant 1$).

We note that for $x \in \mathbb{R}^d$, $x = (x_1, x_2, \ldots, x_d)$ and for the most part $x' = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$, and similarly for a, b. We employ the usual convention that $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ for $\alpha \in \mathbb{R}^d$. We set $h(x^{\frac{1}{b}}) = h(x_1^{\frac{1}{b_1}}, \ldots, x_d^{\frac{1}{b_d}})$. Also if $m, n \in \mathbb{R}^d$ we write $m \cdot n = m_1 n_1 + \cdots + m_d n_d$ and

$m \cdot n \cdot b = \sum_{j=1}^{d} m_j n_j b_j$ and so on. We sometimes use $\vec{\rho}$ in place of ρ and we use $\bar{1}$ in case $\rho_1 = \cdots = \rho_d = 1$.

We use positive constants, denoted by C, in the usual way, indexed if needed.

2. Preliminary estimates

We begin by requiring that $\Phi(x, y)$ is real-valued and for $x, y \geqslant \bar{1}$, that

$$\left|\partial_x^\alpha \partial_y^\beta \Phi(x, y)\right| \leqslant C_{\alpha\beta}, \quad \forall \alpha, \beta \in \mathbb{N}^d, \quad \text{and} \quad \sum_{j=1}^{d} (\alpha_j + \beta_j) \geqslant 1. \tag{2.1}$$

We need that $e^{i\Phi(x,y)} \in S_{0,0}^0$.

However, in this section the results are for operators where the phase $g(x, y) = x^a \cdot y^b$ and one of the variables x_j or y_l has compact support.

Note if $0 \leqslant x_j \leqslant 1$ or $0 \leqslant y_l \leqslant 1$ for any $j, l \in \{1, 2, \ldots, d\}$ then by (1.4) $g(x, y) = x^a \cdot y^b$. However, if $1 < x_1, \ldots, x_d, y_1, \ldots, y_d < 2$ then $\mu_{\bar{1}}(x)\mu_{\bar{1}}(y)$ can be positive and if $\Phi(x, y) \neq 0$, then according to (1.4) we get a more general phase function.

We are led to consider the following operators

$$K_{\vec{\rho}\vec{\eta}} f(x) = \mu_{\vec{\rho}}(x) \int_{\mathbb{R}_+^d} k(x, y)\mu_{\vec{\eta}}(y) f(y)\, dy \tag{2.2}$$

and $\mu_{\vec{\rho}}(x) = \mu_{\rho_1}(x_1) \cdots \mu_{\rho_d}(x_d)$ with $\rho_1, \ldots, \rho_d \in \{0, 1\}$, similarly for $\mu_{\vec{\eta}}(y)$.

Also notice that $(1 - \mu_{\bar{1}}(2y)\mu_{\bar{1}}(2x))$ vanishes if $x, y \geqslant \bar{1}$ and so for the operator $K_{\vec{\rho}\vec{\eta}}((1 - \mu_{\bar{1}}(2y)\mu_{\bar{1}}(2x)) f)(x)$ the phase reduces to $g(x, y) = x^a \cdot y^b$.

We thus prove for the operators in (2.2) where $g(x, y) = x^a \cdot y^b$ and $\vec{\rho}$ or $\vec{\eta}$ contain a zero coordinate with $r = \frac{a_1}{b_1} = \cdots = \frac{a_d}{b_d}$ so that

$$\begin{cases} \text{(a) } \|K_{\vec{\rho}\vec{\eta}} f\|_p \leqslant C\|f\|_p, & \text{for } p = 1 + \frac{1}{r},\ r \geqslant 1, \text{ and} \\ \text{(b) } \|K_{\vec{\rho}\vec{\eta}} f\|_2 \leqslant C\|f\|_2, & \text{in case } r = 1\ (p = 2). \end{cases} \tag{2.3}$$

This extends to d-dimensions some of the 1- and 2-dimensional results from [3] and [7], respectively.

We employ induction on the dimension. Take $k(x', y') = \varphi(x', y') e^{i(x^a \cdot y^b - x_d^{a_d} y_d^{b_d})}$ and assume

$$\begin{cases} r = \frac{a_1}{b_1} = \cdots = \frac{a_{d-1}}{b_{d-1}}, \quad a', b' \geqslant 1,\ \varphi(x', y') \text{ satisfies (1.2) then for } p = 1 + \frac{1}{r} \\ \text{we suppose that } \int_{\mathbb{R}_+^{d-1}} |\int_{\mathbb{R}_+^{d-1}} k(x', y') f(y')\, dy'|^p\, dx' \leqslant C \int_{\mathbb{R}_+^{d-1}} |f(y')|^p\, dy'. \end{cases} \tag{2.4}$$

Remark 2.1. In (2.4) a more general assumption is needed, i.e. x' is any $(d - 1)$-dimensional variable and $r = \frac{a_i}{b_i} = \frac{a_j}{b_j} \geqslant 1$ over the relevant i, j and $k(x', y')$ is as in (2.4) with the more general x', y'. So we assume all of this and denote it as (2.4).

We state the first result

Proposition 2.2. *If* (2.4) *holds,* $g(x, y) = x^a \cdot y^b$, $a, b \geqslant \bar{1}, \varphi(x, y)$ *satisfies* (1.2) *and* $\vec{\rho}$ *or* $\vec{\eta}$ *contain a zero coordinate, i.e.* ρ_i *or* $\eta_j = 0$ *for some* $i, j \in \{1, 2, \ldots, d\}$, *then* (2.3) *holds.*

Proposition 2.3. *Let $k(x, y)$ be as in Proposition* 2.2*,* (2.4) *holds, and suppose $\varphi(x, y)$ does not depend upon one of the variables $x_1, y_1, \dots, x_d, y_d$, and $\varphi(x, y)$ satisfies* (1.2)*, then* (2.3) *holds.*

Proof. It is enough to consider the cases where $\varphi(x, y)$ does not depend upon x_d or y_d. See Remark 2.1.

In case $\varphi(x, y)$ does not depend upon x_d, by Theorem 3.1 of [3] (note we employ [1] in case $b_d = 1$) we get that

$$\int_0^\infty \left| \int_0^\infty e^{i x_d^{a_d} y_d^{b_d}} H(x', y_d)\, dy_d \right|^p dx_d \leqslant C \int_0^\infty \left| H(x', y_d) \right|^p dy_d,$$

where

$$H(x', y_d) = \int_{\mathbb{R}_+^{d-1}} e^{i(x^a \cdot y^b - x_d^{a_d} y_d^{b_d})} \varphi(x', y) f(y)\, dy'.$$

Therefore we get that

$$\int_{\mathbb{R}_+^{d-1}} \left(\int_0^\infty |Kf|^p\, dx_d \right) dx' \leqslant C \int_0^\infty \left(\int_{\mathbb{R}_+^{d-1}} \left| H(x', y_d) \right|^p dx' \right) dy_d \leqslant C \int_{\mathbb{R}_+^d} \left| f(y) \right|^p dy,$$

and the last inequality follows from the induction assumption (2.4) (note Remark 2.1).

Suppose $\varphi(x, y)$ does not depend upon y_d, then by the induction assumption (2.4) we get that

$$\int_{\mathbb{R}_+^{d-1}} \left| \int_{\mathbb{R}_+^{d-1}} e^{i(x^a \cdot y^b - x_d^{a_d} y_d^{b_d})} \varphi(x, y') \left(\int_0^\infty e^{i x_d^{a_d} y_d^{b_d}} f(y)\, dy_d \right) dy' \right|^p dx'$$
$$\leqslant C \int_{\mathbb{R}_+^{d-1}} \left| \int_0^\infty e^{i x_d^{a_d} y_d^{b_d}} f(y)\, dy_d \right|^p dy',$$

and the argument is now completed as above. □

Proposition 2.4. *Suppose that $\varphi(x, y)$ satisfies* (1.2) *and $a, b \geqslant \bar{1}$. Suppose*

(i) *the support of $\varphi(x, y)$ is contained in the set $\{|x - y| \leqslant 1\}$, or*
(ii) *$g(x, y) = x^a \cdot y^b$ and in* (1.1) *we replace $\varphi(x, y)$ by $\mu_0(x_l)\varphi(x, y)$ or by $\mu_0(y_j)\varphi(x, y)$ for any $l, j \in \{1, 2, \dots, d\}$.*

If (2.4) *holds, then* (2.3) *holds.*

Proof. (i) Follows from Schur's lemma.

Before we begin to show (ii) we consider the special case where

$$k(x, y) = e^{i x^a \cdot y^b} \mu_0(x_d) \mu_0(y_d) \varphi(x, y).$$

By Hölder's inequality applied to the inner integral we get that

$$
\begin{aligned}
I(x') &= \int_0^\infty \mu_0(x_d)^p \left| \int_0^\infty e^{i x_d^{a_d} y_d^{b_d}} \mu_0(y_d) H(x, y_d)\, dy_d \right|^p dx_d \\
&\leqslant C \int_0^\infty \mu_0(x_d)^p \left(\int_0^\infty \mu_0(y_d) \left| H(x, y_d) \right|^p dy_d \right) dx_d,
\end{aligned}
$$

$$
H(x, y_d) = \int_{\mathbb{R}_+^{d-1}} e^{(i x^a \cdot y^b - x_d^{a_d} y_d^{b_d})} \varphi(x, y) f(y)\, dy'.
$$

As in Proposition 2.3 we employ the induction assumption (2.4) on the dimension with $\|I(x')\|_1 = \int_{\mathbb{R}_+^{d-1}} |I(x')|\, dx'$ (note Remark 2.1) to get

$$
\left\| I(x') \right\|_1 \leqslant C \int_0^\infty \mu_0(x_d)^p \left(\int_0^\infty \mu_0(y_d) \left(\int_{\mathbb{R}_+^{d-1}} \left| H(x, y_d) \right|^p dx' \right) dy_d \right) dx_d \leqslant C \|f\|_p^p.
$$

Now we are in a position to complete (ii). Thus it suffices to consider the case where $k(x, y) = e^{i x^a \cdot y^b} \mu_0(x_d) \varphi(x, y)$. Write

$$
k(x, y) = e^{i x^a \cdot y^b} \varphi(x, y) \left[\mu_0(x_d) \mu_0\left(\frac{y_d}{4} \right) + \mu_0(x_d) \mu_1\left(\frac{y_d}{4} \right) \right] = k_1(x, y) + k_2(x, y).
$$

Let K_j denote the corresponding operators for $j = 1, 2$.

From the beginning of the argument we get that

$$
\|K_1 f\|_p \leqslant C \left\| I(x') \right\|_1^{\frac{1}{p}} \leqslant C \|f\|_p.
$$

For the operator K_2, we note by Taylor's formula expanded about x_d $(0 \leqslant x_d \leqslant 2)$ that

$$
\varphi(x, y) = \sum_{j=0}^{s} \partial_{x_d}^j \varphi(x', 0, y) \frac{x_d^j}{j!} + R(x, y),
$$

where s is the smallest positive integer greater than $(2d - 3, d - 2, 1)$, d the dimension. Thus we get that

$$
\left| R(x, y) \right| \leqslant \frac{C}{((\xi - y_d)^2 + |x' - y'|^2)^{\frac{s+1}{2}}}
$$

for some ξ between 0 and x_d, since $\varphi(x, y)$ satisfies (1.2). The operator K_2 is the sum of $s + 1$ terms that satisfies the conditions of Proposition 2.3, while

$$
\left| k_2(x, y) - e^{i x^a \cdot y^b} \mu_0(x_d) \mu_1\left(\frac{y_d}{4} \right) \sum_{j=0}^{s} \partial_{x_d}^j \varphi(x', 0, y) \frac{x_d^j}{j!} \right| \leqslant C \frac{\mu_0(x_d) \mu_1(\frac{y_d}{4})}{(1 + y_d^2 + |x' - y'|^2)^{\frac{s+1}{2}}},
$$

and this error term satisfies the conditions of Schur's lemma, and now the proof is complete. □

Remark 2.5. Thus Proposition 2.4(ii) implies Proposition 2.2, and in case $r = \frac{a_1}{b_1} = \cdots = \frac{a_d}{b_d} = 1$ that (2.3)(a) reduces to (2.3)(b).

We are left with showing (2.3) for all the operators $K_{\vec{\rho}\vec{\eta}}$ in which we include the more general phase function. For these operators, because of (1.4) and (2.1), this phase function $g(x, y)$ is more general than in [3] and [7].

3. An L^2-estimate

Take $0 \leqslant \gamma_0, \gamma_1$, $\gamma_0(x) + \gamma_1(x) = 1$, $\gamma_0 \in C^\infty(\mathbb{R}^d)$, $\gamma_0(x) = 1$ if $|x| \leqslant 1$ and $\gamma_0(x) = 0$ if $|x| \geqslant 2$. We consider a cutoff version of the kernel for the operator $K_{\bar{1}\bar{1}}$ as defined in (2.2),

$$Sf(x) = \mu_{\bar{1}}(x) \int\limits_{\mathbb{R}^d_+} \mu_{\bar{1}}(y)\gamma_1(x-y)k(x,y)f(y)\,dy.$$

Also we consider the operators $\mu_{\bar{1}}(2x)K_{\vec{\rho}\vec{\eta}}(\mu_{\bar{1}}(2y)f)$ (and denote their cutoff versions as $S_{\vec{\rho}\vec{\eta}}$) which were excluded in Section 2.

By Proposition 2.4(i) it follows that the operator $\int_{\mathbb{R}^d_+} \gamma_0(x-y)k(x,y)f(y)\,dy$ maps $L^p(\mathbb{R}^d_+)$ into itself for all $1 \leqslant p \leqslant \infty$. Thus the mapping properties of the operators here are not altered by inserting the cutoff function γ_1 into each of the corresponding kernels.

We complete the proof of (2.3)(b), i.e. the case $r = 1$, and show

Theorem 3.1. *Let $a, b \geqslant \bar{1}$, $\varphi(x, y)$ satisfies* (1.2) *and $\Phi(x, y)$ satisfies* (2.1)*, then for $p = 2$ we get that*

$$\begin{cases} \text{(a) } \|K_{\bar{1}\bar{1}}f\|_2 \leqslant C\|f\|_2, \text{ and} \\ \text{(b) } \|\mu_{\bar{1}}(2x)K_{\vec{\rho}\vec{\eta}}(\mu_{\bar{1}}(2y)f)\|_2 \leqslant C\|f\|_2. \end{cases} \tag{3.1}$$

In place of proving an L^2-estimate for these operators, it is enough to show an L^2-estimate for related operators.

Define these operators as follows

$$Lf(x) = \mu_{\bar{1}}\big(x^{\frac{1}{a}}\big) \int\limits_{\mathbb{R}^d_+} e^{i\Phi_*(x,y)}\mu_{\bar{1}}\big(y^{\frac{1}{b}}\big)\gamma_1\big(x^{\frac{1}{a}} - y^{\frac{1}{b}}\big)\varphi\big(x^{\frac{1}{a}}, y^{\frac{1}{b}}\big)f(y)\,dy$$

for S. And for the operators $S_{\vec{\rho}\vec{\eta}}$ set

$$L^*f(x) = \mu_{\bar{1}}\big(2x^{\frac{1}{a}}\big)\mu_{\vec{\rho}}\big(x^{\frac{1}{a}}\big) \int\limits_{\mathbb{R}^d_+} e^{i\Phi_*(x,y)}\mu_{\bar{1}}\big(2y^{\frac{1}{b}}\big)\mu_{\vec{\eta}}\big(y^{\frac{1}{b}}\big)\gamma_1\big(x^{\frac{1}{a}} - y^{\frac{1}{b}}\big)\varphi\big(x^{\frac{1}{a}}, y^{\frac{1}{b}}\big)f(y)\,dy,$$

where $\Phi_*(x, y)$ is defined in (1.4).

We begin with the proposition on p. 282 of [8], which originally appeared in [2].

Theorem A. *If $Pf(x) = \int_{\mathbb{R}^d} \lambda(x, y)e^{ix\cdot y}f(y)\,dy$ and $\lambda(x, y) \in S^0_{0,0}$, then $\|Pf\|_2 \leqslant C\|f\|_2$.*

Theorem 3.2. *Let $\Phi(x,y)$ be real-valued and satisfy* (2.1). *If $\varphi(x,y)$ satisfies* (1.2), *then for $a,b \geqslant \bar{1}$ we get that*

$$\begin{cases} \text{(a) } \|Lf\|_2 \leqslant C\|\mu_{\bar{1}}(y^{\frac{1}{b}})f\|_2, \text{ and} \\ \text{(b) } \|L^*f\|_2 \leqslant C\|\mu_{\bar{1}}(2y^{\frac{1}{b}})f\|_2. \end{cases} \tag{3.2}$$

Proof. Take $\Phi_{**}(x,y) = \mu_{\bar{1}}(x^{\frac{1}{a}})\mu_{\bar{1}}(y^{\frac{1}{b}})\Phi(x,y)$ and so

$$\mu_{\bar{1}}\big(2x^{\frac{1}{a}}\big)\mu_{\bar{1}}\big(2y^{\frac{1}{b}}\big)\mu_{\vec{\rho}}\big(x^{\frac{1}{a}}\big)\mu_{\vec{\eta}}\big(y^{\frac{1}{b}}\big)\gamma_1\big(x^{\frac{1}{a}} - y^{\frac{1}{b}}\big)e^{i\Phi_{**}(x,y)}\varphi\big(x^{\frac{1}{a}}, y^{\frac{1}{b}}\big) \in S^0_{0,0}$$

since $\Phi(x,y)$ satisfies (2.1) and $a,b \geqslant \bar{1}$. The result (3.2) follows from Theorem A. $\square$

We are in a position to prove Theorem 3.1 and some further estimates.

Proof of Theorem 3.1. Using the notation of (2.1) in [5], we set

$$S_{mn}h(x) = \int_{\mathbb{R}^d_+} k_{mn}\big(x^{\frac{1}{a}}, y^{\frac{1}{b}}\big)h(y)\,dy, \quad \text{and} \quad K_{mn}f(x) = \int_{\mathbb{R}^d_+} k_{mn}(x,y)f(y)\,dy,$$

where $k_{mn}(x,y) = \psi_m(x)\psi_n(y)e^{ig(x,y)}\varphi(x,y)\gamma_1(x-y)$ with $\psi_n(x) = \psi_{n_1}(x_1)\cdots\psi_{n_d}(x_d)$, $\psi_{n_l}(x_l) = \psi(2^{-n_l}x_l)$, $l = 1,2,\dots,d$, $0 \leqslant \psi \in C^\infty(\mathbb{R})$, $\operatorname{supp}\psi(t) \subset [\frac{1}{4}, 2]$ so that $\sum_{l=0}^\infty \psi_l(t) = 1$ for $t \geqslant \frac{1}{2}$.

Thus, $Sf(x) = \sum_{m,n} K_{mn}f(x)$. Note a similar decomposition can be given for each of the operators $S_{\vec{\rho}\vec{\eta}}$. Also note that the latter operators come into play only if $\Phi(x,y) \neq 0$.

It follows from (3.2)(a) that with $dv = x^{\frac{1}{a}-\bar{1}}\,dx$ and $du = y^{\bar{1}-\frac{1}{b}}\,dy$

$$\begin{cases} \text{(a) } \int_{\mathbb{R}^d_+} |Lh|^2\,dv \leqslant \|Lh\|_2^2 \leqslant C\int_{\mathbb{R}^d_+} \chi(y \geqslant \bar{1})|h(y)|^2\,dy \leqslant C\int_{\mathbb{R}^d_+} |h(y)|^2\,dy, \text{ and} \\ \text{(b) } \int_{\mathbb{R}^d_+} |S_{mn}h|^2\,dx \leqslant C\int_{\mathbb{R}^d_+} \psi_n(y^{\frac{1}{b}})|h(y)|^2\,dy. \end{cases} \tag{3.3}$$

If we set $h(y) = f(y^{\frac{1}{b}})y^{\frac{1}{b}-\bar{1}}$ we get (3.1)(a) from (3.3)(a) after changing variables ($u = y^{\frac{1}{b}}$ and $v = x^{\frac{1}{a}}$).

While from (3.3)(b) because of $\psi_m(x^{\frac{1}{a}})$ on the left side and $\psi_n(y^{\frac{1}{b}})$ on the right side we get that

$$2^{m\cdot(a-\bar{1})}\int_{\mathbb{R}^d_+} x^{\frac{1}{a}-\bar{1}}\big|S_{mn}h(x)\big|^2\,dx \leqslant C2^{n\cdot(\bar{1}-b)}\int_{\mathbb{R}^d_+} y^{\bar{1}-\frac{1}{b}}\big|h(y)\big|^2\,dy$$

and so we get with $h(y) = f(y^{\frac{1}{b}})y^{\frac{1}{b}-\bar{1}}$ after changing variables that

$$\|K_{mn}f\|_2^2 \leqslant C2^{-(m\cdot(a-\bar{1})+n\cdot(b-\bar{1}))}\|f\|_2^2. \tag{3.4}$$

Next set $v^b = x^a$ and this time we get with $\psi_m(v^{\frac{b}{a}})$ on the left side and $dx = v^{\frac{b}{a}-\bar{1}}\,dv$ that

$$\int_{\mathbb{R}^d_+}\bigg|\int_{\mathbb{R}^d_+} k_{mn}\big(v^{\frac{b}{a}}, y\big)f(y)\,dy\bigg|^2 v^{\frac{b}{a}-\bar{1}}\,dv \leqslant C2^{-(m\cdot(a-\bar{1})+n\cdot(b-\bar{1}))}\|f\|_2^2$$

or, with $U_{mn} f(x) = \int_{\mathbb{R}^d_+} k_{mn}(x^{\frac{b}{a}}, y) f(y)\, dy$ (as in (2.11) of [5]) we get that

$$\|U_{mn} f\|_2 \leqslant C 2^{-\frac{1}{2}[m\cdot a\cdot(\bar{1}-\frac{1}{b})+n\cdot(b-\bar{1})]} \|f\|_2. \tag{3.5}$$

Thus in case that $b_j > 1$ for all $j = 1, 2, \dots, d$ we get that (3.5) sums. Note a similar estimate holds for the operators U_{mn} associated to $S_{\vec{\rho}\vec{\eta}}$. $\square$

4. L^p-estimates

In this section, we complete the proof of (2.3)(a), i.e. we obtain the (p, p) estimates of the operator K in (1.3) for $p = 1 + \frac{1}{r}$ in d-dimensions. We get the full result, i.e. with the phase $\Phi(x, y)$ satisfying (2.1) if either a_j and $b_j > 1$ for all $j \in \{1, 2, \dots, d\}$, or $a_l = b_l \geqslant 1$ for some $l \in \{1, 2, \dots, d\}$. While for all $a, b \geqslant \bar{1}$, we get the (p, p) result for the operator K if $\Phi(x, y) \equiv 0$. This extends the 1-dimensional result in [3] to a class of operators with more general phase functions.

Theorem 4.1. *Let $K_{\vec{\rho}\vec{\eta}}$ be as in* (2.2), $r = \frac{a_1}{b_1} = \cdots = \frac{a_d}{b_d} \geqslant 1$ *and $g(x, y)$ as in* (1.4). *Let $\varphi(x, y)$ satisfy* (1.2), $a, b \geqslant \bar{1}$, *and* $p = 1 + \frac{1}{r}$.

$$\begin{cases} \text{(a) If } \Phi(x, y) \equiv 0, \text{ then (i) } \|K_{\bar{1}\bar{1}} f\|_p \leqslant C\|f\|_p. \\ \text{(b) If instead } \Phi(x, y) \neq 0 \text{ and } \Phi(x, y) \text{ satisfies (2.1), and if } a_j \text{ and } b_j > 1, \\ \quad \forall j \in \{1, 2, \dots, d\} \text{ or } a_l = b_l \geqslant 1 \text{ for some } l \in \{1, 2, \dots, d\}. \text{ Then,} \\ \quad \text{(ii) } \|\mu_{\bar{1}}(2x) K_{\vec{\rho}\vec{\eta}}(\mu_{\bar{1}}(2y) f)\|_p \leqslant C\|f\|_p. \end{cases} \tag{4.1}$$

Proof. In case that $a_l = b_l \geqslant 1$ for some $l \in \{1, 2, \dots, d\}$ then $p = 2$ and the result (b) (as well as (a)) follows from (3.1).

Next to estimate the remaining cases, we begin with the result (a). This proof follows along the lines of Theorem 1.1 of [5], which was done in 2-dimensions. Using the notation from [5], we notice as in Lemmas 2.1 and 2.2 of [5], applied respectively to the operators S_{mn}, U_{mn} we get if $a_j b_j > 1$ for all $j \in \{1, 2, \dots, d\}$ that

$$\begin{cases} \text{(a) } \|S_{mn} f\|_2 \leqslant C d_{mn} \|f\|_2, \text{ and} \\ \text{(b) } \|U_{mn} f\|_2 \leqslant C d_{mn} \|f\|_2, \end{cases} \tag{4.2}$$

where for some constant $\delta > 0$ (dependent only on the choice of $k(x, y)$), $d_{mn} = 2^{-\delta((n+m)\cdot\bar{1})}$, $n_j, m_j \geqslant 1$ for $j \in \{1, 2, \dots, d\}$.

To finish off this argument, it suffices to show (2.16) of [5], this applies to both cases (a) and (b) (of the theorem), i.e.

$$\nu\left(\left\{x\colon \left|J_{mn} f(x)\right| > \lambda\right\}\right) \leqslant \frac{C}{\lambda}\|f\|_1, \tag{4.3}$$

$J_{mn} f(x) = x_1 \cdots x_d U_{mn} f(x)$ and for the measure $\nu(x)$ we take $d\nu(x) = \frac{dx_1 \cdots dx_d}{x_1^2 \cdots x_d^2}$. We observe as we did in (2.16) of [5], that $J_{mn} f$ is supported in $I_{m_1} \times \cdots \times I_{m_d}$ where $I_{m_l} = [(\frac{2^{m_l}}{4})^r, (2 \cdot 2^{m_l})^r]$, $r = \frac{a_1}{b_1} = \cdots = \frac{a_d}{b_d}$.

To show (4.3) we note that $\lambda < x_1 \cdots x_d |U_{mn} f(x)| \leqslant C \psi_m(x^{\frac{b}{a}}) \|f\|_1 x_1 \cdots x_d$, and without any loss we can suppose that $\|f\|_1 > 0$. Thus, $x_d > \frac{\lambda}{\|f\|_1 (x_1 \cdots x_{d-1})}$ and $x_d \in I_{m_d}$. Therefore $\frac{\lambda}{\|f\|_1 (x_1 \cdots x_{d-1})} \leqslant (2 \cdot 2^{m_d})^r$, otherwise $\nu(E_\lambda) = 0$ with $E_\lambda = \{x\colon |J_{mn} f(x)| > \lambda\}$.

Hence we get that

$$\begin{cases} \text{(a)}\ \int_{\frac{\lambda}{\|f\|_1(x_1\cdots x_{d-1})}}^{(2\cdot 2^{m_d})^r} \frac{1}{x_d^2}\,dx_d \leqslant \frac{C\|f\|_1(x_1\cdots x_{d-1})}{\lambda}, \quad \text{if } \frac{\lambda}{\|f\|_1(x_1\cdots x_{d-1})} \in I_{m_d}, \text{ and} \\ \text{(b)}\ \int_{I_{m_d}} \frac{1}{x_d^2}\,dx_d \leqslant \frac{C}{(2^{m_d})^r}, \quad \text{if } \frac{\lambda}{\|f\|_1(x_1\cdots x_{d-1})} < (\frac{2^{m_d}}{4})^r. \end{cases} \tag{4.4}$$

Therefore we get that

$$\nu(E_\lambda) \leqslant \int\limits_{I_{m_1}} \left(\int\limits_{I_{m_2}} \cdots \left(\int\limits_{\frac{\lambda}{\|f\|_1(x_1\cdots x_{d-1})}}^{(2\cdot 2^{m_d})^r} \frac{1}{x_d^2}\,dx_d \right) \cdots \frac{dx_2}{x_2^2} \right) \frac{dx_1}{x_1^2}$$
$$+ \int\limits_{I_{m_1}} \left(\cdots \left(\int\limits_{\frac{\lambda(\frac{4}{2^{m_d}})^r}{\|f\|_1(x_1\cdots x_{d-2})}}^{(2\cdot 2^{m_{d-1}})^r} \left(\int\limits_{I_{m_d}} \frac{dx_d}{x_d^2} \right) \frac{dx_{d-1}}{x_{d-1}^2} \right) \cdots \right) \frac{dx_1}{x_1^2} = I + II.$$

Integrating out the dth iterate, using (4.4)(a), we get

$$I \leqslant \frac{C\|f\|_1}{\lambda} \int\limits_{I_{m_1}} \left(\int\limits_{I_{m_2}} \cdots \left(\int\limits_{I_{m_{d-1}}} x_1 \cdots x_{d-1} \frac{dx_{d-1}}{x_{d-1}^2} \right) \cdots \frac{dx_2}{x_2^2} \right) \frac{dx_1}{x_1^2} \leqslant \frac{C\|f\|_1}{\lambda}.$$

Next integrating out the dth and $(d-1)$st iterate, using (4.4)(b), we get

$$II \leqslant \frac{C\|f\|_1}{\lambda} \left(2^{m_d}\right)^r \left(\frac{1}{2^{m_d}}\right)^r \int\limits_{I_{m_1}} \left(\cdots \left(\int\limits_{I_{m_{d-2}}} x_1 \cdots x_{d-2} \frac{dx_{d-2}}{x_{d-2}^2} \right) \cdots \right) \frac{dx_1}{x_1^2} \leqslant \frac{C\|f\|_1}{\lambda}.$$

This completes our proof of (4.3).

By (4.2)(b) we get that

$$\|J_{mn} f\|_{2,\nu}^2 = \int\limits_{\mathbb{R}_+^d} |U_{mn} f|^2\,dx \leqslant C d_{mn}^2 \|f\|_2^2, \tag{4.5}$$

using (4.3) and (4.5) we get our result (a) by interpolation (just as we did for Theorem 1.1 in [5]).

To obtain the estimate for (b), we use (3.5), but this time $d_{mn} = 2^{-\frac{1}{2}[m\cdot a\cdot(\bar{1}-\frac{1}{b})+n\cdot(b-\bar{1})]}$ in (4.2)(b). Thus these U_{mn} estimates in (3.5) apply to the operators $S_{\vec{\rho}\vec{\eta}}$ and S as defined in Section 3. The result follows by interpolation, but here we need that $a \geqslant b > \bar{1}$ in order for the estimates to sum. This completes the proof. □

Now to complete the induction argument using (2.4) and Remark 2.1.

Proposition 4.2. *Let* $r = \frac{a_1}{b_1} = \cdots = \frac{a_d}{b_d} \geqslant 1$ *and* $a, b \geqslant \bar{1}$. *If* $\varphi(x, y)$ *satisfies* (1.2) *with the phase* $g(x, y) = x^a \cdot y^b$ *then for* K *in* (1.3) *we get that*

$$\|Kf\|_p \leqslant C\|f\|_p, \quad \textit{for } p = 1 + \frac{1}{r}.$$

Proof. To begin the induction argument, we note the result is valid in dimension $d = 1$ for $a, b > 1$ by [3] and in case $a > 1$ and $b \geqslant 1$ by [1]. In case $a = b = \bar{1}$ (then $p = 2$), the result is true in all dimensions by [4]. In d-dimensions with $\vec{\rho} = (\rho_1, \dots, \rho_d)$ (or $\vec{\eta}$) and $\rho_j \in \{0, 1\}$, $\forall j \in \{1, \dots, d\}$, we get that

$$Kf(x) = \sum_{\vec{\rho}, \vec{\eta}} K_{\vec{\rho}\vec{\eta}} f(x). \tag{4.6}$$

By Theorem 4.1(a) we obtain the p-estimate for the term $K_{\bar{1}\bar{1}}$ in (4.6) in d-dimensions. By Proposition 2.2 along with the induction assumption (2.4), we obtain the p-estimate for all the remaining terms in the sum (4.6) (note Remark 2.1, and $\Phi(x, y) \equiv 0$). This completes the induction proof. □

Corollary 4.3. *Let $r = \frac{a_1}{b_1} = \dots = \frac{a_d}{b_d} \geqslant 1$ and either $a, b > \bar{1}$ or $a_l = b_l \geqslant 1$ for some $l \in \{1, 2, \dots, d\}$. If $\varphi(x, y)$ satisfies* (1.2) *and $\Phi(x, y)$ satisfies* (2.1)*, then for K in* (1.3) *we get that*

$$\|Kf\|_p \leqslant C\|f\|_p, \quad \text{for } p = 1 + \frac{1}{r}.$$

Proof. Incorporating the sum (4.6) for Kf we use Theorem 4.1(b) to estimate all the terms $\mu_{\bar{1}}(2x) K_{\vec{\rho}\vec{\eta}}(\mu_{\bar{1}}(2y) f)$, then we are left with the terms $K_{\vec{\rho}\vec{\eta}}((1 - \mu_{\bar{1}}(2x)\mu_{\bar{1}}(2y)) f)$. If $x \geqslant \bar{1}$ and $y \geqslant \bar{1}$, then $1 - \mu_{\bar{1}}(2x)\mu_{\bar{1}}(2y) = 0$ and this operator vanishes. Thus it implies that $\vec{\rho}$ or $\vec{\eta}$ contain a zero coordinate. Then by (1.4) this implies that $g(x, y) = x^a \cdot y^b$ for these operators. For these latter terms Proposition 4.2 applies and gets their p-estimates. Thus we have obtained the p-estimates for all the terms in (4.6), and this gets the result. □

By duality we get from Corollary 4.3,

Corollary 4.4. *Let $r = \frac{a_1}{b_1} = \dots = \frac{a_d}{b_d} \leqslant 1$ and either $a, b > \bar{1}$ or $a_l = b_l \geqslant 1$ for some $l \in \{1, 2, \dots, d\}$. If $\varphi(x, y)$ satisfies* (1.2) *and $\Phi(x, y)$ satisfies* (2.1)*, then the operator K in* (1.3) *satisfies*

$$\|Kf\|_p \leqslant C\|f\|_p, \quad \text{for } p = 1 + \frac{1}{r}.$$

References

[1] L.C. Cheng, Y. Pan, L^p estimates for oscillatory integral operators, Proc. Amer. Math. Soc. 27 (10) (1999) 2995–3002.

[2] A.P. Calderón, R. Vaillancourt, A class of bounded pseudo-differential operators, Proc. Natl. Acad. Sci. USA 69 (1972) 1185–1187.

[3] Y. Pan, G. Sampson, P. Szeptycki, L^2 and L^p estimates for oscillatory integrals and their extended domains, Studia Math. 122 (3) (1997) 201–224.

[4] D.H. Phong, E.M. Stein, Hilbert integrals, singular integrals and Radon transforms I, Acta Math. 157 (1986) 99–157.

[5] G. Sampson, L^p estimates of a class of oscillatory integral operators, Proc. Amer. Math. Soc. 131 (2003) 2727–2732.

[6] G. Sampson, L^p type mapping estimates for oscillatory integrals in higher dimensions, Studia Math. 172 (2) (2006) 101–123.

[7] G. Sampson, P. Szeptycki, The complete (L^p, L^p) mapping properties of some oscillatory integrals in several dimensions, Canad. J. Math. 53 (5) (2001) 1031–1056.

[8] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Univ. Press, Princeton, NJ, 1993.