Atypical values at infinity of a polynomial function on the real plane: an erratum, and an algorithmic criterion

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Abstract

We correct a previously published theorem by proving a criterion to decide whether the fibration given by a real polynomial function \( f : \mathbb{R}^2 \to \mathbb{R} \) is locally trivial at infinity. The algorithmical nature of this criterion provides a test that can be applied to any real polynomial \( f \). © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a nonconstant polynomial function. We are interested in the smallest subset \( S \subset \mathbb{R} \) such that the fibration \( f : \mathbb{R}^2 \setminus f^{-1}[S] \to \mathbb{R} \setminus S \) is locally trivial. The failure of local triviality may be caused by a critical point of \( f \). But the local triviality of \( f \) may also fail to hold at infinity. We say that \( f \) is trivial at infinity over the interval \((\alpha, \beta)\) if there exists a compact subset \( K \) of \( \mathbb{R}^2 \) such that the fibration \( f : f^{-1}[(\alpha, \beta)] \cap [\mathbb{R}^2 \setminus K] \to (\alpha, \beta) \) is trivial. A real number \( \lambda \) will be called a typical value of \( f \) at infinity if \( f \) is trivial at infinity over some open interval containing \( \lambda \). Otherwise, \( \lambda \) will be called an atypical value of \( f \) at infinity; it may or may not be a critical value of \( f \).

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Section 4 in [3] is an attempt to study the atypical values at infinity. Assume that $f(X,Y)$ is monic in $Y$ of positive degree, and let $f_Y$ be its derivative with respect to $Y$. The following definition was given in [3]: $\lambda \in \mathbb{R}$ is said to be a real critical value at infinity for $f$ if there exists a half-branch at infinity $C$ of the zero set $f_Y^{-1}(0)$ along which $f_Y$ changes sign and such that $\lim_{C} f = \lambda$ (the precise meaning of these expressions will be given below). The set of all real critical values at infinity of $f$ is finite (cf. 3.1 and 3.11 [3]). This definition is inspired by the “polar curve criterion” given in [4, Theorem 1.5(iii)] for characterizing the critical values corresponding to singularities at infinity in the complex case. Theorem 4.2 in [3] states that the real critical values at infinity of $f$ coincide with its atypical values at infinity. It is true that every atypical value at infinity is a real critical value at infinity. But the reverse inclusion is false, as showed by an example in [5].

The main result of the paper [5] by Tibar and Zaharia is the following characterization: a regular value $\lambda$ of $f$ is typical at infinity if and only if the Euler characteristic of the level curve $f^{-1}[\lambda]$ is constant for $\lambda$ in a neighborhood of $\lambda$, and there is no connected component of $f^{-1}[\lambda]$ which vanishes at infinity as $\lambda$ tends to $\lambda$. This characterization is actually proved more generally for a one-parameter algebraic family of curves on a smooth, noncompact, affine real algebraic surface.

We present below a corrected theorem which gives another characterization of the atypical values at infinity. It applies only to the case of a polynomial function on the plane. It provides an algorithmical way to determine effectively the atypical values of $f$ at infinity. The arguments are very elementary and have a strong real flavor; they rely on the cylindrical decomposition of semialgebraic sets (see for instance [1, 2.3.1]), and on properties of monotonicity of $f$.

2. Statement of the theorem

Let $\Gamma \subset \mathbb{R}^2$ be a real algebraic curve. For large $r > 0$, the intersection of $\Gamma$ with the complement of the closed disk of radius $r$ centered at the origin has a fixed number of connected components, each one homeomorphic to a line. The germ at infinity of such a connected component will be called half-branch at infinity of $\Gamma$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function and $C$ a half-branch at infinity of a real algebraic curve in $\mathbb{R}^2$. As indicated after 1.1 [3], $f$ is monotonic along $C$, i.e., either strictly increasing, or strictly decreasing or constant. Let us be more precise. We may and will always assume throughout the paper that $C$ is not asymptotic to any vertical line in the plane $\mathbb{R}^2$. Hence, the first coordinate $x$ tends to $+\infty$ or $-\infty$ along $C$. We say that $C$ is a right half-branch in the first case and a left half-branch in the second case. If $C$ is a right (resp. left) half-branch, there exist $M \in \mathbb{R}$ and a Nash (i.e. analytic algebraic) function $g : (M, +\infty) \to \mathbb{R}$ such that $C$ is the germ of the curve $(x = t, y = g(t))$ (resp. $(x = -t, y = g(t))$) as $t \to +\infty$. The function $t \mapsto f(t, g(t))$ (resp. $t \mapsto f(-t, g(t))$) is strictly increasing, or strictly decreasing or constant for large $t$. Hence, it has a limit $\lambda = \lim_{C} f$ in $\mathbb{R} \cup \{+\infty, -\infty\}$. We will denote the three possibilities (increasing,
decreasing or constant) by \( f \nearrow C \), \( f \searrow C \), and \( f = C \), respectively. Note that \( g \) (or rather its germ at \(+\infty\)) is given by a real algebraic Puiseux series in \( 1/t \).

From now on, we assume that \( f \) is monic of positive degree in \( Y \). This ensures that no half-branch of \( f_Y^{-1}[0] \) is asymptotic to a vertical line. We shall be interested in the half-branches \( C \) of \( f_Y^{-1}[0] \) along which \( f_Y \) changes sign. If \( C \) is parametrized by \((x = t, y = g(t))\) (resp. \((x = -t, y = g(t))\), this means that, for all sufficiently large \( t \) we have \( f_Y(t, g(t) + \varepsilon) f_Y(t, g(t) - \varepsilon) < 0 \) (resp. \( f_Y(-t, g(t) + \varepsilon) f_Y(-t, g(t) - \varepsilon) < 0 \)) if \( \varepsilon > 0 \) is small enough. This is equivalent to the fact that the half-branch \( C \) corresponds to a root of odd multiplicity of \( f_Y \) in the field of Puiseux series in \( 1/x \). In particular, if \( f_Y \) is a square-free polynomial, \( f_Y \) changes sign along all half-branches of \( f_Y^{-1}[0] \) at infinity.

If \( M > 0 \) is large enough, there are Nash functions \( g_1 < \cdots < g_p \) \((M, +\infty) \rightarrow \mathbb{R} \) and \( h_1 < \cdots < h_q \) \((M, +\infty) \rightarrow \mathbb{R} \) such that the right half-branches \( C_1, \cdots, C_p \) (resp. the left half-branches \( D_1, \cdots, D_q \) of \( f_Y^{-1}[0] \) along which \( f_Y \) changes sign are the germs at infinity of the curves \((x = t, y = g_i(t))\) for \( i = 1, \ldots, p \) (resp. \((x = -t, y = h_j(t))\) for \( j = 1, \ldots, q \)). In this way we put an order \( C_1 < \cdots < C_p \) (resp. \( D_1 < \cdots < D_q \)) on the set of right (resp. left) half-branches of \( f_Y^{-1}[0] \) along which \( f_Y \) changes sign.

**Definition 1.** Let \( C_1 < \cdots < C_p \) be the right half-branches of \( f_Y^{-1}[0] \) along which \( f_Y \) changes sign. A sequence of consecutive half-branches \( C_k < C_{k+1} < \cdots < C_{\ell} \) is said to be a right critical cluster belonging to \( \lambda \) if there is a symbol \( \succ \) in \( \{\nearrow, \searrow, =\} \) such that:

1. for every \( i = k, \ldots, \ell \), one has \( f \succ C_i \lambda \),
2. \( f \succ C_{k-1} \lambda \) does not hold (or \( k = 1 \)),
3. \( f \succ C_{\ell+1} \lambda \) does not hold (or \( \ell = p \)).

The left critical clusters are defined in the same way; they consist of left half-branches of \( f_Y^{-1}[0] \) along which \( f_Y \) changes sign. It can be noted that all half-branches in a critical cluster are asymptotic to one of them ([3], 2.3).

**Theorem 1.** The real number \( \lambda \) is an atypical value of \( f \) at infinity if and only if there exists a critical cluster belonging to \( \lambda \) consisting of an odd number of half-branches of \( f_Y^{-1}[0] \) along which \( f_Y \) changes sign.

The point which was wrong with Theorem 4.2 in [3] is that it was implicitly assumed that a critical cluster contains only one half-branch.

### 3. Proof of the theorem

We shall now analyze more closely the critical clusters. This analysis will give the proof of the theorem and will also explain what kind of phenomena (vanishing, cleaving, ...) occur with critical clusters. In order to make the exposition more readable, we only consider right half-branches and right critical clusters.
Let $C_1 < \cdots < C_p$ be the right half-branches of $f_Y^{-1}[0]$ along which $f_Y$ changes sign. They are the germs at infinity of Nash curves $(x = t, y = g_i(t))$, for $i = 1, \ldots, p$, with $g_1 < \cdots < g_p : (M, +\infty) \to \mathbb{R}$. By abuse of notation, we shall identify the half-branches with these curves. We define the bands

\begin{align*}
(-\infty, C_1) &= \{(x, y) \in (M, +\infty) \times \mathbb{R}; \ y < g_1(x)\}, \\
(C_i, C_{i+1}) &= \{(x, y) \in (M, +\infty) \times \mathbb{R}; \ g_i(x) < y < g_{i+1}(x)\} \quad \text{for } i = 1, \ldots, p - 1, \\
(C_p, +\infty) &= \{(x, y) \in (M, +\infty) \times \mathbb{R}; \ g_p(x) < y\}.
\end{align*}

In case $p = 0$, there is just one band $(-\infty, +\infty)$. For simplicity, we set $-\infty = C_0$ and $+\infty = C_{p+1}$, although $C_0$ and $C_{p+1}$ are not half-branches. If $M$ is large enough, the derivative $f_Y$ is everywhere $\geq 0$ or everywhere $\leq 0$ on a band $(C_i, C_{i+1})$, and the sign alternates when one passes from a band to the next one. The function $f$ is strictly monotone on each vertical segment contained in a band. All these properties either are immediate or follow easily from the decomposition of semialgebraic sets (cf. Theorem 2.3.1 in [1]).

**Definition 2.** A half-branch at infinity $C_i$ of $f_Y^{-1}[0]$ along which $f_Y$ changes sign is called a valley if $f_Y \leq 0$ on the band $(C_{i-1}, C_i)$, and a crest if $f_Y \geq 0$ on $(C_{i-1}, C_i)$.

Crests and valleys alternate, just as in nature.

Taking $M$ large enough, we can also assume that, for $i = 1, \ldots, p$, $f(x, g_i(x))$ is either strictly monotone or constant on $(M, +\infty)$.

**Lemma 1.** Let $C_i < C_{i+1}$ be consecutive right half-branches of $f_Y^{-1}[0]$ along which $f_Y$ changes sign. Suppose that $\lim_{C_i} f = \lim_{C_{i+1}} f = \lambda \in \mathbb{R}$.

(a) If $f ↘ C_i \lambda$ or $f = C_i \lambda$, and $f ↗_{C_{i+1}} \lambda$ or $f = C_{i+1} \lambda$, then $f_Y \geq 0$ on $(C_i, C_{i+1})$.

(b) If $f ↘_{C_i} \lambda$ or $f = C_i \lambda$, and $f ↗_{C_{i+1}} \lambda$ or $f = C_{i+1} \lambda$, then $f_Y \leq 0$ on $(C_i, C_{i+1})$.

Moreover, in either case (a) or (b), $f = C_i \lambda$ and $f = C_{i+1} \lambda$ cannot hold simultaneously and, if neither $f = C_i \lambda$ nor $f = C_{i+1} \lambda$, a half-branch at infinity of the level curve $f^{-1}[\lambda]$ is contained in the band $(C_i, C_{i+1})$.

The proof of this lemma is very easy. One has for instance to realize that Fig. 1 is impossible: $f$ would have to increase along the circuit indicated by the arrows, but it has the same limit $\lambda$ at both ends.
Notation for all figures. The half-branches $C_i$ are dashed and the level curves of $f$ are solid. An arrow on a half-branch or a vertical segment indicates the sense along which $f$ grows on the half-branch or the segment.

Lemma 2 (Basic possible situations). Let $C_i$ be a right half-branch of $f_Y^{-1}[0]$ at infinity along which $f_Y$ changes sign. We assume $\lim_{C_i} f = \lambda \in \mathbb{R}$. If $f|_{C_i, \lambda}$ (resp. $f|_{C_i, \lambda}$), for every $\varepsilon > 0$ sufficiently small, there is a unique $x > M$ such that $f(x, g_i(x)) = \lambda + \varepsilon$ (resp. $f(x, g_i(x)) = \lambda - \varepsilon$). Then:

(a) If $f|_{C_i, \lambda}$ and $C_i$ is a crest, there exists $\delta_+$ and $\delta_-$ in $[M, x)$ such that the intersection of the level curve $f^{-1}[\lambda + \varepsilon]$ with the band $(C_i, C_{i+1})$ (resp. $(C_{i-1}, C_i)$) is the graph of a continuous function $(\delta_+, x) \to \mathbb{R}$ (resp. $(\delta_-, x) \to \mathbb{R}$).

(b) If $f|_{C_i, \lambda}$ and $C_i$ is a valley, there exists $\delta_+$ and $\delta_-$ in $(x, +\infty]$ such that the intersection of the level curve $f^{-1}[\lambda + \varepsilon]$ with the band $(C_i, C_{i+1})$ (resp. $(C_{i-1}, C_i)$) is the graph of a continuous function $(x, \delta_+) \to \mathbb{R}$ (resp. $(x, \delta_-) \to \mathbb{R}$).

(c) If $f|_{C_i, \lambda}$ and $C_i$ is a crest, there exists $\delta_+$ and $\delta_-$ in $(x, +\infty]$ such that the intersection of the level curve $f^{-1}[\lambda - \varepsilon]$ with the band $(C_i, C_{i+1})$ (resp. $(C_{i-1}, C_i)$) is the graph of a continuous function $(x, \delta_+) \to \mathbb{R}$ (resp. $(x, \delta_-) \to \mathbb{R}$).

(d) If $f|_{C_i, \lambda}$ and $C_i$ is a valley, there exists $\delta_+$ and $\delta_-$ in $[M, x)$ such that the intersection of the level curve $f^{-1}[\lambda - \varepsilon]$ with the band $(C_i, C_{i+1})$ (resp. $(C_{i-1}, C_i)$) is the graph of a continuous function $(\delta_+, x) \to \mathbb{R}$ (resp. $(\delta_-, x) \to \mathbb{R}$).

A sketch of Lemma 2 is found in Fig. 2. The proof of this lemma is again very easy, taking into account the fact that $f$ is monotone along vertical segments contained in the bands and also monotone or constant along the half-branches of $f_Y^{-1}[0]$.

Let $C_i$ and $C_{i+1}$ be consecutive half-branches at infinity of $f_Y^{-1}[0]$ along which $f_Y$ changes sign, such that $\lim_{C_i} f = \lim_{C_{i+1}} f = \lambda \in \mathbb{R}$ and neither $f = C_i \lambda$ nor $f = C_{i+1} \lambda$. Based on Lemmas 1 and 2, Fig. 3 displays the only possible configurations of the level curves $f^{-1}[\lambda \pm \varepsilon]$ having nonempty intersection with the band $(C_i, C_{i+1})$, for $\varepsilon > 0$ small enough.

We shall now fix some notation for the proof of the theorem.

Recall that, in the terminology of [3], $v \in \mathbb{R}$ is a real critical value at infinity of $f$ if there exists a half-branch at infinity $C$ of $f_Y^{-1}[0]$ along which $f_Y$ changes sign such
that \( \lim_C f = v \). Clearly, the set of real critical values at infinity of \( f \) is finite. Hence, given \( \lambda \in \mathbb{R} \), there is a bounded open interval \((x, \beta)\) containing \( \lambda \) but no real critical values at infinity of \( f \) different from \( \lambda \). We fix \( \lambda \), \( x \), and \( \beta \) for the following.

Applying 2.3.1 [1], we can assume that \( M \) is sufficiently large so that every connected component \( \Omega \) of \( f^{-1}((x, \beta)] \cap ((M, +\infty) \times \mathbb{R}) \) is of the form

\[
(\xi_-, \xi_+) = \{(x, y) \in (M, +\infty) \times \mathbb{R}; \, \xi_- (x) < y < \xi_+ (x)\},
\]

where \( \xi_- < \xi_+ : (M, +\infty) \to \mathbb{R} \) are continuous functions. We identify \( \xi_- \) and \( \xi_+ \) with their graphs. Moreover, we can assume that every half-branch at infinity \( C_i \) of \( f_Y^{-1}[0] \) along which \( f_Y \) changes sign is either contained in or disjoint from \( \Omega \). By the choice of \( x \) and \( \beta \), if \( C_i \) is contained in \( \Omega \) then \( \lim_{C_i} f = \lambda \) and every other \( C_j \) belonging to the same critical cluster as \( C_i \) is also contained in \( \Omega \). We can also assume that similar properties hold for the connected components of \( f^{-1}((x, \beta)] \cap ((-\infty, -M) \times \mathbb{R}) \).

Recall that \( f \) is always assumed to be monic in \( Y \). Using this fact, we can choose \( N > 0 \) large enough so that, setting \( K = [-M, M] \times [-N, N] \), we have that every connected component of \( f^{-1}((x, \beta)] \setminus K \) is contained in \((M, +\infty) \times \mathbb{R}\) or in \((-\infty, -M) \times \mathbb{R}\).

**Proof of the theorem.** We decompose the proof into several claims. We only treat right critical clusters. The other case is similar.

**Claim 1.** \( \lambda \) is an atypical value at infinity if there exists a half-branch at infinity \( C_i \) of \( f_Y^{-1}[0] \) along which \( f_Y \) changes sign and \( f = C_i \lambda \).
Say $f_Y \geq 0$ on the band $(C_{i-1}, C_i)$. Then, for every $z \in C_i$, there is a neighborhood $U$ of $z$ such that $f(z') \leq \lambda$ for every $z' \in U$; hence, for every $\varepsilon > 0$, the level curve $f^{-1}[\lambda + \varepsilon]$ does not meet $U$. So we cannot have triviality at infinity over an open interval containing $\lambda$. If, on the other hand, $f_Y \leq 0$ on $(C_{i-1}, C_i)$, a similar reasoning applies.

Note that in this case $C_i$ is a curve of critical points of $f$. Note also that, by Lemma 1, $\{C_i\}$ is a critical cluster.

**Claim 2.** $\lambda$ is an atypical value at infinity of $f$ if there is a critical cluster $C_k \prec \cdots \prec C_{k+2\ell}$ belonging to $\lambda$ consisting of an odd number of half-branches at infinity of $f_Y^{-1}[0]$ along which $f_Y$ changes sign.

By Claim 1, we can assume that, for all $i = 1, \ldots, p$, it is not the case that $f =_{C_i} \lambda$.

The critical cluster $C_k \prec \cdots \prec C_{k+2\ell}$ is contained in a connected component $\Omega = (\xi_-, \xi_+)$ of $f^{-1}([\alpha, \beta]) \setminus K$.

(a) Suppose that $f \nearrow_{C_k} \lambda$.

(ai) Suppose in addition that $C_k$ is a crest (see Fig. 4). Then the $C_{k+\text{odd}}$’s are valleys and the $C_{k+\text{even}}$’s are crests. Let $\varepsilon > 0$ be small enough. By Lemma 2, the level curve $f^{-1}[\lambda - \varepsilon]$ zigzags as shown on Fig. 4 when intersecting the curves $C_i$ with $k \leq i \leq k + 2\ell$. By Lemma 1, the hypothesis on $\alpha$ and $\beta$ and the fact that $C_k$ and $C_{k+2\ell}$ are crests, $\xi_-$ (resp. $\xi_+$) must be contained in the band $(C_{k-1}, C_k)$ (resp. $(C_{k+2\ell}, C_{k+2\ell+1})$), and both $\xi_-$ and $\xi_+$ are contained in the level curve $f^{-1}[\alpha]$. It follows that $\Omega$ contains exactly one connected component of $f^{-1}[\lambda - \varepsilon]$ which is homeomorphic to a line for $\varepsilon > 0$ sufficiently small and that $f < \lambda$ on $\Omega$. Hence, $\lambda$ is an atypical value at infinity. In the terminology of [5], a connected component of $f^{-1}[v]$ vanishes at infinity as $v \nearrow \lambda$. Notice that such a vanishing makes the Euler characteristic decrease by 1.

(aii) Suppose instead that $C_k$ is a valley (see Fig. 5). Then the $C_{k+\text{odd}}$’s are crests and the $C_{k+\text{even}}$’s are valleys. Let $\varepsilon > 0$ be small enough. By Lemma 2, the level curve $f^{-1}[\lambda - \varepsilon]$ zigzags as shown on Fig. 5 when intersecting the curves $C_i$ with $k \leq i \leq k + 2\ell$. Either $C_{k+2\ell+1}$ is contained in $\Omega$, and then $f \searrow_{C_{k+2\ell+1}} \lambda$, or $\xi_+$ is contained in $(C_{k+2\ell}, C_{k+2\ell+1})$, and then $\xi_+$ is also contained in the level curve $f^{-1}[\beta]$. 
In both cases the band \((C_{k+2'}, C_{k+2'+1})\) contains a unique half-branch at infinity of the level curve \(f^{-1}[\lambda]\). In the same way, we conclude that the band \((C_{k-1}, C_k)\) contains a unique half-branch at infinity of the level curve \(f^{-1}[\lambda]\). Fix any \(x \in (M, +\infty)\). For every \(\epsilon > 0\) sufficiently small, there exist unique \((x, y'(x)) \in (C_{k-1}, C_k)\) and \((x, y''(x)) \in (C_{k+2'}, C_{k+2'+1})\) with \(f(x, y'(x)) = f(x, y''(x)) = \lambda - \epsilon\). So, for every compact subset \(L\) of \(\mathbb{R}^2\) containing \(K\), we can find two points on the same connected component of \(f^{-1}[\lambda - \epsilon]\) \(\setminus L\) which tend, as \(\epsilon\) tends to 0, to points on different connected components of \(f^{-1}[\lambda]\) \(\setminus L\). It follows that \(\lambda\) is an atypical value at infinity. We can say that a connected component of \(f^{-1}[v]\) cleaves at infinity as \(v \to \lambda\). We avoid the verb “splits” because it is used with a different meaning in [5]. Notice that such a cleaving at infinity does not necessarily increase the number of connected components of the fiber, but increases by 1 the Euler characteristic of the fiber.

(b) Suppose that \(f^{-1}_{\setminus C_k} \lambda\).

(bi) If \(C_k\) is a crest, a similar argument as in case (aii) shows that a connected component of \(f^{-1}[v]\) cleaves at infinity as \(v \setminus \lambda\).

(bii) If \(C_k\) is a valley, a similar argument as in case (ai) shows that a connected component of \(f^{-1}[v]\) vanishes at infinity as \(v \setminus \lambda\).

**Claim 3.** If all critical clusters belonging to \(\lambda\) consist of an even number of half-branches, \(\lambda\) is a typical value at infinity (this includes the case that \(\lambda\) is not a real critical value at infinity).

By Lemma 1, there is no half-branch at infinity \(C_i\) of \(f_Y^{-1}[0]\) along which \(f_Y\) changes sign such that \(f = f_{C_i} \lambda\). Let \(\Omega = (\xi_-, \xi_+\) be a connected component of \(f^{-1}[(\alpha, \beta)] \setminus K\). If \(\Omega\) contains no critical cluster belonging to \(\lambda\), then \(f\) is monotone on each vertical segment contained in \(\Omega\). Hence, \(f : \Omega \to (\alpha, \beta)\) is a trivial fibration. Suppose now that \(\Omega\) contains a critical cluster \(C_k < \cdots < C_{k+2'+1}\). Suppose \(f^{-1}_{\setminus C_i} \lambda\) and \(C_k\) is a crest. By Lemma 1, \(\xi_+\) must be contained in the band \((C_{k-1}, C_k)\), and also in the level curve \(f^{-1}[\alpha]\). On the other side of the cluster, \(C_{k+2'+1}\) is a valley. There are two cases:

(a) \(\xi_-\) is not contained in the band \((C_{k+2'+1}, C_{k+2'+2})\). Then there is another critical cluster, \(C_{k+2'+2} < \cdots < C_{k+2m+1}\) belonging to \(\lambda\) contained in \(\Omega\). By Lemma 1, there
is a unique half-branch at infinity of $f^{-1}[\lambda]$ contained in the band $(C_{k+2m+1}, C_{k+2m+2})$. Since $C_{k+2m+1}$ is a valley and $\lambda \notin C_{k+2m+1}$, $\zeta_\pm$ must be contained in the band $(C_{k+2m+1}, C_{k+2m+2})$. By Lemma 2, for $\varepsilon > 0$ sufficiently small, the level curves $f^{-1}[\lambda - \varepsilon]$ and $f^{-1}[\lambda + \varepsilon]$ zigzag as shown on Fig. 6. Hence, there are $\alpha'$ and $\beta'$ such that $\alpha < \alpha' < \lambda < \beta' < \beta$ and $f : \Omega' \to (\alpha', \beta')$ is a trivial fibration, where $\Omega'$ is the connected component of $f^{-1}((\alpha', \beta')) \setminus K$ contained in $\Omega$.

(b) $\xi_\pm$ is contained in the band $(C_{k+2m+1}, C_{k+2m+2})$. Then $\xi_\pm$ is also contained in the level curve $f^{-1}[\beta]$. In this case there is $\alpha'$ such that $\alpha < \alpha' < \lambda$ and $f : \Omega' \to (\alpha', \beta)$ is a trivial fibration, where $\Omega'$ is the connected component of $f^{-1}((\alpha', \beta')) \setminus K$ contained in $\Omega$.

The other possibilities ($f \nearrow C_k \lambda$ and $C_k$ is a valley, $f \searrow C_k \lambda$ and $C_k$ is a crest, $f \searrow C_k \lambda$ and $C_k$ is a valley) can be dealt with in a similar way, and we conclude that $f$ is a trivial fibration at infinity over some open interval containing $\lambda$. This completes the proof of the theorem. □

Note that we recover the result of Tibar and Zaharia for the case of a polynomial in two variables: if $\lambda$ is not a critical value, then it is typical at infinity if and only if there is no vanishing nor cleaving at infinity as $v \to \lambda$. This is equivalent to the fact that there is no vanishing at infinity and the Euler characteristic does not change.

4. Algorithmic aspect and examples

The theorem above provides an algorithmical method to find all atypical values at infinity of a given $f$ and to precise the kind of phenomena which occur for a given atypical value: curve of critical points, vanishing, cleaving. The method is as follows: Assume that $f$ is a polynomial with coefficients in $\mathbb{Q}$.

- If $f$ is not monic in $Y$, perform a linear change of coordinates so that it becomes monic in $Y$.
• If \( f_Y \) is not squarefree, factor \( f_Y = \varphi \psi^2 \), where \( \varphi \) and \( \psi \) are polynomials with coefficients in \( \mathbb{Q} \) and \( \varphi \) is squarefree. This can be done by gcd computations.

• Compute the Puiseux expansions of the solutions for \( Y \) of the equation \( f_Y = 0 \) (or \( \varphi = 0 \) if \( f_Y \) is not squarefree), as \( X \to \infty \). This can be done using rational Puiseux expansions over \( \mathbb{Q} \) (cf. [2]), and we get solutions of the form \( (x = a/t^r, \ y = \sigma(t)) \), where \( a \) is a nonzero constant in \( \mathbb{Q} \), \( r \) a positive integer and \( \sigma \) a power series in \( t \) with coefficients in a finite algebraic extension of \( \mathbb{Q} \). These coefficients are polynomial expressions in a primitive element of the extension, given by its minimal polynomial.

• Establish the ordered lists of all right and left half-branches of \( f_Y^{-1}[0] \) along which \( f_Y \) changes sign.
  - First, one has to find the real branches at infinity, which means all Puiseux expansions with real coefficients. This is done by determining the real roots of the minimal polynomials mentioned above.
  - Then, one has to decide whether a real branch gives one left half-branch and one right half-branch, or two right half-branches, or two left half-branches. This depends on the parity of \( r \) and the sign of \( a \) in \( x = a/t^r \).
  - Finally, one has to order the left half-branches and the right half-branches. This is done by comparing the leading terms in the real Puiseux expansions. The comparison of coefficients involving real roots of minimal polynomials can be made by isolating these roots in intervals with rational endpoints.
  - Since \( f \) is monic in \( Y \), the top right (resp. left) half-branch is a valley. Valleys and crests alternate as one goes down in the ordered lists.

• For each half-branch \( C \), compute \( \lambda = \lim_C f \) and, if \( \lambda \neq \pm \infty \), determine whether \( f/C \lambda \) is a \( \lambda \)-pole of \( f \). This can be done by substituting the Puiseux expansions in the polynomial \( f \). The case of \( f = \lambda \) can be detected by computing the gcd of \( f - \lambda \) and \( f_Y \) (or \( \varphi \)). In the other cases, one has to obtain the first term after the constant in the expansion of \( f \).

One has then all the information needed to apply the theorem.

The computations can be performed with the software Maple, using the command “puiseux” of the package “algcurves” for the rational Puiseux expansions and the library “realroot” for isolating real roots.

Let us apply our theorem to the Example 3.4 in [5]. The polynomial is

\[
f := 2y^5 + 4xy^4 + (2x^2 - 9)y^3 - 9xy^2 + 12y,
\]

and its derivative with respect to \( y \) is denoted here by \( fy \). First we compute the Puiseux expansions of the solutions of \( fy = 0 \) and put them in a list. We have reordered the Maple output in order to have the Puiseux expansions written as usual.

\[
> \text{PE} := \text{convert} (\text{puiseux}(fy,x=\text{infinity},y,7,t),\text{list});
\]

\[
\text{PE} := \left[ \frac{1}{x} = t, \ y = \left( 2t^2 - \frac{10}{3}t^4 + \frac{80}{9}t^6 \right) x \right],
\]
\[
\begin{align*}
\frac{1}{x} &= t, y = \left( -1 - \frac{9}{4} t^2 + \frac{177}{32} t^4 - \frac{3591}{128} t^6 \right) x, \\
\frac{1}{x} &= t, y = \left( -\frac{3}{5} - \frac{3}{4} t^2 - \frac{35}{96} t^4 + \frac{1575}{128} t^6 \right) x, \\
\frac{1}{x} &= t, y = \left( t^2 - \frac{11}{6} t^4 + \frac{247}{36} t^6 \right) x
\end{align*}
\]

Then we substitute these expansions in \( f \).

\[
\begin{align*}
> \text{series(algsubs(x=1/t,algsubs(PE[1,2],f)),t=0,3)}; & \quad 4t + O(t^3) \\
> \text{series(algsubs(x=1/t,algsubs(PE[2,2],f)),t=0,3)}; & \quad -\frac{15}{8} t^{-1} - \frac{135}{32} t + O(t^3) \\
> \text{series(algsubs(x=1/t,algsubs(PE[3,2],f)),t=0,3)}; & \quad -\frac{216}{3125} t^{-5} - \frac{162}{125} t^{-3} + O(t^{-1}) \\
> \text{series(algsubs(x=1/t,algsubs(PE[4,2],f)),t=0,3)}; & \quad 5t + O(t^3).
\end{align*}
\]

From these computations we obtain the following table of ordered left and right half-branches. The disposition in the table reflects the disposition of the half-branches in the real plane. The abbreviation v. stands for valley, and c. for crest. The notation PE\([n]\) indicates that the half-branch comes from the \(n\)th Puiseux expansion in the list PE above. Finally, we indicate for each half-branch the behavior of \( f \) along this half-branch.

<table>
<thead>
<tr>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>v. PE[2] \↗ +∞</td>
<td>v. PE[1] \↘ 0</td>
</tr>
<tr>
<td>c. PE[3] \↗ +∞</td>
<td>c. PE[4] \↘ 0</td>
</tr>
<tr>
<td>left: v. PE[4] \↗ 0</td>
<td>right: v. PE[3] \↘ -∞</td>
</tr>
<tr>
<td>c. PE[1] \↗ 0</td>
<td>c. PE[2] \↘ -∞</td>
</tr>
</tbody>
</table>

So 0 is a real critical value. Since both critical clusters belonging to 0 have two half-branches, then 0 is a typical value at infinity.

Let us see another example: the Example 3.1 of [5]. Here

\[
f := x^2 y^3 (y^2 - 25)^2 + 2 x y (y^2 - 25) (y + 25) - y^4 - y^3 + 50 y^2 + 51 y - 575.
\]

We perform the substitution \( x = x + y \) in order to obtain a polynomial \( g \) which is monic of degree 9 in \( y \), and its derivative with respect to \( y \) is here denoted by \( g_y \).

We compute the Puiseux expansions for the solutions of \( g_y = 0 \) and put them in a list (we have shortened and rearranged the actual output of the computation).

\[
> \text{PE} := \left[ \frac{1}{x} = t, y = \left( 5 t - \frac{3}{25} t^2 + \cdots \right) x \right],
\]

\[
> \text{series(algsubs(x=1/t,algsubs(PE[1,1],f)),t=0,3)}; & \quad 3 t + O(t^3)
\]

\[
> \text{series(algsubs(x=1/t,algsubs(PE[2,1],f)),t=0,3)}; & \quad -\frac{12}{8} t^{-4} - \frac{162}{125} t^{-2} + O(t^{-1})
\]

\[
> \text{series(algsubs(x=1/t,algsubs(PE[3,1],f)),t=0,3)}; & \quad -\frac{216}{3125} t^{-5} - \frac{162}{125} t^{-3} + O(t^{-1})
\]

\[
> \text{series(algsubs(x=1/t,algsubs(PE[4,1],f)),t=0,3)}; & \quad 5 t + O(t^3).
\]
\[
\begin{align*}
\left[ \frac{1}{x} = t, y = (-1 + t^4 + \cdots)x \right], & \quad \left[ \frac{1}{x} = t, y = \left( -5t + \frac{2}{25}t^2 + \cdots \right)x \right], \\
\left[ \frac{1}{x} = \frac{2}{3}t^2, y = \left( \frac{4}{9}t^3 + \frac{8}{675}t^4 + \cdots \right)x \right], & \quad \left[ \frac{1}{x} = t, y = \left( -\frac{7}{9} - \frac{200}{49}t^2 + \cdots \right)x \right], \\
\left[ \frac{1}{x} = t, y = \left( t\%1 + \frac{14951}{7350} + \frac{7}{450}\%1 \right)t^2 + \cdots \right)x \right] \\
\text{v. PE[2], } \downarrow -\infty & \quad \text{v. PE[1], } \nearrow 0 \\
\text{c. PE[5], } \nearrow +\infty & \quad \text{c. PE[6], } \sqrt{\frac{75}{7}} \nearrow +\infty \\
\text{v. PE[1], } \nearrow 0 & \quad \text{v. PE[4], } t > 0 \nearrow -\infty \\
\text{c. PE[6], } \sqrt{\frac{75}{7}} \nearrow +\infty & \quad \text{c. PE[4], } t < 0 \nearrow +\infty \\
\text{v. PE[6], } -\sqrt{\frac{75}{7}} \nearrow -\infty & \quad \text{v. PE[6], } -\sqrt{\frac{75}{7}} \nearrow -\infty \\
\text{c. PE[3], } \nearrow 0 & \quad \text{c. PE[3]} \nearrow 0 \\
\text{c. PE[2] } \nearrow -\infty & \quad \text{c. PE[2] } \nearrow -\infty
\end{align*}
\]

Then we substitute these expansions in \( g \). Here also we shorten some of the outputs.

\[
\begin{align*}
> \text{series(algsubs(x=1/t,algsubs(PE[1,2],g)),t=0,3);} \\
& \quad -\frac{216}{125}t^2 + O(t^3) \\
> \text{series(algsubs(x=1/t,algsubs(PE[2,2],g)),t=0,3);} \\
& \quad -t^{-4} + t^{-3} + 50t^{-2} + O(t^{-1}) \\
> \text{series(algsubs(x=1/t,algsubs(PE[3,2],g)),t=0,3);} \\
& \quad -\frac{64}{125}t^2 + O(t^3) \\
> \text{series(algsubs(x=3/(2^*t^2),algsubs(PE[4,2],g)),t=0,6);} \\
& \quad -\frac{2500}{3}t^{-1} - \frac{1825}{3} + \cdots + O(t^5) \\
> \text{series(algsubs(x=1/t,algsubs(PE[5,2],g)),t=0,3);} \\
& \quad -\frac{3294172}{387420489}t^{-9} + \frac{3361400}{4782969}t^{-7} + O(t^{-5}) \\
> \text{series(algsubs(x=1/t,simplify(algsubs(PE[6,2],g),RootOf)),t=0,3);} \\
& \quad \frac{750000}{343}\text{RootOf}(7 Z^2 - 75)t^{-2} + \cdots + O(t)
\end{align*}
\]

From these computations we obtain the following table of ordered left and right half-branches. There is no need to use "realroot" in order to deal with RootOf\( (7 Z^2 - 75) \) in PE [6]. Note that the branch PE [4] with \( x=3/(2t^2) \) gives two right half-branches.
We find that 0 is an atypical value. There are two cleavings and two vanishings as \( v \to 0 \). This agrees, of course, with the analysis of [5], although we obtain here information about what happens at infinity only.

References