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Global existence, asymptotic behavior and blowup of solutions for a class of nonlinear wave equations with dissipative term ☆

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Abstract

The paper studies the global existence, asymptotic behavior and blowup of solutions to the initial boundary value problem for a class of nonlinear wave equations with dissipative term. It proves that under rather mild conditions on nonlinear terms and initial data the above-mentioned problem admits a global weak solution and the solution decays exponentially to zero as $t \to +\infty$, respectively, in the states of large initial data and small initial energy. In particular, in the case of space dimension $N = 1$, the weak solution is regularized to be a unique generalized solution. And if the conditions guaranteeing the global existence of weak solutions are not valid, then under the opposite conditions, the solutions of above-mentioned problem blow up in finite time. And an example is given.

Keywords: Global solution; Asymptotic behavior; Blowup of solutions; Nonlinear wave equation; Initial boundary value problem

1. Introduction

In this paper, we study the global existence, the asymptotic behavior of weak solutions and the blowup of solutions to the initial boundary value problem for a class of nonlinear wave equations with dissipative term:

$$u_{tt} + \Delta^2 u + \lambda u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

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$$u|_{\partial\Omega} = 0, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \quad \text{on } [0, +\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega, \Delta$ is the Laplace operator, $\frac{\partial u}{\partial n}|_{\partial\Omega}$ indicates derivative of u in outward normal direction of $\partial\Omega, \sigma_i(s)$ $(i = 1, \ldots, N)$ are given nonlinear functions and $\lambda \geqslant 0$ is a real number.

In the case of $N = 1$, without loss of generality we assume that $\Omega = (0, 1)$, problem (1.1), (1.2) becomes

$$u_{tt} + u_{xxxx} + \lambda u_t = \sigma(u_x)_x \quad \text{in } (0,1) \times (0, +\infty), \tag{1.3}$$

$$u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0, \quad t \geqslant 0,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 \leqslant x \leqslant 1. \tag{1.4}$$

Equations of type (1.3) are a class of essential nonlinear evolution equations appearing in the elasto-plastic-microstructure models. They describe the longitudinal motion of an elasto-plastic bar and the anti-plane shearing, see [2]. When $\lambda = 0$, under the assumption "$\sigma(s) = as^2$, where $a < 0$ is a real number", the authors [2] showed that the interaction between the lower-order nonlinear terms that change type and the small, higher order dispersive microstructure terms leads to the equations that have a soliton structure locally. The competition of the focusing effect of the nonlinearity and the spreading effect of the dispersive microstructure terms leads to a well-posed but growing 'jump' profile. And for general Eq. (1.3), with $\lambda = 0$, under the assumption "$\sigma \in C^2(\mathbf{R}), \sigma''(s)$ satisfies local Lipschitz condition and $\sigma'(s)$ is bounded below", the authors [5] proved that corresponding problem (1.3), (1.4) admits a unique generalized solution and gave some sufficient conditions which make the solutions of problem (1.3), (1.4) blow up in finite time. But if $\sigma'(s)$ is not bounded below, does problem (1.3), (1.4) admit any global solution? When the space dimension $N \geqslant 2$, does problem (1.1), (1.2) admit any global solution? These questions are still open.

In real process, the linear damping, as well as dissipation, plays an important role. Therefore, the study of nonlinear evolution equations with linear damping or dissipative term has recently attracted the attention of many mathematicians and engineers, and there have been a lot of impressive literature, see [3,4,7–9].

In the present paper, on the one hand, by a Galerkin approximation scheme, as well as combining it with the potential well method, we proved that

1. If $\sigma_i \in C^1(\mathbf{R}), \sigma_i(s)$ are of polynomial growth order, either $\sigma_i(s)s \geqslant 0$ or $\sigma_i'(s) \geqslant C_0, s \in \mathbf{R}, i = 1, \ldots, N$, where and in the sequel C_0 is a constant, then problem (1.1), (1.2) admits a global weak solution u as long as initial data $u_0 \in H_0^2(\Omega), u_1 \in L_2(\Omega)$. And if $\int_0^s \sigma_i(\tau)\,d\tau \leqslant \sigma_i(s)s, s \in \mathbf{R},\ i = 1, \ldots, N,$ then when

$\lambda > 0$, the solution features the asymptotic behavior

$$||u_t(t)||^2_{L_2(\Omega)} + ||\Delta u(t)||^2_{L_2(\Omega)} \leqslant ME(0)e^{-\delta t}, \quad t > 0, \tag{1.5}$$

where $E(0)$ is as shown in (2.6), M and δ are positive constants (see Theorem 2.2).

2. Even if the above-mentioned conditions "either $\sigma_i(s)s \geqslant 0$ or $\sigma_i'(s) \geqslant C_0$, $s \in \mathbf{R}, i = 1, \ldots, N$" are not valid, problem (1.1), (1.2) admits a global weak solution u as long as initial data $u_0 \in W$ (potential well), $u_1 \in L_2(\Omega)$ such that the initial energy $E(0) > 0$ is properly small, and when $\lambda > 0$,

$$||u_t(t)||^2_{L_2(\Omega)} + ||\Delta u(t)||^2_{L_2(\Omega)} + ||\nabla u(t)||^{m+1}_{L_{m+1}}(\Omega) \leqslant ME(0)e^{-\delta_1 t}, \quad t > 0, \tag{1.6}$$

where δ_1 is a positive constant (see Theorem 2.1). Eqs. (1.5) and (1.6) show that the additionally dissipative term u_t makes the weak solutions decay exponentially. In particular, in the case of space dimension $N = 1$, the weak solutions can be regularized to be a unique generalized solution (see Theorem 2.3).

On the other hand, by an energy method, we prove that if the above-mentioned conditions guaranteeing the global existence of weak solutions are not valid, then under the opposite assumptions similar to thresholds, the solutions of problems (1.1)–(1.4) blow up in finite time (see Theorem 2.4).

The plan of the paper is as follows. The main results concerning the global existence, the asymptotic behavior of weak solutions and the blowup of solutions are stated in Section 2. The proofs of global existence and asymptotic behavior of weak solutions are given in Section 3. In the case of $N = 1$, the weak solution of problem (1.3), (1.4) is regularized to be a unique generalized solution in Section 4. In Section 5, the proof of a blowup theorem is given and an example shown.

2. Statement of main results

We first introduce the following abbreviations:

$$Q_T = \Omega \times (0, T), \quad L_p = L_p(\Omega), \quad W^{m,p} = W^{m,p}(\Omega),$$

$$W_0^{m,p} = W_0^{m,p}(\Omega), \quad C^k = C^k(\Omega), \quad C_0^\infty = C_0^\infty(\Omega),$$

$$H^m = W^{m,2}, \quad H_0^m = W_0^{m,2}, \quad ||\cdot||_p = ||\cdot||_{L_p}, \quad ||\cdot|| = ||\cdot||_{L_2}.$$

Let $(\cdot,\cdot)$ denote the L_2-inner product and $p' = p/(p-1)$ for any real number $p > 1$.

Define the potential well

$$W = \{u \in H_0^2 \ |I(u) = ||\Delta u||^2 - b||\nabla u||^{m+1}_{m+1} > 0\} \cup \{0\}, \tag{2.1}$$

where and in the sequel $m > 1$ and $b > 0$ are real numbers.

Lemma 2.1 (Adams [1] and Ladyzhenskaya [6]). *For any* $u \in H_0^2$, $||\Delta u||$ *is equivalent to* $||u||_{H^2}$.

Lemma 2.2. *Let* $m+1 \leqslant \frac{2N}{N-2}$ *if* $N>2$. *Then* W *is a neighborhood of* 0 *in* H_0^2.

Proof. By the Sobolev embedding theorem,

$$H_0^2 \hookrightarrow W_0^{1,m+1}. \tag{2.2}$$

For any $u \in H_0^2$, if $||\Delta u|| = 0$, obviously $u \in W$; if $||\Delta u|| > 0$, (2.2) and the Poincaré inequality yield

$$b||\nabla u||_{m+1}^{m+1} \leqslant C_* b||\Delta u||^{m-1}||\Delta u||^2 < ||\Delta u||^2 \tag{2.3}$$

as long as $||\Delta u|| < (1/C_* b)^{\frac{1}{m-1}}$, where and in the sequel C_* denotes embedding constant from H_0^2 to $W_0^{1,m+1}$. Eq. (2.3) implies the conclusion of Lemma 2.2. Lemma 2.2 is proved. □

For later purpose we introduce the functional J defined by

$$J(u) := \frac{1}{2}||\Delta u||^2 - \frac{b}{m+1}||\nabla u||_{m+1}^{m+1} \tag{2.4}$$

for suitable u. Obviously, we have

$$J(u) = \frac{1}{2}I(u) + d_1||\nabla u||_{m+1}^{m+1} = \frac{1}{m+1}I(u) + \frac{d_1}{b}||\Delta u||^2 \tag{2.5}$$

for all such u, where and in the sequel $d_1 = \frac{(m-1)b}{2(m+1)}$.

Now we state the main results of the paper. (To simplify notation we shall not introduce the range of summation if it is extending from $1, \dots, N$.)

Theorem 2.1. *Assume that*

(i) $\sigma_i \in C^1(\mathbf{R})$, $|\sigma_i(s)| \leqslant b|s|^m, s \in \mathbf{R}$, $i = 1, \dots, N$, *and if* $N>2$, *also* $m+1 \leqslant \frac{2N}{N-2}$.
(ii) $u_0 \in W, u_1 \in L_2$ *such that*

$$\begin{aligned} 0 < E(0) &= \frac{1}{2}||u_1||^2 + \frac{1}{2}||\Delta u_0||^2 + \sum_i \int_\Omega \int_0^{u_{0x_i}} \sigma_i(s)\, ds\, dx \\ &< \frac{m-1}{4(m+1)}\left(\frac{1}{C_* b}\right)^{\frac{2}{m-1}}. \end{aligned} \tag{2.6}$$

Then for any $T>0$, *problem* (1.1), (1.2) *admits a weak solution* $u \in L_\infty([0,T]; H_0^2) \cap W^{1,\infty}([0,T]; L_2)$, *and when* $\lambda > 0$, (1.7) *holds.*

Theorem 2.2. *Assume that*

(i) *Assumption* (i) *of Theorem* 2.1 *holds, and either* $\sigma_i(s)s \geqslant 0$ *or* $\sigma_i'(s) \geqslant C_0, s \in \mathbf{R}, \ i = 1, \dots, N$, *where* C_0 *is a constant.*

(ii) $u_0 \in H_0^2, u_1 \in L_2$.

Then for any $T > 0$, *problem* (1.1), (1.2) *admits a weak solution* $u \in L_\infty([0, T]; H_0^2) \cap W^{1,\infty}([0, T]; L_2)$. *And if*

(iii) $\int_0^s \sigma_i(\tau)\, d\tau \leqslant \sigma_i(s)s, s \in \mathbf{R}, i = 1, \dots, N$.

Then when $\lambda > 0$, (1.6) *holds.*

Theorem 2.3. *Assume that*

(i) $\sigma \in C^3(\mathbf{R}), \sigma'''(s)$ *is locally Lipschitz continuous*, $\sigma'(0) = \sigma''(0) = 0, |\sigma(s)| \leqslant b|s|^m$, $s \in \mathbf{R}$.

(ii) $u_0 \in W \cap H^4, u_1 \in H_0^2$ *such that*

$$0 < E(0) = \frac{1}{2}||u_1||^2 + \frac{1}{2}||u_{0xx}||^2 + \int_\Omega \int_0^{u_{0x}} \sigma(s)\, ds\, dx$$
$$< \frac{m-1}{4(m+1)}\left(\frac{1}{C_* b}\right)^{\frac{2}{m-1}}. \tag{2.7}$$

Then for any $T > 0$, *problem* (1.3), (1.4) *admits a unique generalized solution* $u \in C([0, T]; H^4 \cap H_0^2) \cap C^1([0, T]; H_0^2) \cap C^2([0, T]; L_2)$, *and when* $\lambda > 0$,

$$||u_t(t)||^2 + ||u_{xx}(t)||^2 + ||u_x(t)||_{m+1}^{m+1} \leqslant ME(0)e^{-\delta_1 t}, \quad t > 0. \tag{2.8}$$

Theorem 2.4. *Assume that*

(i) $\sigma_i \in C(\mathbf{R}), \sigma_i(s)s \leqslant k \int_0^s \sigma_i(\tau)\, d\tau \leqslant -k\beta|s|^{m+1}, s \in \mathbf{R}, i = 1, \dots, N$, *where* $k > 2$ *and* $\beta > 0$ *are constants, and if* $\lambda > 0$, *also* $1 < m \leqslant 3$.

(ii) $u_0 \in H_0^2, u_1 \in L_2$ *such that* $E(0) < 0$, *where* $E(0)$ *is as shown in* (2.6).

Then the solution u of problem (1.1), (1.2) *blows up in finite time* $\tilde{T}$, *i.e. when* $\lambda > 0, 1 < m \leqslant 3$,

$$||u_t(t)||^2 + \int_0^t ||u(\tau)||^2\, d\tau \to +\infty \quad \text{as} \quad t \to \tilde{T}^-, \tag{2.9}$$

and when $\lambda = 0$,

$$||u_t(t)|| + ||u(t)|| \to +\infty \quad \text{as} \quad t \to \tilde{T}^-, \tag{2.10}$$

where $\tilde{T}$ *is different for different conditions.*

3. Global existence and asymptotic behavior of weak solutions

Proof of Theorem 2.1. We look for approximate solutions $u^n(t)$ of problem (1.1), (1.2) of the form

$$u^n(t) := \sum_{j=1}^{n} T_{jn}(t) w_j, \tag{3.0}$$

where $\{w_j\}_{j=1}^{\infty}$ is an orthogonal basis in H_0^2, and also in L_2, and the coefficients $\{T_{jn}\}_{j=1}^{n}$ satisfy $T_{jn}(t) = (u^n(t), w_j)$ with

$$\begin{aligned}&(u_{tt}^n(t), w_j) + (\Delta^2 u^n(t), w_j) + \lambda(u_t^n(t), w_j)\\ &\quad = \sum_i \left(\frac{\partial}{\partial x_i} \sigma_i(u_{x_i}^n(t)), w_j\right), \quad t>0,\ j = 1, \ldots, n,\end{aligned} \tag{3.1}$$

$$u^n(0) = u_0^n, \quad u_t^n(0) = u_1^n. \tag{3.2}$$

Since C_0^∞ is dense in H_0^2 and L_2, we choose $u_0^n, u_1^n \in C_0^\infty$ such that

$$u_0^n \to u_0 \quad \text{in } H_0^2, \qquad u_1^n \to u_1 \quad \text{in } L_2 \quad \text{as } n \to \infty. \tag{3.3}$$

Replacing w_j in (3.1) by $u_t^n(t)$ gets

$$\frac{d}{dt} E_n(t) + \lambda ||u_t^n(t)||^2 = 0, \tag{3.4}$$

$$E_n(t) + \lambda \int_0^t ||u_t^n(\tau)||^2 \, d\tau = E_n(0), \quad t>0, \tag{3.5}$$

where

$$E_n(t) = \frac{1}{2}||u_t^n(t)||^2 + \frac{1}{2}||\Delta u^n(t)||^2 + \sum_i \int_\Omega \int_0^{u_{x_i}^n} \sigma_i(s) \, ds \, dx.$$

Obviously,

$$E_n(t) \geqslant \tfrac{1}{2}||u_t^n(t)||^2 + J(u^n(t)), \quad t>0, \tag{3.6}$$

$$E_n(0) = \frac{1}{2}||u_1^n||^2 + \frac{1}{2}||\Delta u_0^n||^2 + \sum_i \int_\Omega \int_0^{u_{0x_i}^n} \sigma_i(s) \, ds \, dx. \tag{3.7}$$

By integral mean value theorem, assumption (i), (2.2) and (3.4),

$$\left|\int_\Omega \int_{u_{0x_i}}^{u^n_{0x_i}} \sigma_i(s)\,ds\,dx\right| \leqslant \int_\Omega |\sigma_i(\xi_i)(u^n_{0x_i}-u_{0x_i})|\,dx$$
$$\leqslant ||\sigma_i(\xi_i)||_{(m+1)'}||u^n_{0x_i}-u_{0x_i}||_{m+1}$$
$$\leqslant b||\xi_i||^m_{m+1}||u^n_{0x_i}-u_{0x_i}||_{m+1}\to 0 \quad \text{as } n\to\infty, \tag{3.8}$$

where $\xi_i = u_{0x_i} + \theta_i u^n_{0x_i}, 0<\theta_i<1,\ i=1,\dots,N$. So $E_n(0)\to E(0)(>0)$ as $n\to\infty$, where $E(0)$ is as shown in (2.6). Without loss of generality, we assume that $E_n(0)<2E(0)$ for all n. And thus (3.5) implies, for all n,

$$E_n(t)<2E(0), \quad t>0. \tag{3.9}$$

Since $u_0\in W$, combining (3.3) with (2.2) yields $I(u^n_0)\to I(u_0)>0$ as $n\to\infty$. Let, without loss of generality, $I(u^n_0)>0$, i.e. $u^n_0\in W$ for all n. Hence, for all n,

$$u^n(t)\in W, \quad t>0. \tag{3.10}$$

In fact, if there exists a $T>0$ such that $u^n(t)\in W, t\in[0,T)$, while $u^n(T)\in\partial W$, i.e. $I(u^n(T))=0$ for some n, then $||\Delta u^n(T)||\neq 0$ (or else by Lemma 2.2, $u^n(T)$ is an inner point of W), and by (2.3), (2.5), (3.6), (3.9) and (2.6),

$$b||\nabla u^n(t)||^{m+1}_{m+1}\leqslant C_* b(2E(0)b/d_1)^{\frac{m-1}{2}}||\Delta u^n(t)||^2<||\Delta u^n(t)||^2,$$
$$0\leqslant t\leqslant T. \tag{3.11}$$

Eq. (3.11) implies $I(u^n(T))>0$, which is a contradiction. So (3.10) is valid. And (3.10) implies that (3.11) holds for $t>0$.

It follows from (3.5), (3.6), (3.9), (3.10) and (2.5) that

$$\frac{1}{2}||u^n_t(t)||^2 + \frac{d_1}{2}\left(\frac{1}{b}||\Delta u^n(t)||^2 + ||\nabla u^n(t)||^{m+1}_{m+1}\right)$$
$$+\lambda\int_0^t ||u^n_t(\tau)||^2\,d\tau\leqslant 2E(0), \quad t>0. \tag{3.12}$$

By (3.12), on the one hand, we have

$$|(\sigma_i(u^n_{x_i}), w_{jx_i})|\leqslant ||\sigma_i(u^n_{x_i})||_{(m+1)'}||w_{jx_i}||_{m+1}$$
$$\leqslant M||u^n_{x_i}(t)||^m_{m+1}||\Delta w_j||\leqslant M,$$
$$t>0,\ \ i=1,\dots,N,\ \ j=1,\dots,n,$$

where and in the sequel we denote by M and $C_i(i=1,2,\dots)$ various positive constants independent of n and t, i.e. for any $T>0$, the nonlinear terms in system of equations (3.1) are uniformly bounded on $[0, T]$. So the solution $u^n(t)$ of problem (3.1), (3.2) exists on $[0, T]$ for each n. On the other hand, we can extract a subsequence from $\{u^n\}$, still denoted by $\{u^n\}$, such that for any $T>0$,

$$\begin{aligned} &u^n \to u \quad \text{weak* in } L_\infty([0,T]; H_0^2), \\ &u_t^n \to u_t \quad \text{weak* in } L_\infty([0,T]; L_2), \end{aligned} \tag{3.13}$$

and for any $t>0$,

$$\begin{aligned} &u^n(t) \to u(t) \quad \text{weak* in } H_0^2, \\ &u_t^n(t) \to u_t(t) \quad \text{weak* in } L_2 \end{aligned} \tag{3.14}$$

as $n \to \infty$. By (3.14), the Sobolev embedding theorem and the continuity of $\sigma_i(s)$, for any $t>0$,

$$\begin{aligned} &\nabla u^n(t) \to \nabla u(t) \quad \text{strongly in } L_2 \text{ and a.e. on } \Omega, \\ &\sigma_i(u^n_{x_i}(t)) \to \sigma_i(u_{x_i}(t)) \quad \text{a.e. on } \Omega, \quad i = 1, \dots, N \end{aligned} \tag{3.15}$$

as $n \to \infty$. Integrating (3.1) over $(0, t)$ gets

$$\begin{aligned} &(u_t^n(t), w_j) + \int_0^t (\Delta u^n(\tau), \Delta w_j)\, d\tau + \lambda \int_0^t (u_t^n(\tau), w_j)\, d\tau \\ &\quad = -\sum_i \int_0^t (\sigma_i(u_x^n(\tau)), w_{jx_i})\, d\tau + (u_1^n, w_j), \quad t>0. \end{aligned} \tag{3.16}$$

Since

$$\int_0^t (\Delta u^n(\tau), \Delta w_j)\, d\tau \leqslant \int_0^t ||\Delta u^n(\tau)||\, ||\Delta w_j|| d\tau \leqslant MT, \quad t \in [0, T], \tag{3.17}$$

$$\begin{aligned} \int_0^t (\sigma_i(u^n_{x_i}(\tau)), w_{jx_i})\, d\tau &\leqslant \int_0^t ||\sigma_i(u^n_{x_i}(\tau))||_{(m+1)'} ||w_{jx_i}||_{m+1}\, d\tau \\ &\leqslant M \int_0^t ||u^n_{x_i}(\tau)||^m_{m+1} ||\Delta w_j||\, d\tau \\ &\leqslant MT, \quad t \in [0, T], \end{aligned} \tag{3.18}$$

$i = 1, \dots, N$, $j = 1, \dots, n$, letting $n \to \infty$ in (3.16) and making use of (3.14), (3.15), (3.17), (3.18) and the Lebesgue-dominated convergence theorem yields

$$\begin{aligned} &(u_t(t), w_j) + \int_0^t (\Delta u, \Delta w_j)\, d\tau + \lambda \int_0^t (u_t, w_j)\, d\tau + \sum_i \int_0^t (\sigma_i(u_{x_i}), w_{jx_i})\, d\tau \\ &\quad = (u_1, w_j), \quad t>0, \ j = 1, 2, \dots . \end{aligned} \tag{3.19}$$

Since $\{w_j\}$ is dense in H_0^2, differentiating (3.19) gets, for any $v \in H_0^2$,

$$(u_{tt}(t) - \Delta^2 u(t) + \lambda u_t(t) - \sum_i \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}(t)), v) = 0, \quad t \in [0, T]. \tag{3.20}$$

By (3.13)

$$(u^n(t), w_j) \to (u(t), w_j) \quad \text{weak* in } W^{1,\infty}[0, T] \quad \text{as } n \to \infty,$$

$$j = 1, 2, \dots . \tag{3.21}$$

Since $W^{1,\infty}[0, T] \hookrightarrow C[0, T]$,

$$(u^n(0), w_j) \to (u(0), w_j) \quad \text{as } n \to \infty, \quad j = 1, 2, \dots . \tag{3.22}$$

Letting $t \to 0^+$ in (3.19) gets

$$(u_t(0), w_j) = (u_1, w_j), \quad j = 1, 2, \dots . \tag{3.23}$$

Combining (3.22), (3.23) with (3.3) gets

$$u(0) = u_0 \quad \text{in } H_0^2, \qquad u_t(0) = u_1 \quad \text{in } L_2. \tag{3.24}$$

Eqs. (3.20) and (3.24) imply that $u \in L_\infty([0, T]; H_0^2) \cap W^{1,\infty}([0, T]; L_2)$ is a global weak solution of problem (1.1), (1.2).

Now, we discuss the asymptotic behavior of the above-mentioned weak solutions.

Replacing w_j in (3.1) by $u^n(t)$ gets

$$\begin{aligned} 0 &= \frac{d}{dt}(u_t^n, u^n) - ||u_t^n(t)||^2 + ||\Delta u^n(t)||^2 + \sum_i (\sigma_i(u_{x_i}^n), u_{x_i}^n) + \frac{\lambda}{2}\frac{d}{dt}||u^n(t)||^2 \\ &\geqslant \frac{d}{dt}(u_t^n, u^n) - ||u_t^n(t)||^2 + I(u^n(t)) + \frac{\lambda}{2}\frac{d}{dt}||u^n(t)||^2, \quad t > 0. \end{aligned} \tag{3.25}$$

By (3.11),

$$b||\nabla u^n(t)||_{m+1}^{m+1} \leqslant \gamma ||\Delta u^n(t)||^2, \quad t > 0,$$

where $\gamma = C_* b(2E(0)b/d_1)^{\frac{m-1}{2}} < 1$. Therefore,

$$I(u^n(t)) = ||\Delta u^n(t)||^2 - b||\nabla u^n(t)||_{m+1}^{m+1} \geqslant b\left(\frac{1}{\gamma} - 1\right)||\nabla u^n(t)||_{m+1}^{m+1}, \tag{3.26}$$

$$I(u^n(t)) \geqslant (1 - \gamma)||\Delta u^n(t)||^2, \tag{3.27}$$

$$||\nabla u^n(t)||_{m+1}^{m+1} + ||\Delta u^n(t)||^2 \leqslant \frac{1}{1-\gamma}\left(\frac{\gamma}{b} + 1\right) I(u^n(t)), \quad t > 0. \tag{3.28}$$

Multiplying (3.4) by $e^{\delta t}$ gives

$$\frac{d}{dt}(e^{\delta t}E_n(t)) + \lambda e^{\delta t}||u_t^n(t)||^2 = \delta e^{\delta t}E_n(t), \quad t>0. \tag{3.29}$$

Integrating (3.29) over $(0,t)$ and using (3.9), (3.28) and (3.25) we obtain

$$\begin{aligned}
&e^{\delta t}E_n(t) + \lambda\int_0^t e^{\delta\tau}||u_t^n(\tau)||^2\,d\tau \\
&\leqslant E_n(0) + \delta\int_0^t e^{\delta\tau}\left(\frac{1}{2}||u_t^n(\tau)||^2 + \frac{1}{2}||\Delta u^n(\tau)||^2 + \frac{b}{m+1}||\nabla u^n(\tau)||_{m+1}^{m+1}\right)d\tau \\
&\leqslant 2E(0) + \frac{\delta}{2}\int_0^t e^{\delta\tau}||u_t^n(\tau)||^2\,d\tau + C_1\delta\int_0^t e^{\delta\tau}I(u^n(\tau))\,d\tau \\
&\leqslant 2E(0) + \left(\frac{1}{2}+C_1\right)\delta\int_0^t e^{\delta\tau}||u_t^n(\tau)||^2\,d\tau \\
&\quad - C_1\delta\left[e^{\delta t}(u_t^n(t),u^n(t)) - (u_1^n,u_0^n) - \delta\int_0^t e^{\delta\tau}(u_t^n(\tau),u^n(\tau))d\tau\right] \\
&\quad - \frac{1}{2}\lambda C_1\delta\left[e^{\delta t}||u^n(t)||^2 - ||u_0^n||^2 - \delta\int_0^t e^{\delta\tau}||u^n(\tau)||^2\,d\tau\right] \\
&\leqslant 2E(0) + \left(\frac{1}{2}+C_1\right)\delta\int_0^t e^{\delta\tau}||u_t^n(\tau)||^2\,d\tau \\
&\quad + C_1\delta\left[e^{\delta t}\left(\frac{1}{2}||u_t^n(t)||^2 + \frac{1}{2}(1+\lambda)||u^n(t)||^2\right) + \frac{1}{2}||u_1^n||^2 + \frac{1}{2}(1+\lambda)||u_0^n||^2\right] \\
&\quad + C_2\delta^2\int_0^t e^{\delta\tau}(||(u_t^n(\tau)||^2 + ||u^n(\tau)||^2)\,d\tau \\
&\leqslant 2E(0) + \left(\frac{1}{2}+C_1\right)\delta\int_0^t e^{\delta\tau}||u_t^n(\tau)||^2\,d\tau + C_3\delta e^{\delta t}E_n(t) \\
&\quad + \frac{1}{2}C_1\delta(||u_1^n||^2 + (1+\lambda)||u_0^n||^2) \\
&\quad + C_4\delta^2\int_0^t e^{\delta\tau}E_n(\tau)\,d\tau, \quad t>0.
\end{aligned} \tag{3.30}$$

Take δ: $0<\delta<\min\{(2C_3)^{-1},(2C_4)^{-1},\lambda/(1+2C_1)\}$, we deduce from (3.30) that

$$\begin{aligned}
&e^{\delta t}E_n(t) + \lambda\int_0^t e^{\delta\tau}||u_t^n(\tau)||^2\,d\tau \\
&\quad \leqslant ME(0) + 2C_4\delta^2\int_0^t e^{\delta\tau}E_n(\tau)\,d\tau, \quad t>0.
\end{aligned} \tag{3.31}$$

Applying the Gronwall inequality to (3.31) yields

$$\frac{1}{2}||u_t^n(t)||^2 + \frac{d_1}{2}\left(\frac{1}{b}||\Delta u^n(t)||^2 + ||\nabla u^n(t)||_{m+1}^{m+1}\right) \leqslant E_n(t) \leqslant ME(0)e^{-\delta_1 t}, \quad t>0, \tag{3.32}$$

where $\delta_1 = (1-2C_4\delta)\delta > 0$. Letting $n \to \infty$ in (3.32), from the sequential weak* lower semi-continuity of the norm in $L_\infty([0,T];H_0^2) \cap W^{1,\infty}([0,T];L_2)$ we deduce that

$$\begin{aligned} &||u_t(t)||^2 + ||\Delta u(t)||^2 + ||\nabla u(t)||_{m+1}^{m+1} \\ &\leqslant \lim_{n\to\infty} \inf(||u_t^n(t)||^2 + ||\Delta u^n(t)||^2 + ||\nabla u^n(t)||_{m+1}^{m+1}) \\ &\leqslant ME(0)e^{-\delta_1 t}, \quad t>0, \end{aligned} \tag{3.33}$$

where $u \in L_\infty([0,T];H_0^2) \cap W^{1,\infty}([0,T];L_2)$ is a weak solution of problem (1.1), (1.2). Theorem 2.1 is proved. □

Proof of Theorem 2.2. We still look for approximate solutions $u^n(t)$ of problem (1.1), (1.2) as shown in (3.0).

Case 1. If $\sigma_i(s)s \geqslant 0, s \in \mathbf{R}, i=1,\dots,N$, note that $\int_0^s \sigma_i(\tau)\,d\tau \geqslant 0$ $(i=1,\dots,N)$ at this time, repeating the proof of Theorem 2.1 and exploiting (3.5) and (3.9) gets

$$||u_t^n(t)||^2 + ||\Delta u^n(t)||^2 + 2\lambda \int_0^t ||u_t^n(\tau)||^2 d\tau \leqslant 4E(0), \quad t>0. \tag{3.34}$$

Replacing w_j in (3.1) by $u^n(t)$ gets

$$\begin{aligned} &\frac{d}{dt}(u_t^n, u^n) - ||u_t^n(t)||^2 + ||\Delta u^n(t)||^2 + \frac{\lambda}{2}\frac{d}{dt}||u^n(t)||^2 \\ &\quad + \sum_i (\sigma_i(u_{x_i}^n), u_{x_i}^n) = 0, \quad t>0. \end{aligned} \tag{3.35}$$

Eq. (3.4) $+\,\varepsilon \times$ (3.35) gives

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}||u_t^n(t)||^2 + \frac{1}{2}||\Delta u^n(t)||^2 + \sum_i \int_\Omega \int_0^{u_{x_i}^n} \sigma_i(s)\,ds\,dx \right. \\ &\left. \quad + \varepsilon\left(\frac{\lambda}{2}||u^n(t)||^2 + (u_t^n, u^n)\right)\right] + (\lambda-\varepsilon)||u_t^n(t)||^2 + \varepsilon||\Delta u^n(t)||^2 \\ &\quad + \varepsilon \sum_i (\sigma_i(u_{x_i}^n), u_{x_i}^n) = 0, \quad t>0. \end{aligned} \tag{3.36}$$

Note that $|(u_t^n, u^n)| \leqslant \frac{\lambda}{2}||u^n(t)||^2 + \frac{1}{2\lambda}||u_t^n(t)||^2$, taking $\varepsilon = \lambda/2$, multiplying (3.36) by $e^{\delta t}$ and integrating the resulting expression over $(0, t)$ gets

$$
\begin{aligned}
& e^{\delta t}\left[\frac{1}{4}||u_t^n(t)||^2 + \frac{1}{2}||\Delta u^n(t)||^2 + \sum_i \int_\Omega \int_0^{u^n_{x_i}} \sigma_i(s)\, ds\, dx\right] \\
& \quad + \frac{\lambda}{2}\int_0^t e^{\delta\tau}\left[||u_t^n(\tau)||^2 + ||\Delta u^n(\tau)||^2 + \sum_i\, (\sigma_i(u^n_{x_i}(\tau)), u^n_{x_i}(\tau))\right] d\tau \\
& \leqslant \frac{1}{2}(||u_1^n||^2 + ||\Delta u_0^n||^2) + \sum_i \int_\Omega \int_0^{u^n_{0x_i}} \sigma_i(s)\, ds\, dx + \frac{\lambda^2}{4}||u_0^n||^2 \\
& \quad + \frac{\lambda}{2}(u_1^n, u_0^n) + \delta \int_0^t e^{\delta\tau} \sum_i\, (\sigma_i(u^n_{x_i}), u^n_{x_i})\, d\tau \\
& \quad + C_5\delta \int_0^t e^{\delta\tau}(||u_t^n(\tau)||^2 + ||\Delta u^n(\tau)||^2)\, d\tau, \quad t > 0, \qquad (3.37)
\end{aligned}
$$

where assumption (iii) of Theorem 2.2 has been used. Take δ: $0 < \delta < \min\{\lambda/2, \lambda/2C_5\}$. From (3.37) we have

$$\tfrac{1}{4}||u_t^n(t)||^2 + \tfrac{1}{2}||\Delta u^n(t)||^2 \leqslant ME_n(0)e^{-\delta t}, \quad t > 0. \qquad (3.38)$$

By (3.34), repeating the arguments of the proof of Theorem 2.1 gives that there exists a subsequence of $\{u^n\}$, still denoted by $\{u^n\}$, such that (3.13)–(3.14) hold and the limiting function $u \in L_\infty([0, T]; H_0^2) \cap W^{1,\infty}([0, T]; L_2)$ is a weak solution of problem (1.1), (1.2).

Letting $n \to \infty$ in (3.38), from (3.13) and the sequential weak* lower semi-continuity of the norm in $L_\infty([0, T]; H_0^2) \cap W^{1,\infty}([0, T]; L_2)$ we obtain

$$
\begin{aligned}
||u_t(t)||^2 + ||\Delta u(t)||^2 & \leqslant \lim_{n\to\infty} \inf\, (||u_t^n(t)||^2 + ||\Delta u^n(t)||^2) \\
& \leqslant ME(0)e^{-\delta t}, \quad t > 0. \qquad (3.39)
\end{aligned}
$$

Case 2. If $\sigma_i'(s) \geqslant C_0, s \in \mathbf{R}, i = 1, \ldots, N$, let $\tilde{\sigma}_i(s) = \sigma_i(s) - k_0 s - \sigma_i(0)$, where $k_0 = \min\{C_0, 0\} \leqslant 0$, $i = 1, \ldots, N$. Obviously $\tilde{\sigma}_i(0) = 0, \tilde{\sigma}_i'(s) = \sigma_i'(s) - k_0 \geqslant 0$, $\tilde{\sigma}_i(s)s \geqslant 0$, $s \in \mathbf{R}$, $i = 1, \ldots, N$, and if assumption (iii) of Theorem 2.2 holds, then a simple calculation shows $\int_0^s \tilde{\sigma}_i(\tau)\, d\tau \leqslant \tilde{\sigma}_i(s)s, s \in \mathbf{R}$, $i = 1, \ldots, N$. Therefore, substituting $\sigma_i(s) = \tilde{\sigma}_i(s) + k_0 s + \sigma_i(0)$ into (3.1) and repeating the proof in Case 1 gets the conclusions of Theorem 2.2. Theorem 2.2 is proved. □

4. The case in one dimension

In order to prove Theorem 2.3, we first quote a lemma.

Lemma 4.1 (Zhou and Fu [10]). *Assume that $G(z_1, \dots, z_h)$ is a k-times continuously differentiable function with respect to variables $z_1, \dots, z_h$ and $z_i \in L_\infty([0,T]; H^k(\Omega))$ $(i = 1, \dots, h)$. Then*

$$\left|\left|\frac{\partial^k}{\partial x^k} G(z_1(\cdot,t), \dots, z_h(\cdot,t))\right|\right|^2 \leqslant C(\bar{M}, k, h) \sum_{i=1}^{h} ||z_i(t)||^2_{H^k}$$

where $\bar{M} = \max_{1 \leqslant i \leqslant h} \max_{(x,t) \in \bar{Q}_T} |z_i(x,t)|$, $C(\bar{M}, k, h)$ is a positive constant depending only on $\bar{M}, k$ and h.

Proof of Theorem 2.3. We still start with approximate solutions $u^n(t)$ of problem (1.3), (1.4) of form (3.0), where $\{w_j\}_{j=1}^{\infty}$ is an orthonormal basis in $H^4 \cap H_0^2$, and the coefficients $\{T_{jn}\}_{j=1}^{n}$ satisfy $T_{jn}(t) = (u^n(t), w_j)$ with

$$(u_{tt}^n(t), w_j) + (u_{x^4}^n(t), w_j) + \lambda(u_t^n(t), w_j) = (\sigma(u_x^n(t))_x, w_j),$$

$$t > 0, \quad j = 1, \dots, n, \tag{4.1}$$

$$u^n(0) = u_0^n, \quad u_t^n(0) = u_1^n, \tag{4.2}$$

where and in the sequel $u_{x^k} = \frac{\partial^k u}{\partial x^k}$, $u_0^n, u_1^n \in C_0^\infty$ and $u_0^n \to u_0$ in $H^4 \cap H_0^2$, $u_1^n \to u_1$ in H_0^2 as $n \to \infty$. Repeating the arguments of the proof of Theorem 2.1 gets (3.12) (replacing Δu^n and ∇u^n there by u_{xx}^n and u_x^n, respectively), and by (3.12) and the Sobolev embedding theorem

$$||u^n(t)||_{C^1} \leqslant M, \quad t > 0. \tag{4.3}$$

Replacing w_j in (4.1) by $u_{x^4t}^n(t)$, integrating by parts and utilizing (4.3) and Lemma 4.1 gets

$$\frac{1}{2}\frac{d}{dt}(||u_{xxt}^n(t)||^2 + ||u_{x^4}^n(t)||^2 + ||u^n(t)||^2) + \lambda||u_{xxt}^n(t)||^2$$

$$= (\sigma(u_x^n)_{x^3}, u_{xxt}^n) + (u^n, u_t^n)$$

$$\leqslant M(||u^n(t)||^2_{H^4} + ||u_{xxt}(t)||^2), \quad t > 0. \tag{4.4}$$

Applying the Gronwall inequality to (4.4) gives

$$||u_t^n(t)||_{H^2}^2 + ||u^n(t)||_{H^4}^2 + 2\lambda \int_0^t ||u_{xxt}^n(\tau)||^2\, d\tau \leqslant M(T),$$
$$||u_t^n(t)||_{C^1} + ||u^n(t)||_{C^3} \leqslant M(T), \quad 0 \leqslant t \leqslant T, \tag{4.5}$$

where and in the sequel we denote by $M(T)$ various positive constants depending only on T. Replacing w_j in (4.1) by $u_{tt}^n(t)$ and making use of the Hölder inequality gets

$$||u_{tt}^n(t)||^2 \leqslant (||u_{x^4}^n(t)|| + \lambda ||u_t^n(t)|| + ||\sigma(u_x^n(t))_x||)||u_{tt}^n(t)||,$$
$$||u_{tt}^n(t)|| \leqslant M(T), \quad 0 \leqslant t \leqslant T. \tag{4.6}$$

Let $v^n(t) = u^n(t) - u^{n-1}(t)$, then $v^n(t)$ satisfy

$$(v_{tt}^n(t), w_j) + (v_{x^4}^n(t), w_j) + \lambda(v_t^n(t), w_j)$$
$$= (\sigma(u_x^n(t))_x - \sigma(u_x^{n-1}(t))_x, w_j), \quad t > 0, \ j = 1, \dots, n, \tag{4.7}$$

$$v^n(0) = u_0^n - u_0^{n-1}, \quad v_t^n(0) = u_1^n - u_1^{n-1}. \tag{4.8}$$

Replacing w_j in (4.7) by $v_{x^4t}^n(t)$ and exploiting the Lagrange mean value theorem and (4.5) gets

$$\frac{1}{2}\frac{d}{dt}(||v_{xxt}^n(t)||^2 + ||v_{x^4}^n(t)||^2 + ||v^n(t)||^2) + \lambda ||v_{xxt}^n(t)||^2$$
$$= ((\sigma(u_x^n) - \sigma(u_x^{n-1}))_{x^3}, v_{xxt}^n) + (v^n, v_t^n)$$
$$\leqslant M(T)(||v^n(t)||_{H^4}^2 + ||v_{xxt}^n(t)||^2), \quad 0 \leqslant t \leqslant T. \tag{4.9}$$

Applying the Gronwall inequality to (4.9) yields

$$||v_{xxt}^n(t)||^2 + ||v^n(t)||_{H^4}^2 + 2\lambda \int_0^t ||v_{xxt}^n(\tau)||^2\, d\tau$$
$$\leqslant (||v_{1xx}^n||^2 + ||v_0^n||_{H^4}^2)e^{M(T)T} \to 0 \tag{4.10}$$

uniformly on $[0, T]$ as $n \to \infty$. Replacing w_j in (4.7) by $v_{tt}^n(t)$ and making use of the Hölder inequality, (4.5) and (4.10) gets

$$||v_{tt}^n(t)|| \leqslant ||v_{x^4}^n(t)|| + \lambda ||v_t^n(t)|| + ||\sigma'(u_x^n(t))v_{xx}^n(t) + \sigma'(\xi_n(t))u_{xx}^{n-1}(t)v_x^n(t)||$$
$$\leqslant M(T)(||v^n(t)||_{H^4} + ||v_t^n(t)||) \to 0 \tag{4.11}$$

uniformly on $[0, T]$ as $n \to \infty$, where $\xi_n = u_x^n + \theta_n u_x^{n-1}, 0 < \theta_n < 1$. From (4.10) and (4.11) we deduce that

$$||u^n(t) - u^m(t)||_{H^4} + ||u_t^n(t) - u_t^m(t)||_{H^2} + ||u_{tt}^n(t) - u_{tt}^m(t)|| \to 0$$

uniformly on $[0, T]$ as $m, n \to \infty$, i.e. $\{u^n\}$ is a Cauchy sequence in $C([0, T]; H^4 \cap H_0^2) \cap C^1([0, T]; H_0^2) \cap C^2([0, T]; L_2)$. Therefore,

$$u^n \to u \quad \text{in } C([0, T]; \; H^4 \cap H_0^2) \cap C^1([0, T]; H_0^2) \cap C^2([0, T]; L_2) \tag{4.12}$$

as $n \to \infty$. Letting $n \to \infty$ in (4.1), (4.2) gives that $u \in C([0, T]; \; H^4 \cap H_0^2) \cap C^1([0, T]; H_0^2) \cap C^2([0, T]; L_2)$ is a generalized solution of problem (1.3), (1.4).

Let $u, v \in C([0, T]; H^4 \cap H_0^2) \cap C^1([0, T]; H_0^2) \cap C^2([0, T]; L_2)$ are two generalized solutions of problem (1.3), (1.4), $w = u - v$. Then we have

$$w_{tt}(t) + w_{xxxx}(t) + \lambda w_t(t) = \sigma(u_x(t))_x - \sigma(v_x(t))_x, \quad t \in (0, T], \tag{4.13}$$

$$w(0) = 0, \quad w_t(0) = 0. \tag{4.14}$$

Taking the L_2-inner product of (4.13) with w_t gives

$$\frac{d}{dt}(||w_t(t)||^2 + ||w_{xx}(t)||^2) + 2\lambda \int_0^t ||w_t(\tau)||^2 \, d\tau$$

$$= 2(\sigma'(u_x)w_{xx} + \sigma''(\xi)v_{xx}w_x, w_t)$$

$$\leqslant M(T)(||w_t(t)||^2 + ||w_{xx}(t)||^2), \quad 0 < t \leqslant T, \tag{4.15}$$

where $\xi = u_x + \theta v_x, 0 < \theta < 1$. Applying the Gronwall inequality to (4.15) gets

$$||w_t(t)|| = ||w_{xx}(t)|| = 0, \quad t \in [0, T]. \tag{4.16}$$

Hence $w(t) = 0$, i.e. $u(t) = v(t), t \in [0, T]$.

Repeating the arguments of the proof of Theorem 2.1 gets (2.8). Theorem 2.3 is proved. $\square$

5. Blowup of solutions

Proof of Theorem 2.4. Taking the L_2-inner product of (1.1) with u_t yields

$$\dot{E}(t) + \lambda ||u_t(t)||^2 = 0, \quad E(t) \leqslant E(0) < 0, \quad t \geqslant 0, \tag{5.1}$$

where and in the sequel $\cdot = \frac{d}{dt}$, and

$$E(t) = \frac{1}{2}(||u_t(t)||^2 + ||\Delta u(t)||^2) + \sum_i \int_\Omega \int_0^{u_{x_i}} \sigma_i(s)\, ds\, dx, \quad t \geqslant 0. \tag{5.2}$$

Let

$$F(t) = ||u(t)||^2 + \lambda \int_0^t ||u(\tau)||^2\, d\tau, \tag{5.3}$$

where $\lambda \geqslant 0$ as shown in (1.1). Then

$$\dot{F}(t) = 2(u, u_t) + \lambda ||u(t)||^2, \tag{5.4}$$

$$\begin{aligned}
\ddot{F}(t) &= 2\left(||u_t(t)||^2 - ||\Delta u(t)||^2 - \sum_i \int_\Omega \sigma_i(u_{x_i}) u_{x_i}\, dx \right) \\
&\geqslant 2\left(||u_t(t)||^2 - ||\Delta u(t)||^2 - k \sum_i \int_\Omega \int_0^{u_{x_i}} \sigma_i(s)\, ds\, dx \right) \\
&\geqslant 2\left(2||u_t(t)||^2 - (k-2) \sum_i \int_\Omega \int_0^{u_{x_i}} \sigma_i(s)\, ds\, dx - 2E(0) \right) \\
&\geqslant 2(2||u_t(t)||^2 + (k-2)\beta ||\nabla u(t)||_{m+1}^{m+1} - 2E(0)), \quad t > 0,
\end{aligned} \tag{5.5}$$

where assumption (i) of Theorem 2.4 and the fact

$$\begin{aligned}
k \sum_i \int_\Omega \int_0^{u_{x_i}} \sigma_i(s)\, ds\, dx \leqslant {} & 2E(0) - ||u_t(t)||^2 + ||\Delta u(t)||^2 \\
& + (k-2) \sum_i \int_\Omega \int_0^{u_{x_i}} \sigma_i(s)\, ds\, dx
\end{aligned}$$

have been used. By (5.5),

$$\dot{F}(t) \geqslant 2(k-2)\beta \int_0^t ||\nabla u(\tau)||_{m+1}^{m+1}\, d\tau - 4E(0)t + \dot{F}(0), \quad t > 0, \tag{5.6}$$

$$\begin{aligned}
\ddot{F}(t) + \dot{F}(t) \geqslant {} & 4||u_t(t)||^2 + 2(k-2)\beta \left(||\nabla u(t)||_{m+1}^{m+1} + \int_0^t ||\nabla u(\tau)||_{m+1}^{m+1}\, d\tau \right) \\
& - 4E(0)(1+t) + \dot{F}(0) \\
= {} & g(t), \quad t > 0.
\end{aligned} \tag{5.7}$$

Take $p=\frac{m+3}{2}$, obviously $2<p<m+1$ and $p'=\frac{m+3}{m+1}\,(<2)$. By the Young inequality and the Sobolev–Poincaré inequality,

$$\begin{aligned}|(u,u_t)| &= \frac{1}{p}||u(t)||_p^p+\frac{1}{p'}||u_t(t)||_{p'}^{p'}\\ &\leqslant C_6[(||\nabla u(t)||_{m+1}^{m+1})^{\mu}+(||u_t(t)||^2)^{\mu}],\end{aligned}$$

$$|(u,u_t)|^{\frac{1}{\mu}}\leqslant C_7[||\nabla u(t)||_{m+1}^{m+1}+||u_t(t)||^2],\quad t>0, \tag{5.8}$$

where and in the sequel C_j $(j=6,7,\ldots)$ denote positive constants independent of t, $\mu=\frac{m+3}{2(m+1)}$ (<1). By the Hölder inequality,

$$||\nabla u(t)||_{m+1}^{m+1}\geqslant C_8(||u(t)||^2)^{\frac{m+1}{2}},\quad t>0, \tag{5.9}$$

$$\int_0^t ||\nabla u(\tau)||_{m+1}^{m+1}d\tau\geqslant C_9 t^{\frac{1-m}{2}}\left(\int_0^t ||u(\tau)||^2\,d\tau\right)^{\frac{m+1}{2}},\quad t>0. \tag{5.10}$$

(1) If $\lambda>0$, then by (5.8)–(5.10),

$$\begin{aligned}g(t) &\geqslant C_{10}\left(2||\nabla u(t)||_{m+1}^{m+1}+||u_t(t)||^2+\int_0^t ||\nabla u(\tau)||_{m+1}^{m+1}\,d\tau\right)\\ &\quad -\ 4E(0)t+\dot F(0)\\ &\geqslant C_{11}\left[|(u,u_t)|^{\frac{1}{\mu}}+(||u(t)||^2)^{\frac{m+1}{2}}+t^{\frac{1-m}{2}}\left(\int_0^t ||u(\tau)||^2\,d\tau\right)^{\frac{m+1}{2}}\right]\\ &\quad -\ 4E(0)t+\dot F(0)\\ &\geqslant C_{11}t^{\frac{1-m}{2}}\left[|(u,u_t)|^{\alpha}+(||u(t)||^2)^{\alpha}+\left(\int_0^t ||u(\tau)||^2\,d\tau\right)^{\alpha}\right]\\ &\quad -\ 4E(0)t+\dot F(0)-C_{11}t^{\frac{1-m}{2}},\quad t\geqslant 1,\end{aligned} \tag{5.11}$$

where and in the sequel $\alpha=1/\mu>1$. Since $-4E(0)t+\dot F(0)-C_{11}t^{\frac{1-m}{2}}\to+\infty$ as $t\to+\infty$, there must be a $t_0\geqslant 1$ such that

$$-4E(0)t+\dot F(0)-C_{11}t^{\frac{1-m}{2}}\geqslant 0\quad \text{as}\quad t\geqslant t_0. \tag{5.12}$$

Let $y(t)=\dot F(t)+F(t)$. Then from (5.6) and (5.3) we obtain $y(t)>0$ as $t\geqslant t_0$. And thus (5.11) and (5.12) imply

$$g(t)\geqslant C_{12}t^{\frac{1-m}{2}}y^{\alpha}(t),\quad t\geqslant t_0, \tag{5.13}$$

where the inequality $(a_1 + \cdots + a_l)^n \leqslant 2^{(n-1)(l-1)}(a_1^n + \cdots + a_l^n)$, here $a_i \geqslant 0$ $(i = 1, \dots, l)$ and $n > 1$ are all real numbers, has been used. Combining (5.7) with (5.13) gives

$$\dot{y}(t) \geqslant C_{12} t^{\frac{1-m}{2}} y^{\alpha}(t), \quad t \geqslant t_0. \tag{5.14}$$

Therefore,

$$t \leqslant \tilde{T} = \begin{cases} \dfrac{3-m}{2}\left[t_0^{\frac{3-m}{2}} + \dfrac{1}{C_{12}(\alpha-1)y^{\alpha-1}(t_0)}\right]^{\frac{2}{3-m}}, & m < 3, \\ t_0 + \exp\dfrac{1}{C_{12}(\alpha-1)y^{\alpha-1}(t_0)}, & m = 3, \end{cases} \tag{5.15}$$

and

$$y(t) \to +\infty \quad \text{as } t \to \tilde{T}^-. \tag{5.16}$$

By (5.3), (5.4) and (5.16),

$$\begin{aligned} &(2+\lambda)\|u(t)\|^2 + \|u_t(t)\|^2 + \lambda \int_0^t \|u(\tau)\|^2 \, d\tau \\ &\quad \geqslant \dot{F}(t) + F(t) \to +\infty \quad \text{as} \quad t \to \tilde{T}^-. \end{aligned} \tag{5.17}$$

And (5.17) implies

$$\|u_t(t)\|^2 + \int_0^t \|u(\tau)\|^2 \, d\tau \to +\infty \quad \text{as} \quad t \to \tilde{T}^-. \tag{5.18}$$

In fact, if

$$\sup_{0 \leqslant t < \tilde{T}} \left(\|u_t(t)\|^2 + \int_0^t \|u(\tau)\|^2 \, d\tau \right) < M, \tag{5.19}$$

then

$$\begin{aligned} \|u(t)\|^2 &= \int_0^t \frac{d}{d\tau} \|u(\tau)\|^2 \, d\tau + \|u_0\|^2 \\ &\leqslant \int_0^t (\|u(\tau)\|^2 + \|u_t(\tau)\|^2) \, d\tau + \|u_0\|^2 \\ &\leqslant (1+\tilde{T})M + \|u_0\|^2, \quad 0 \leqslant t < \tilde{T}, \end{aligned} \tag{5.20}$$

which contradicts (5.17). Therefore (5.18) holds.

(2) If $\lambda = 0$, then by (5.8) and (5.9),

$$g(t) \geqslant C_{10}(2||\nabla u(t)||_{m+1}^{m+1} + ||u_t(t)||^2 + 1) - 4E(0)t + \dot{F}(0)$$

$$\geqslant C_{11}[|(u, u_t)|^\alpha + (||u(t)||^2)^\alpha] - 4E(0)t + \dot{F}(0), \quad t > 0. \tag{5.21}$$

By the same method used in deriving (5.14), there must be a $t_1 \geqslant 0$ such that $-4E(0)t + \dot{F}(0) > 0$ and $y(t) = \dot{F}(t) + F(t) > 0$ as $t \geqslant t_1$. So combining (5.7) with (5.21) yields

$$\dot{y}(t) \geqslant C_{13} y^\alpha(t), \quad t \geqslant t_1. \tag{5.22}$$

Eq. (5.22) implies that there exists a positive constant $\tilde{T} = t_1 + [C_{13}(\alpha - 1)y^{\alpha-1}(t_1)]^{-1}$ such that $y(t) \to +\infty$ as $t \to \tilde{T}^-$. Since $y(t) \leqslant ||u_t(t)||^2 + 2||u(t)||^2$,

$$||u_t(t)|| + ||u(t)|| \to +\infty \quad \text{as} \quad t \to \tilde{T}^-. \tag{5.23}$$

Theorem 2.4 is proved. □

Example. Take $\sigma_i(s) = a|s|^{m-1}s, \ i = 1, \ldots, N$, where $a \neq 0, m > 1$ are all real numbers. Obviously, $\sigma_i \in C^1(\mathbf{R}), |\sigma_i(s)| = |a|\,|s|^m, i = 1, \ldots, N$.

(1) If $a > 0$, and if $N > 2$, also $m + 1 \leqslant \frac{2N}{N-2}$, then $\sigma_i(s)s \geqslant 0$ and

$$\int_0^s \sigma_i(\tau)\,d\tau = \frac{a}{m+1}|s|^{m+1} \leqslant a|s|^{m+1} = \sigma_i(s)s, \quad s \in \mathbf{R}, \ i = 1, \ldots, N,$$

i.e. assumptions (i) and (iii) of Theorem 2.2 hold. So by Theorem 2.2, corresponding problem (1.1), (1.2) admits a global weak solution $u \in L_\infty([0,T]; H_0^2) \cap W^{1,\infty}([0,T]; L_2)$ as long as initial data $u_0 \in H_0^2, u_1 \in L_2$. And when $\lambda > 0$, the solution has asymptotic behavior (1.6), where $E(0)$ as shown in (5.24).

(2) In the case of $a < 0$.

(a) If $u_0 \in W, u_1 \in L_2$ such that

$$0 < E(0) = \frac{1}{2}||u_1||^2 + \frac{1}{2}||\Delta u_0||^2 + \frac{a}{m+1}||\nabla u_0||_{m+1}^{m+1}$$

$$< \frac{m-1}{4(m+1)}\left(\frac{1}{C_*|a|}\right)^{\frac{2}{m-1}}, \tag{5.24}$$

and if $N > 2$, also $m + 1 \leqslant \frac{2N}{N-2}$, then assumptions (i) and (ii) of Theorem 2.1 hold. So by Theorem 2.1, corresponding problem (1.1), (1.2) still admits a global weak solution $u \in L_\infty([0,T]; H_0^2) \cap W^{1,\infty}([0,T]; L_2)$. And when $\lambda > 0$, the solution has decay property (1.7).

(b) If $u_0 \in H_0^2, u_1 \in L_2$ such that $E(0) < 0$, and if $\lambda > 0$, also $1 < m \leqslant 3$, then a simple verification shows that

$$\sigma_i(s)s = k \int_0^s \sigma_i(\tau)\, d\tau = -k\beta|s|^{m+1}, \quad s \in \mathbf{R}, \ i = 1, \ldots, N,$$

where $k = m + 1 > 2, \beta = -a/k > 0$, i.e. assumptions (i) and (ii) of Theorem 2.4 hold. So by Theorem 2.4, the solution u of corresponding problem (1.1), (1.2) blows up in finite time, see (2.9) and (2.10).

(3) In the case of $N = 1, \sigma(s) = a|s|^{m-1}s$. If $m > 3$, a direct verification shows that $\sigma \in C^3(\mathbf{R}), \sigma'''(s)$ is locally Lipschitz continuous, $\sigma'(0) = \sigma''(0) = 0$. So by Theorem 2.3, corresponding problem (1.3), (1.4) admits a unique generalized solution $u \in C([0,T]; H^4 \cap H_0^2) \cap C^1([0,T]; H_0^2) \cap C^2([0,T]; L_2)$, and when $\lambda > 0, u$ has decay property (2.8) as long as initial data $u_0 \in W \cap H^4, u_1 \in H_0^2$ such that

$$0 < E(0) = \frac{1}{2}||u_1||^2 + \frac{1}{2}||u_{0xx}||^2 + \frac{a}{m+1}||u_{0x}||_{m+1}^{m+1} < \frac{m-1}{4(m+1)}\left(\frac{1}{C_*|a|}\right)^{\frac{2}{m-1}}.$$

Remark. The example shows that there exist some clear condition boundaries similar to thresholds among the sign of a, the states of initial energy $E(0)$ and the existence, asymptotic behavior and nonexistence of global solutions of problem (1.1), (1.2).

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