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Capacities of certain Cantor sets

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ABSTRACT

We prove a Nevanlinna-like criterion for positive capacity of Cantor-type sets K. Using this criterion, examples are constructed of such K with capacity zero such that the projections of the square $K \times K$ in all but two directions have positive capacity. We also construct a set of positive capacity whose projections in infinitely many directions have capacity zero.

1. INTRODUCTION

The present investigation was motivated by the question whether a function on $\mathbf{C}^2$ which is real analytic and subharmonic in each variable separately has to be subharmonic. In this context it would have been helpful if products of sets $K \subset \mathbf{R}$ of logarithmic capacity zero would have to have capacity zero, see [Bł]. We will show that this is not the case by estimating the capacity of certain Cantor sets, see Section 6.

An earlier version of this paper appeared as Part C in my Ph.D. thesis [Mo].

The standard Cantor set is obtained in the following way. Start with the unit interval $[0, 1]$ and remove the middle one third, retaining the two subintervals $E_1^1 = [0, \frac{1}{3}]$ and $E_1^2 = [\frac{2}{3}, 1]$. In the second step, remove middle thirds and retain the two subintervals in each E_1^j of length $\frac{1}{9}$, which have the left-hand or right-hand end-point in common with an E_1^j. This process is repeated indefinitely. In each step, a part is removed from each interval $E_n^j = [a, a + 1/3^n]$, and only the two intervals $E_n^{2j-1} = [a, a + 1/3^{n+1}]$ and $E_n^{2j} = [a + 1/3^n - 1/3^{n+1}, a + 1/3^n]$ are retained $(1 \leq j \leq 2^n)$. The limit set is the Cantor set; it is a closed set with linear Lebesgue measure zero, but positive logarithmic capacity. In the sequel, capacity will always mean logarithmic capacity.

One may consider more general Cantor sets, all homeomorphic to the standard one, in a number of ways, e.g. the ratio between the lengths of the intervals appearing in consecutive steps may be chosen different from $\frac{1}{3}$. Suppose that in the kth step, the length of the new intervals is $1/s_k$ times the length of the old intervals. Denote the limit set obtained in this way by $K(s_1, s_2, \ldots)$. The following theorem was proved by Nevanlinna [Ne, Section V.6.6], cf. also Adams and Hedberg [AH]:

Theorem 1.1. *The capacity of $K(s_1, s_2, \ldots)$ is positive if and only if the series*

$$\sum_k \frac{\log s_k}{2^k}$$

converges.

One may also consider the more general situation of planar sets instead of linear sets. Here one starts with a closed disc of which one retains a number of mutually disjoint connected compact subsets E_1^j; in the second step one retains smaller subsets E_2^k within these subsets (the same number inside every E_1^j), and this process is continued indefinitely.

Many authors have considered the question of positivity of the capacity of generalized Cantor sets, e.g. Tsuji [Ts2], Ohtsuka [Oh] and Riiber [Ri1]. The same question has also been considered for α-capacity, see [Oh] and [Ri2]. Many more references can be found in Carleson's book [Ca].

The main result is Theorem 3.2. It states a necessary and a sufficient condition for positive capacity of generalized Cantor sets and is slightly more general than the results obtained by Riiber [Ri1]. The proof is based on estimating Robin constants and is a modification of a method of Nevanlinna.

It will turn out that in some cases where Theorem 3.2 is inconclusive, it is possible to choose the subsets E_n^j in a different way (leading to the same K) so that the theorem becomes applicable. Thus, given some K, it may be important to choose the sets E_n^j carefully.

Section 4 contains some interesting special cases in which the conditions for positive capacity are both necessary and sufficient. Relevant examples may be found in Section 5.

In Sections 6 and 7 we construct examples of Cantor sets K of capacity zero, such that certain projections of their squares $K \times K$ have positive capacity. These Cantor sets are such that almost all projections of their squares are again Cantor sets – or contain Cantor sets.

In Section 8, an example is given of a set K of positive capacity, such that the capacities of the projections in an infinite (but countable) number of directions are zero. Let Δ be the set of directions with this property and call the set of all directions S. In an earlier version of this paper I asked if $\#\Delta$ could be more than countable. Dijkstra and van Mill have proved that this is automatically true if Δ is a dense subset of S [DM, Proposition 2]. (The latter is actually the case for the example in Section 8.) The reason why I asked the question is that a positive answer would confirm that in a certain sense, projection makes the set much smaller.

We will make use of some basic notions of potential theory, cf. [Ts1], which are recalled here. Let A be a compact set in $\mathbf{C}$.
For a positive measure ν with support in A, the logarithmic potential of ν is

$$U^{\nu}(z) = \int_A \log \frac{1}{|z-w|}\, d\nu(w), \quad z \in \mathbf{C}.$$

If U^{ν} is bounded from above for some probability measure ν on A, the *Robin constant* $\gamma(A)$ is defined as

$$\gamma(A) = \inf_{\mu \in P} \sup_{z \in A} U^{\mu}(z),$$

where $P = P(A)$ is the set of all probability measures μ with support in A. In this case, the capacity is

$$\operatorname{cap} A = e^{-\gamma(A)}.$$

Otherwise $\operatorname{cap} A = 0$. The capacity of a disc with radius R is R.

The equilibrium distribution ω of A is the unique probability measure on A such that

$$U^{\omega}(z) = \int_A \log \frac{1}{|z-w|}\, d\omega(w) = \gamma(A)$$

for all $z \in A$ outside a set of capacity 0.

The equality holds everywhere on A if A is a continuum or a union of finitely many continua.

For $a > 0$ and $b \in \mathbf{C}$, $\gamma(aA + b) = \gamma(A) - \log a$, in other words $\operatorname{cap}(aA + b) = a \operatorname{cap} A$. Furthermore, $A_1 \subset A_2$ implies $\operatorname{cap} A_1 \leq \operatorname{cap} A_2$. If $\{E_n\}$ is a sequence of compact sets with $E_{n+1} \subset E_n$ for all n, then

$$\operatorname{cap}\left(\bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} \operatorname{cap} E_n,$$

cf. [He, Theorem 7.20].

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2. GENERAL CANTOR SETS

We use the convention that a sum with upper index one less than the lower index is 0, and a product with such indices is 1.

Starting with the disc $D = \{z \in \mathbf{C} : |z| \leq \frac{1}{2}\}$ we choose $p_1 > 1$ mutually disjoint continua $E_1^1, \ldots, E_1^{p_1}$ in D and set

$$E_1 = \bigcup_{j=1}^{p_1} E_1^j.$$

In the next step, $p_2 > 1$ disjoint continua E_2^i are chosen in each E_1^j and

$$E_2 = \bigcup_{i=1}^{p_1 p_2} E_2^i.$$

This process is continued indefinitely. In general

$$E_n = \bigcup_{j=1}^{p_1 \cdots p_n} E_n^j,$$

a union of $p_1 \times \cdots \times p_n$ mutually disjoint continua in E_{n-1} (where $E_0 = D$ and every $p_k > 1$). Thus $E_n \subset E_{n-1}$.
Let

$$(2.1) \qquad d_n = \max_{1 \le j \le p_1 \cdots p_n} \operatorname{diam} E_n^j, \quad n \in \mathbf{N}.$$

If $\lim_{n \to \infty} d_n > 0$ then K contains a continuum, therefore $\operatorname{cap} K > 0$ ([Ts1, Theorem III.5]). We will concentrate on the interesting remaining case and assume that $d_n \to 0$.
The Cantor set K associated with the family $\{E_n^j\}$ is defined by

$$(2.2) \qquad K = K(\{E_n^j\}) = \bigcap_{n=1}^{\infty} E_n.$$

The set K is homeomorphic to the standard Cantor set.

Some more sequences associated with K will be needed.
For convenience set $d_0 = \operatorname{diam} D = 1$. Introduce

$$(2.3) \qquad s_n = d_{n-1}/d_n \ge 1, \quad n \in \mathbf{N},$$

so that

$$d_n = \frac{1}{s_1 \cdots s_n}.$$

Let

$$(2.4) \qquad \alpha_n = s_1 \cdots s_{n-1} \min_{j \ne k} d(E_n^j, E_n^k) = \frac{1}{d_{n-1}} \min_{j \ne k} d(E_n^j, E_n^k) < 1,$$

where the minima are taken over all pairs of indices j and k such that E_n^j and E_n^k are contained in the *same component* E_{n-1}^i of E_{n-1}.
Furthermore define

$$(2.5) \qquad d'_n = \min_{1 \le j \le p_1 \cdots p_n} \operatorname{diam} E_n^j, \quad n \in \mathbf{N}.$$

This quantity is positive for each n.
Note that

$$(2.6) \qquad \alpha_{n+1} = s_1 \cdots s_n \min_{j \ne k} d(E_{n+1}^j, E_{n+1}^k) \le s_1 \cdots s_n d'_n,$$

where the minimum is taken as in (2.4).

For a continuum E it is well known that

$$\operatorname{cap} E \geq \frac{\operatorname{diam} E}{4},$$

cf. Corollary 5 in [Ts1, Section III.9]. Hence we have

$$(2.7) \qquad \gamma(E_n^j) \leq \log \frac{4}{d_n'}, \quad 1 \leq j \leq p_1 \cdots p_n,$$

and using (2.6),

$$(2.8) \qquad \gamma(E_n^j) \leq \log s_1 \cdots s_n + \log(4/\alpha_{n+1}).$$

3. A CRITERION FOR POSITIVE CAPACITY OF CANTOR SETS

We will prove Theorem 3.2 below for Cantor sets as defined in Section 2 by estimating the Robin constant of E_n and letting $n \to \infty$. With the (mild) extra assumption that $s_n > c_2 > 1$ where c_2 is independent of n, the result could also be derived from Riiber's work [Ri, Satz 1 and 2] where a different method was used.

We start with an auxiliary result.

If μ is an arbitrary probability measure with support in A and ω is the equilibrium distribution of A, then

$$\int U^\mu \, d\omega = \int U^\omega \, d\mu = \gamma(A).$$

Thus we arrive at the following lemma, cf. [Ts1, Theorem III.15].

Lemma 3.1. *Let μ be any probability measure with support in A. Then*

$$\inf_A U^\mu(z) \leq \gamma(A) \leq \sup_A U^\mu(z).$$

Theorem 3.2. *Let K be associated with a sequence $\{E_n^j\}$ as in Section 2, with $1 \leq j \leq p_1 \cdots p_n$. Let s_n be as in (2.3) and α_n as in (2.4).*
(i) *If* $\operatorname{cap} K > 0$, *then the following series converges*:

$$(3.1) \qquad \sum_k \frac{\log s_1 \cdots s_k}{p_1 \cdots p_k}.$$

(ii) *If both series*

$$(3.2) \qquad \sum_k \frac{\log s_1 \cdots s_k}{p_1 \cdots p_k} \quad \text{and} \quad \sum_k \frac{\log 1/\alpha_k}{p_1 \cdots p_{k-1}}$$

converge, then $\operatorname{cap} K > 0$.

Proof. Fix $n > 1$. For $1 \leq j \leq p_1 \cdots p_n$ let μ_j be the equilibrium distribution of E_n^j and set

$$\mu = \frac{1}{p_1 \cdots p_n} (\mu_1 + \cdots + \mu_{p_1 \cdots p_n}),$$

the average of the equilibrium distributions of the sets E_n^k, $1 \le k \le p_1 \cdots p_n$. Taking $z \in E_n^j$, we will estimate

$$U^\mu(z) = \frac{1}{p_1 \cdots p_n} \left\{ \int_{E_n^j} \log \frac{1}{|z-w|}\, d\mu_j(w) + \sum_{\substack{l=1 \\ l \neq j}}^{p_1 \cdots p_n} \int_{E_n^l} \log \frac{1}{|z-w|}\, d\mu_l(w) \right\}.$$

Note that the first integral is equal to $\gamma(E_n^j)$. Since E_n^j is contained in a disc of radius d_n,

$$\gamma(E_n^j) \ge \log(1/d_n) = \log s_1 \cdots s_n.$$

Now consider the $p_n - 1$ sets E_n^k, $k \neq j$, contained in the same E_{n-1}^i as E_n^j. For w in such an E_n^k,

$$\text{(3.3)} \qquad \log s_1 \cdots s_{n-1} \le \log \frac{1}{|z-w|} \le \log \frac{s_1 \cdots s_{n-1}}{\alpha_n}.$$

More generally, there are $(p_m - 1) p_{m+1} \cdots p_n$ sets E_n^k contained in the same E_{m-1}^i as E_n^j but (in case $m > 1$) not in the same E_m^i. For w in such an E_n^k,

$$\text{(3.4)} \qquad \log s_1 \cdots s_{m-1} \le \log \frac{1}{|z-w|} \le \log \frac{s_1 \cdots s_{m-1}}{\alpha_m}.$$

The first inequality in (3.3) together with the first inequality in (3.4) gives

$$\text{(3.5)} \qquad \begin{cases} U^\mu(z) \ge \dfrac{1}{p_1 \cdots p_n} \\ \qquad \times \left(\log s_1 \cdots s_n + \displaystyle\sum_{m=2}^{n} (p_m - 1)(p_{m+1} \cdots p_n) \log s_1 \cdots s_{m-1} \right) \\ \qquad \ge \dfrac{\log s_1 \cdots s_n}{p_1 \cdots p_n} + \dfrac{1}{2} \displaystyle\sum_{k=1}^{n-1} \dfrac{\log s_1 \cdots s_k}{p_1 \cdots p_k} \end{cases}$$

since $p_m - 1 \ge \frac{1}{2} p_m$. By Lemma 3.1 the same lower bound holds for $\gamma(E_n)$.

If $n \to \infty$, then $E_n \downarrow K$, hence $\gamma(E_n) \to \gamma(K)$. If the series $\sum_k ((\log s_1 \cdots s_k)/(p_1 \cdots p_k))$ diverges, then $\lim_{n\to\infty} \gamma(E_n) = \infty$, hence cap $K = 0$ and part (i) of the theorem is proved.

We now estimate $U^\mu(z)$ from above using the other inequalities in (3.3) and (3.4). Also using (2.6)–(2.8) we get

$$\text{(3.6)} \qquad U^\mu(z) \le \sum_{k=1}^{n} \frac{\log s_1 \cdots s_k}{p_1 \cdots p_k} + \sum_{m=1}^{n+1} \frac{\log 1/\alpha_m}{p_1 \cdots p_{m-1}} + \frac{\log 4}{p_1 \cdots p_n}.$$

We see that if $\sum_k ((\log s_1 \cdots s_k)/(p_1 \cdots p_k))$ and $\sum_m ((\log 1/\alpha_m)/(p_1 \cdots p_{m-1}))$ both converge, then $\gamma(K) = \lim_{n\to\infty} \gamma(E_n) < \infty$ so cap $K > 0$, and part (ii) is also proved. □

The following proposition clarifies the relation between Theorem 3.2 and certain results of Riiber [Ri, Satz 1 and 2], cf. also Nevanlinna [Ne, Section V.6.10].

Proposition 3.3. *In Theorem* 3.2, *the product* $s_1 \cdots s_k$ *may be replaced by* s_k: *if one* *of the series* $\sum_k ((\log s_1 \cdots s_k)/(p_1 \cdots p_k))$, $\sum_k ((\log s_k)/(p_1 \cdots p_k))$ *converges, so does the other.*

Proof. Suppose that $\sum_k ((\log s_1 \cdots s_k)/(p_1 \cdots p_k))$ converges. Then obviously also $\sum_k ((\log s_k)/(p_1 \cdots p_k))$ converges. On the other hand, using the fact that $p_n \geq 2$ for all n one obtains

$$\sum_{k=1}^{n} \frac{\log s_1 \cdots s_k}{p_1 \cdots p_k} = \sum_{k=1}^{n} \sum_{l=1}^{k} \frac{\log s_l}{p_1 \cdots p_k} = \sum_{l=1}^{n} \sum_{k=l}^{n} \frac{\log s_l}{p_1 \cdots p_k} \leq \sum_{l=1}^{n} \frac{2 \log s_l}{p_1 \cdots p_l}.$$

It follows that the converse also holds true. □

From (3.5) and (3.6) one may obtain estimates for the Robin constant of K and hence also for the capacity.

Remark 3.4. Since $\alpha_k < 1$, $\log 1/\alpha_k > 0$ and in order to prove cap $K > 0$ it is sufficient to estimate α_k from below.

Obviously, one set K may be defined in terms of different families of E_n^j's. The following example shows that a 'good' choice of E_n^j's is important when one wants to apply Theorem 3.2.

Example 3.5. Let K be a Cantor set in **R** for which

(3.7) $\qquad p_n \equiv 2, \qquad s_n = e^{2^n}, \qquad E_1^1 = [0, 1/s_1],\ E_1^2 = [\frac{1}{2} - 1/s_1, \frac{1}{2}],$

all E_n^j's have the same length and each E_n^j has a common end-point with an E_{n-1}^i $(n > 1)$. Then cap $K = 0$ according to part (i) of Theorem 3.2.
Instead of E_n^j one could have used

(3.8) $\qquad \tilde{E}_n^j = \{x + iy \mid x \in E_n^j,\ |y| \leq c_n\},$

where c_n is such that the diameter $\tilde{d}_n$ of $\tilde{E}_n^j$ is $1/(n+1)!$ and consequently $\tilde{s}_n = \tilde{d}_{n-1}/\tilde{d}_n = n + 1$. The Cantor set associated with $\{\tilde{E}_n^j\}$ is again K but now

$$\sum_n \frac{\log \tilde{s}_1 \cdots \tilde{s}_n}{p_1 \cdots p_n} = \sum_n \frac{\log(n+1)!}{2^n}$$

converges and Theorem 3.2 cannot be used to show that cap $K = 0$. (Apparently the series $\sum_k ((\log 1/\alpha_k)/(p_1 \cdots p_{k-1}))$ becomes divergent, otherwise we would have a contradiction to part (ii) of Theorem 3.2!)

4. SOME SPECIAL CASES

Theorem 3.2 gives rise to the question what happens if the series (3.1) converges but the second series in (3.2) does not. In this section we will see that in this case the capacity may be positive but it may also be zero. Special additional assumptions on the position of the E_n^j ensure that convergence of (3.1) is a necessary and sufficient condition for positive capacity. Under other additional assumptions the convergence of both series in (3.2) is necessary and sufficient for positive capacity, but Example 3.5 shows that the way in which K is defined is important.

If $p_k \equiv p$ and $\alpha_k \geq \alpha > 0$, $\sum_k ((\log 1/\alpha_k)/(p_1 \cdots p_{k-1}))$ is obviously convergent, so in that case, $\operatorname{cap} K > 0$ is equivalent to $\sum_{k=1}^{\infty} (\log s_k/p^k) < \infty$. This is true for the standard Cantor set, where $p = 2$, $s_k \equiv 3$ and $\alpha_k \equiv \frac{1}{3}$. On the other hand, if K is as in Example 3.5 (defined using (3.7)), $\alpha_k = 1 - 2/e^{2^k} > 1 - 2/e$ (cf. (2.4)) and $\operatorname{cap} K = 0$.

Let $K \subset \mathbf{R}$. The next theorem exhibits a class of Cantor sets where convergence of (3.1) is necessary and sufficient for positivity of the capacity.

Theorem 4.1. *Let $K \subset \mathbf{R}$ be as in* (2.2), *with $\alpha_n \geq c/p_n$ for some c between* 0 *and* 1. *Assume also that* $(\log p_{n+1})/(p_1 \cdots p_n)$ *is bounded. Then* $\operatorname{cap} K > 0$ *if and only if* $\sum_k ((\log s_k)/(p_1 \cdots p_k))$ *converges.*

The proof will be similar to the one of Theorem 3.2. Instead of simply estimating $|z - w| \geq (\alpha_n)/(s_1 \cdots s_{n-1})$ for $z \in E_n^j$ and $w \in E_n^i$ $(i \neq j)$, we will more precisely take into account the position of the various E_n^i's.

Proof. Let $n > 1$. First consider subsets in a fixed E_{n-1}^i. Renumber the E_n^m's such that $j = 1$, the distance between E_n^1 and E_n^i is a nondecreasing function of i and $E_n^1, \ldots, E_n^{p_n}$ are all in the same E_{n-1}^i. Since all E_n^m's are contained in $\mathbf{R}$, the two closest to E_n^1 will be at distance $\geq (\alpha_n)/(s_1 \cdots s_{n-1})$, the next pair will be at distance $\geq 2((\alpha_n)/(s_1 \cdots s_{n-1}))$ etc., but after say, the kth pair, $0 \leq k \leq (p_n - 1)/2$, there are no pairs anymore. Thus

$$(4.1)\qquad \left\{ \begin{aligned} \sum_{l=2}^{p_n} \int_{E_n^l} \log \frac{1}{|z-w|}\, d\mu_l(w) &\leq 2 \sum_{m=1}^{k} \log \frac{s_1 \cdots s_{n-1}}{m\alpha_n} + \sum_{m=k+1}^{p_n - 1 - k} \log \frac{s_1 \cdots s_{n-1}}{m\alpha_n} \\ &\leq 2 \sum_{m=1}^{p_n} \log \frac{s_1 \cdots s_{n-1}}{m\alpha_n}. \end{aligned} \right.$$

If $n > 2$, we next consider the $p_n(p_{n-1} - 1)$ E_n^k's that are in the same E_{n-2}^k as E_n^1, but not in the same E_{n-1}^i. Grouping the subsets per E_{n-1}^l, one sees that their contribution in $U^\mu(z)$ is less than or equal to

$$\frac{1}{p_1 \cdots p_n} 2 p_n \sum_{m=1}^{p_{n-1}} \log \frac{s_n \cdots s_{n-2}}{m\alpha_{n-1}}.$$

Continuing in this way we obtain (cf. (3.6))

$$(4.2)\qquad U^\mu(z) \leq \frac{\log 4p_{n+1}s_1 \cdots s_n/c}{p_1 \cdots p_n} + 2 \sum_{k=1}^{n} \frac{1}{p_1 \cdots p_k} \sum_{m=1}^{p_k} \log \frac{p_k s_1 \cdots s_{k-1}}{cm},$$

where we have used that $\alpha_n \geq c/p_n$.
Since

$$(4.3)\qquad \log n! \geq \log \left(\frac{n}{e}\right)^n \geq n \log n - n \log e,$$

one may estimate

$$(4.4)\qquad \begin{cases} \sum_{m=1}^{p_k} \log \frac{ap_k}{m} = p_k \log ap_k - \log p_k! \\ \qquad\qquad \le p_k \log ap_k - p_k \log p_k + p_k \log e = p_k \log(ae). \end{cases}$$

Substitution of $a = (s_1 \cdots s_{k-1})/c$ gives

$$(4.5)\qquad \begin{cases} U^\mu(z) \le \frac{\log 4p_{n+1}s_1 \cdots s_n/c}{p_1 \cdots p_n} + 2\sum_{k=1}^{n} \frac{1}{p_1 \cdots p_k} p_k \log \frac{es_1 \cdots s_{k-1}}{c} \\ \le \frac{\log 4p_{n+1}}{p_1 \cdots p_n} - \frac{\log c}{p_1 \cdots p_n} + 2\sum_{k=2}^{n+1} \frac{\log s_1 \cdots s_{k-1}}{p_1 \cdots p_{k-1}} + 2\sum_{k=1}^{n} \frac{\log e/c}{p_1 \cdots p_{k-1}}. \end{cases}$$

It follows that $\gamma(K)$ is finite if $\sum_k ((\log s_1 \cdots s_k)/(p_1 \cdots p_k))$ converges and $(\log p_{n+1})/(p_1 \cdots p_n)$ is bounded. □

To obtain a class of Cantor sets for which the convergence of both series in (3.2) is necessary and sufficient for positive capacity we impose the following conditions:

(i) There is a constant $C > 0$ such that

$$|z - w| \le \frac{C\alpha_n^\delta}{s_1 \cdots s_{n-1}} = C\alpha_n^\delta d_{n-1}, \quad n = 1, 2, \ldots$$

for $z \in E_n^j$ and $w \in E_n^k$ where E_n^j and E_n^k are neighboring subsets of E_n.

(ii) $\sum_k ((\log p_k)/(p_1 \cdots p_{k-1}))$ converges.

Under these conditions we get the following equivalence involving also the series with $\log(1/\alpha_k)$.

Theorem 4.2. *Assume that $K \subset \mathbf{R}$ and that the conditions* (i) *and* (ii) *above are satisfied. Then* cap $K > 0$ *if and only if both series*

$$(4.6)\qquad \sum_k \frac{\log s_k}{p_1 \cdots p_k} \quad \text{and} \quad \sum_k \frac{\log 1/\alpha_k}{p_1 \cdots p_{k-1}}$$

converge.

Proof. By (ii) of Theorem 3.2, (4.6) implies that cap $K > 0$. Now assume that cap $K > 0$. By (i) of Theorem 3.2 we know then that the first series in (4.6) converges. It remains to prove the convergence of the second series. We again take some $n > 1$ and $z \in E_n^j$. The sets E_n^i in the same E_{n-1}^l as z can be renumbered in such a way that $j = 1$ and the distance between E_n^1 and E_n^l becomes a nondecreasing function of l. For $m < n$ we renumber the E_m^i and may assume that $z \in E_m^1$ for every m. Then by (i) for some $C > 0$ independent of m,

$$(4.7)\qquad |z - w| \le \frac{lC\alpha_m^\delta}{s_1 \cdots s_{m-1}}, \quad z \in E_m^1,\ w \in E_m^l,\ l = 2, \ldots, p_m,\ \delta = \min(\delta_1, \delta_2).$$

For technical reasons it is convenient to take $C > 1$. As in the proofs of Theorems 3.2 and 4.1 we estimate $|z - w|$ for $z \in E_n^j$ and w in E_n^i, depending on the largest m such that z and w are in the same E_m^k. Thus instead of (3.5) we now get (cf. (3.6))

$$(4.8)\quad \begin{cases} U^\mu(z) \ge \dfrac{1}{p_1 \cdots p_n}\left(\log s_1 \cdots s_n + \sum\limits_{m=2}^{n} (p_{m+1}) \cdots (p_n) \sum\limits_{l=2}^{p_m} \log \dfrac{s_1 \cdots s_{m-1}}{lC\alpha_m^\delta}\right) \\ \qquad \ge \dfrac{\log s_1 \cdots s_n}{p_1 \cdots p_n} + \dfrac{1}{2}\sum\limits_{k=1}^{n-1} \dfrac{\log s_1 \cdots s_k}{p_1 \cdots p_k} + \sum\limits_{m=2}^{n} \sum\limits_{l=2}^{p_m} \dfrac{\log(1/lC\alpha_m^\delta)}{p_1 \cdots p_m}. \end{cases}$$

(For $m = n$ the product $(p_{m+1}) \cdots (p_n) = 1$ by definition.) Stirling's formula leads to an estimate similar to (4.3):

$$\log p! \le \log((p/e)^p \sqrt{3p\pi}) \le p \log(p(3p\pi)^{1/2p}).$$

This gives

$$(4.9)\quad \begin{cases} \sum\limits_{m=2}^{n} \sum\limits_{l=2}^{p_m} \dfrac{\log(1/lC\alpha_m^\delta)}{p_1 \cdots p_m} \\ \quad = -\sum\limits_{m=2}^{n} \dfrac{\log p_m! + (p_m - 1)\log C}{p_1 \cdots p_m} + \sum\limits_{m=2}^{n} \dfrac{(p_m - 1)\delta \log 1/\alpha_m}{p_1 \cdots p_m} \\ \quad \ge -\sum\limits_{m=2}^{n} \dfrac{\log p_m (3p_m\pi)^{1/(2p_m)} + \log C}{p_1 \cdots p_{m-1}} + \dfrac{1}{2}\sum\limits_{m=2}^{n} \dfrac{\delta \log 1/\alpha_m}{p_1 \cdots p_{m-1}}. \end{cases}$$

By (ii) the first series on the right-hand side of (4.9) converges. Thus if the final series would diverge, it would follow as in the proof of Theorem 3.2 that $\gamma(E_n) \to \infty$ (cf. (4.8)), hence cap $K = 0$. This contradiction completes the proof. □

The idea of taking into account the position of the E_n^j's as in Theorem 4.1 can also be made to work in the case of planar sets. For convenience we let the E_n^j be discs in order to obtain a lower bound for $d(E_n^1, E_n^k)$, and hence an upper bound for $\log(1/|z - w|)$, when $z \in E_n^1$ and $w \in E_n^k$ $(k \ne 1)$. Instead of the condition $\alpha_n > c/p_n$ we now need $\alpha_n > c/\sqrt{p_n}$, and instead of p_k/m in the last sum in (4.2) we get $\sqrt{p_k/m}$.

Theorem 4.3. *Let all sets E_n^j be closed discs with radius $d_n/2$. Suppose that there is a constant $c > 0$ such that $\alpha_n > c/\sqrt{p_n}$ for all n. Then* cap $K > 0$ *is equivalent to convergence of* $\sum_k ((\log s_k)/(p_1 \cdots p_k))$.

Proof. Fix $n \in \mathbf{N}$ and renumber the E_n^i's as in the proof of Theorem 4.2 (so that in particular $z \in E_n^1$). It is easy to verify that then

$$(4.10)\quad d(E_n^1, E_n^k) > \frac{c}{6}\sqrt{k/p_n}\, d_{n-1} = \frac{(c/6)\sqrt{k/p_n}}{s_1 \cdots s_{n-1}}, \quad k = 2, \ldots, p_n.$$

Indeed, this follows easily if $k \le 36$. Otherwise fix $k > 36$ and assume that (4.10) is false. Consider discs $D_j = D(z_j, r_0)$ with radius

$$(4.11)\quad r_0 = \frac{d_n}{2} + \frac{\alpha_n}{2s_1\cdots s_{n-1}} = \frac{1+\alpha_n s_n}{2s_1\cdots s_n}.$$

These are mutually disjoint (except possibly for one boundary point). After some calculations one gets a contradiction to $k > 36$ since the sum of areas of D_j's is bounded above by a constant times the area of a disc $D_0 = D(z_1, r)$ with radius

$$(4.12)\quad r = (c/6)\sqrt{k/p_n}d_{n-1} + d_n = \frac{(c/3)\sqrt{k/p_n}s_n + 2}{2s_1\cdots s_n}.$$

Recall that $z \in E_n^1$. In the same vein as before we can estimate

$$\begin{aligned} U^\mu(z) &\leq \frac{1}{p_1\cdots p_n}\left(\gamma(E_n^1) + \sum_{m=1}^{n}(p_{m+1})\cdots p_n \sum_{k=2}^{p_m}\log\frac{\sqrt{p_m}s_1\cdots s_{m-1}}{(c/6)\sqrt{k}}\right) \\ &\leq \frac{\log 2s_1\cdots s_n}{p_1\cdots p_n} + \sum_{m=1}^{n}\frac{\log s_1\cdots s_{m-1}}{p_1\cdots p_{m-1}} + \sum_{m=1}^{n}\frac{\log 6/c}{p_1\cdots p_{m-1}} \\ &\quad + \frac{1}{2}\sum_{m=1}^{n}\frac{1}{p_1\cdots p_m}\sum_{k=2}^{p_m}\log\frac{p_m}{k}. \end{aligned}$$

By (4.4) the last repeated sum is a convergent sequence for $n \to \infty$. Thus the convergence of $\sum((\log s_1\cdots s_k)/(p_1\cdots p_k))$ implies cap $K > 0$. The proof is completed by Proposition 3.3. □

5. EXAMPLES

In this section we will show that the classes of Cantor sets to which the theorems in Section 4 can be applied are relatively rich.

For convenience we will denote the series

$$\sum_k \frac{\log s_1\cdots s_k}{p_1\cdots p_k} \quad \text{by } (s)$$

and

$$\sum_k \frac{\log 1/\alpha_k}{p_1\cdots p_{k-1}} \quad \text{by } (\alpha).$$

Lemma 5.1. *Convergence of (s) and convergence of (α) are independent.*

Proof. The constants α_n in (2.4) may be chosen arbitrarily small, thus making (α) divergent independent of the choice of (s). On the other hand, taking $p_n \equiv 2$ and $s_n = e^{2^n}$, we may clearly let $\alpha_n \equiv \alpha$. Then (s) diverges while (α) converges. □

A set K with $p_n \equiv 2$, $s_n = e^{2^n}$ and $\alpha_n \equiv \alpha > 0$ satisfies the conditions $\alpha_n \geq c/p_n$ and $(\log p_{n+1})/(p_1\cdots p_n) \leq M$ of Theorem 4.1 with fixed c and M. It is also possible to construct a set with these values of the constants using discs. It then also satisfies the condition $\alpha_n > c/\sqrt{p_n}$ of Theorem 4.3 for some $c > 0$. Thus we have examples with divergent (s) for these theorems.

The following example shows that there are sets for which the positivity of the capacity follows from Theorem 4.1 but not from Theorem 3.2.

Example 5.2. Construction of K as in Theorem 4.1 with (s) convergent and (α) divergent.

Let $p_1 = 2$ and for all $n > 1$, $p_n = e^{p_1 \cdots p_{n-1}}$, and $s_n = 2p_n$. Let the intervals E_n^j be distributed in such a way that for each n, the distances between consecutive E_n^j which are contained in the same E_{n-1}^i are equal, and such that the endpoints of E_{n-1}^i are also endpoints of E_n^j's. Then $\alpha_n \leq 1/p_n$ and

$$(5.1) \qquad \frac{\log 1/\alpha_n}{p_1 \cdots p_{n-1}} \geq \frac{\log p_n}{p_1 \cdots p_{n-1}} = 1.$$

Thus (α) diverges. On the other hand (s) converges.

Distributing the E_n^j as in Example 5.2 but choosing

$$p_n = n+1, \quad s_n = 2n+1 \qquad \text{so that } \alpha_n = \frac{1}{s_n} = \frac{1}{2n+1},$$

one obtains an example where all conditions of Theorem 4.2 are satisfied, and (s) and (α) both converge.

Remark. If $\alpha_n = 1/s_n$, then $\min_{j \neq k} d(E_n^j, E_n^k) = d_n$, see (2.3) and (2.4). Similarly, if $p_n = n+1$ and $s_n = e^{n!}$ we have

$$\alpha_n = \frac{1 - p_n/s_n}{p_n - 1} \geq \frac{1}{2n}, \quad n > 1.$$

Thus (s) diverges while (α) converges.

Example 5.3. Construction of K as in Theorem 4.2 with (s) convergent and (α) divergent.

Let all E_n^j's have length d_n. It is clearly possible to have

$$p_n = (n+2)^2, \qquad s_n = e^{(n+1)!^2}, \qquad \text{and} \qquad \alpha_n = \frac{1}{s_n}.$$

Furthermore we may choose the E_n^j's in such a way that the distances between consecutive ones (inside one E_{n-1}^i) are all the same. Then conditions (i) and (ii) of Theorem 4.2 are satisfied.

The required properties clearly follow.

Note that the present E_n^j's are concentrated on a very small part of the E_{n-1}^i's.

Remark 5.4. The sets E_n^j in Example 5.3 can be adjusted in such a way that the (s)-series becomes divergent, but K remains the same. Indeed, let $\tilde{E}_n^j$ be the smallest interval containing

$$\bigcup_{m:E^m_{n+1}\subset E^j_n} E^m_{n+1}, \quad n\in\mathbf{N},\ 1\le j\le p_1\cdots p_n.$$

Then if $\tilde{d}_n = \operatorname{diam} \tilde{E}^j_n$ and $\tilde{s}_n = \tilde{d}_{n-1}/\tilde{d}_n$, $(\tilde{s})$ diverges. This shows once again (cf. Example 3.5) that it depends on the choice of the E^j_n whether Theorem 3.2 is applicable.

The next example shows that Theorem 4.3 may be used to prove positivity of the capacity of certain sets where Theorem 3.2 fails to be conclusive.

We first state a simple auxiliary result which will allow us to choose discs E^j_n at a sufficiently large distance from each other.

Lemma 5.5. *For every p there are p mutually disjoint closed discs with radius* $\ge 1/(5\sqrt{p})$ *inside the disc* $D = \{|z|\le\frac{1}{2}\}$.

Example 5.6. Construction of K as in Theorem 4.3 with (s) convergent and (α) divergent.

Let $p_1 = 2$ and for all $n > 1$, $p_n = e^{p_1\cdots p_{n-1}}$ as in Example 5.2. We will show that for an appropriate choice of positive constants c_1 and c_2 it is possible to take

$$s_n = c_1 p_n \qquad \text{and} \qquad \alpha_n = \frac{1}{\sqrt{c_2 p_n}}$$

with suitable c_1 and c_2 independent of n. It then follows as in Example 5.2 that (s) converges and (α) diverges, cf. (5.1).

By (2.4) it is sufficient to show that for s_n and α_n as indicated it is possible to choose p_n mutually disjoint discs with diameter

$$(5.2) \qquad d_n + \alpha_n d_{n-1} = \left(\frac{1}{s_n} + \alpha_n\right) d_{n-1}$$

inside a disc with diameter d_{n-1}. This follows from Lemma 5.5 if c_1 and c_2 are sufficiently large.

6. A CANTOR SET K OF CAPACITY 0 SUCH THAT PROJECTIONS OF K^2 HAVE POSITIVE CAPACITY

In this section we will show that the set K of Example 3.5, which has capacity 0, is such that for all lines l through 0 in $\mathbf{C}$ except the real and imaginary axes, the orthogonal projection of $K^2 = K\times K$ onto l has positive capacity.

Recall that $p_n \equiv 2$ and $s_n = e^{2^n}$, $E^1_1 = [0, 1/s_1]$, $E^2_1 = [\frac{1}{2} - 1/s_1, \frac{1}{2}]$ and each interval E^j_n has a common end-point with an interval E^i_{n-1} (for $n > 1$).

By Theorem 3.2 cap $K = 0$.

For $\theta\in(0,\pi/2)$, let l_θ be the line given by $y = -x\tan\theta$ and let P_θ be the orthogonal projection onto l_θ (fig. 1). We will prove that

$$(6.1) \qquad \operatorname{cap} P_\theta(K^2) > 0, \quad 0 < \theta \le \pi/4.$$

Note that by symmetry it also follows that cap $P_\theta(K^2) > 0$ for other θ except when l_θ is the real or imaginary axis.

First let $\theta = \pi/4$. Observe that $P_{\pi/4}(K^2)$ is a Cantor set with $p_n \equiv 3$. It has the same sequence $\{s_n\}$ as K (except for s_1) and an increasing sequence $\{\alpha_n\}$ ($\alpha_n = \frac{1}{2}(1 - 3/s_n)$ for $n \geq 2$). Thus α_n can be estimated from below by a positive constant. By these facts, both series in (3.2) converge. This implies that cap $P_{\pi/4}(K^2) > 0$.

Now let $\pi/4 > \theta > \arctan 2/(s_1 - 4)$. In this case $P_\theta(K^2)$ *contains* a Cantor set K' with $p_n \equiv 3$, the same s_n as before (again except for s_1) and increasing α_n, see figure 1. (In every step, one may limit oneself to the projections of the three subsquares closest to l_θ.) It follows from Theorem 3.2 that cap $K' > 0$ and consequently cap $P_\theta(K^2) > 0$.

If $0 < \theta \leq \arctan 2/(s_1 - 4)$, we get overlap in the projection of K^2, but in this case the same argument as before can be applied to a small part of K^2, ignoring the rest. Indeed, choose m so large that

$$\tan\theta > \frac{1}{s_{m+1} - 2} = \frac{1}{e^{2^{m+1}} - 2}.$$

Then $E_m^1 \cap K$ is a Cantor set of the desired form: the projection P_θ of its square contains a Cantor set with $p_n \equiv 3$ ($n \geq m$). The old sequence is replaced by

$$s_n' = s_{n+m} = e^{2^{n+m}}.$$

Hence the previous argument shows that

$$\text{cap } P_\theta((E_m^1 \cap K)^2) > 0,$$

and thus also cap $P_\theta(K^2) > 0$.

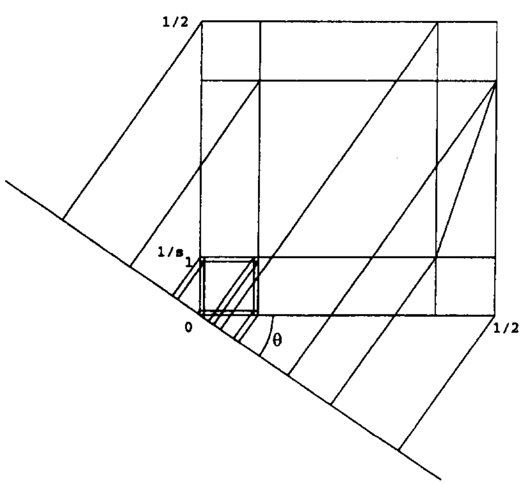

Fig. 1.

7. A NONLINEAR CANTOR SET K OF CAPACITY 0 SUCH THAT PROJECTIONS OF K^2 ONTO 2-DIMENSIONAL SUBSPACES HAVE POSITIVE CAPACITY

We will construct a set $K \subset \mathbf{R}^2$ and consider its square $K^2 = K \times K$ in $\mathbf{R}^4$. We then project K^2 onto two-dimensional subspaces W of $\mathbf{R}^4$. Such spaces W can be identified with $\mathbf{C}$ (or $\mathbf{R}^2$) and thus the capacity of the projection of K^2 onto W makes sense. For all except 6 such subspaces, this orthogonal projection will have positive capacity, while cap $K = 0$.

We first construct an auxiliary linear Cantor set K_0 in exactly the same way as in Section 6, but now with $s_n = e^{4^n}$. Define

(7.1) $\quad K = K_0^2.$

One should actually translate it over $-\frac{1}{4} - \frac{1}{4} i$ to obtain a set inside $\{|z| \leq \frac{1}{2}\}$, but this is irrelevant.

Theorem 7.1. *The Cantor set K defined above has capacity zero. For all except* 6 *two-dimensional subspaces W, the orthogonal projection of $K \times K$ onto W has positive capacity.*

Proof. It follows from Theorem 3.2 that cap $K = 0$.

Now look at the argument of Section 6 again. In order to prove the positivity of the capacity of a projection of K^2, the fact was used that the four corners of a square are projected onto at least three different points whenever the direction of projection is not parallel to the real or imaginary axis. Let V_2 be the set of corners of the square $[0,1]^2$. (We may take this square instead of $[0, \frac{1}{2}]^2$.) Using the notation of Section 6, $\#P_{\pi/4}(V_2) = 3$ and $\#P_\theta(V_2) = 4$ if $0 < \theta < \pi/4$. We could deal with overlap in $P_\theta(K^2)$ by 'omitting' a sufficiently large number of steps at the beginning, considering only a small part of K^2.

The same can be done in the present case. Let Y be a 2-dimensional subspace of $\mathbf{R}^4$ and let P_Y be the projection *along* Y onto $W = Y^\perp$ and consider $P_Y(K^2)$.

Let $V = \{(0,0,0,0), (1,0,0,0), (0,1,0,0), \ldots\}$ be the set of vertices of $[0,1]^4$ ($\#V = 16$). As in Section 6, one can prove that cap $P_Y(K^2) > 0$ whenever $\#P_Y(V) > 4$: in that case $P_Y(K^2)$ will contain a Cantor set with the same sequence $\{s_n\}$ except for s_1 and with $p_n \equiv 5$, so that

$$\sum_k \frac{\log s_k}{p_1 \cdots p_k} = \sum_k \left(\frac{4}{5}\right)^k$$

converges.

The quantities α_k can again be estimated from below by a positive constant.

It follows that it is sufficient to prove that there are only 6 subspaces Y for which $\#P_Y(V) \leq 4$. Note that there are 6 pairs of standard basis vectors e_i and e_j in $\mathbf{R}^4$. Thus our result will be a consequence of

Lemma 7.2. *Let V, Y and P_Y be as above. Then $\#P_Y(V) \geq 4$. Furthermore, if $\#P_Y(V) = 4$, then there are* 2 *standard basis vectors e_i and e_j such that $Y =$* span(e_i, e_j).

Proof. Suppose for a moment that $\#P_Y(V) \leq 3$. Then 5 or more points are projected along Y onto one point x. By reflection in hypersurfaces given by $\{x_j = \frac{1}{2}\}$, we may assume that 0 is one of those 5 points. It follows that our (new) Y contains 5 or more points of V. We will first prove that this cannot happen: Suppose $v_j \in (V \cap Y)\setminus\{0\}, j = 1,2,3,4$, $Y = \text{span}(v_1, v_2)$. Then $v_3 = \lambda v_1 + \mu v_2$. $\lambda = \mu = 1$ implies $v_3 = v_1 + v_2$, $\lambda = 1$, $\mu = -1$ implies $v_3 = v_1 - v_2$ so $v_1 = v_2 + v_3$, the case $\lambda = -1$, $\mu = 1$ is similar and there are no other possibilities.

In the same way v_4 is the sum or difference of the 2 elements $\{v_1, v_2\}$.

By renumbering $\{v_1, v_2, v_3\}$ if necessary, we may assume $v_3 = v_1 + v_2$. Then $v_4 = \pm(v_1 - v_2)$. It would follow that $v_3 = v_4 + 2v_2$ or $v_3 = v_4 + 2v_1$, but a coordinate ≥ 2 is impossible. This contradiction shows that Y cannot contain more than 4 points of V.

It follows that $\#P_Y(V) \geq 4$.

Suppose that $\#P_Y(V) = 4$. Then four points of V are projected onto 0. In other words, Y contains three nonzero elements v_1, v_2 and v_3 of V. As before we may assume that $Y = \text{span}(v_1, v_2)$ and $v_3 = v_1 + v_2$. Then v_1 and v_2 are sums of different sets of standard basis vectors. (Otherwise $v_1 + v_2$ would have at least one coordinate equal to 2.)

By permutation of the coordinates we can always obtain one of the following cases:

(i) $Y = \text{span}(e_1, e_2)$.
(ii) $Y = \text{span}(e_1, e_2 + e_3)$.
(iii) $Y = \text{span}(e_1, e_2 + e_3 + e_4)$.
(iv) $Y = \text{span}(e_1 + e_2, e_3 + e_4)$.

In the second case, $P(e_2) \neq P(e_3)$ and both are nonzero, that is, unequal to $P(e_1)$. Adding e_4, we get 3 more points $e_1 + e_4$, $e_2 + e_4$ and $e_3 + e_4$ in V with 3 more different images and $\#P_Y(V) > 4$. In the third and fourth case one similarly finds that $\#P_Y(V) > 4$. □

8. A CANTOR SET OF POSITIVE CAPACITY SUCH THAT MANY OF ITS PROJECTIONS ONTO LINES HAVE CAPACITY ZERO

The example constructed in this section illustrates how one might go about trying to find a set of positive capacity, such that the capacities of its orthogonal projections onto lines are zero. It turns out to be possible to adjust the example so as to make the capacity of more than a countable number of projections zero, see [DM]. On the other hand it is known that for almost all directions, the projection has positive capacity, see [DM] and cf. Remark 4.12 in Mattila [Ma].

A direction will be an element of the unit circle in **C**. A direction $e^{i\theta}$ ($\theta \in [0, 2\pi)$) will be called *dyadic* if it can be represented in the form

$$(8.1) \qquad \theta = \pi \sum_{k=0}^{m} \frac{a_k}{2^k}, \quad m = m(\theta) \geq 0,\ a_k \in \{0,1\},\ a_m = 1 \text{ if } \theta \neq 0.$$

The set to be constructed will have projections of capacity zero in all dyadic directions.

Let $\{e^{i\theta_j} \mid j \in \mathbf{N}\}$ be an enumeration of all dyadic directions as follows.

First take the angles for which $m = 0$ in (8.1): $\theta_1 = 0$, $\theta_2 = \pi$. Then take those for which $m = 1$, in increasing order: $\theta_3 = \pi/2$, $\theta_4 = 3\pi/2$. The enumeration continues in this way, e.g. $\theta_5 = \pi/4$, $\theta_6 = 3\pi/4$.

Starting with the (closed) disc $D = D(0, \frac{1}{2})$ in $\mathbf{C}$, choose two disjoint discs E_1^1 and E_1^2 in D, each with diameter $d_1 = 1/s_1$, touching the boundary respectively in $\frac{1}{2}e^{i\theta_1}$ and $\frac{1}{2}e^{i\theta_2}$. Then choose $p_2 = 3$ new discs in each of these discs, touching the boundaries in the points corresponding to $\{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}\}$, each with diameter $d_2 = 1/(s_1 s_2)$. In general, $p_n = n + 1$, cf. figure 2. In the n-th step of the process, $(n+1)!$ discs called $E_n^1, \ldots, E_n^{(n+1)!}$ are chosen, each with diameter d_n, $n + 1$ in each of the old discs, touching the boundary of a disc $E_{n-1}^l = D(a, r)$ in $a + re^{i\theta_j}$, $j = 1, \ldots, n+1$, respectively. It is clear that for sufficiently small diameters $d_n = 1/(s_1 \cdots s_n)$ the sets E_n^j are mutually disjoint. As in Section 2, let

$$E_n = \bigcup_{j=1}^{(n+1)!} E_n^j \qquad \text{and} \qquad K = \bigcap_{n=1}^{\infty} E_n.$$

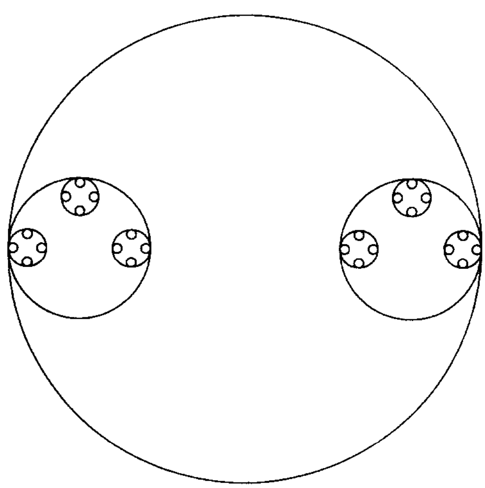

Fig. 2.

In order to get a set of positive capacity, s_k must not grow too fast: recall that $\operatorname{cap} K > 0$ if $\sum_k ((\log s_k)/(p_1 \cdots p_k))$ and $\sum_k ((\log 1/\alpha_k)/(p_1 \cdots p_{k-1}))$ converge. In other words, d_k must not approach 0 too fast. On the other hand, however, we want projections of capacity zero, so s_k must not grow too slowly. The choice of s_k in (8.2) below will turn out to satisfy our requirements.

Theorem 8.1. *Let K be a Cantor set as described above and set*

(8.2) $$s_k = e^{(k+1)!/k^2}.$$

Then cap $K > 0$ *and for every projection P_θ in a dyadic direction $e^{i\theta}$,*

$$\text{cap } P_\theta(K) = 0.$$

Proof. We first show that cap $K > 0$. Since $p_1 \cdots p_k = (k+1)!$,

$$\sum_k \frac{\log s_k}{p_1 \cdots p_k} = \sum_k \frac{1}{k^2}$$

and this series converges. We next show that the (α)-series for K converges too.

Indeed, let $t(n) = \min_{1 \le j < k \le n} |\theta_j - \theta_k|$. Suppose $n = 2^m + k$, $0 \le k < 2^m$. Then

(8.3) $$t(n) = t(2^m + k) \ge \frac{2\pi}{2^{m+1}} \ge \frac{2\pi}{2n} = \frac{\pi}{n}.$$

Since the construction of E_n involves the angles $\theta_1, \ldots, \theta_{n+1}$, this implies

(8.4) $$\alpha_n \ge \sin \frac{\pi}{2(n+1)} - \frac{2}{s_n}.$$

Because $s_n = e^{(n+1)!/n^2}$, it follows that for some $C > 0$ we have $\alpha_n > C/n$. Thus the sets E_n^j are mutually disjoint and

$$\sum_n \frac{\log 1/\alpha_n}{p_1 \cdots p_{n-1}} = \sum_n \frac{\log 1/\alpha_n}{n!}$$

converges and cap $K > 0$ by Theorem 3.2.

It is convenient to define $E_0^1 = D(0, \frac{1}{2})$. We now need a lemma.

Lemma 8.2. *For every dyadic direction $e^{i\theta}$ there is a constant $N_0 \ge 0$ such that for any pair E_n^j, E_n^k, $(j \ne k, n \ge N_0)$, both contained in the same E_{n-1}^i, there are only two possibilities: either $P_\theta(E_n^j) = P_\theta(E_n^k)$ or $P_\theta(E_n^j) \cap P_\theta(E_n^k) = \emptyset$.*

Proof. Fix θ and let $N_0 \ge 2^{m(\theta)+1}$, though the lemma could be true for smaller values of N_0. Then the set $\{e^{i\theta_j} \mid 1 \le j \le N_0\}$ contains $\pm e^{i\theta}$ and $\pm ie^{i\theta}$. Consider E_{n-1}^1 (see fig. 3, we could have taken any other E_{n-1}^j). The radius of E_{n-1}^1 is $d_{n-1}/2$. Denote its center by z. Let A_1 be the $E_n^{i_1}$ which touches the boundary of E_{n-1}^1 at $z + (d_{n-1}/2)(-ie^{i\theta})$. Furthermore let A_2 be (one of) the sets $E_n^i \subset E_{n-1}^1$ which is closest to A_1. (In fig. 3, A_2 could also be on the other side of A_1.)

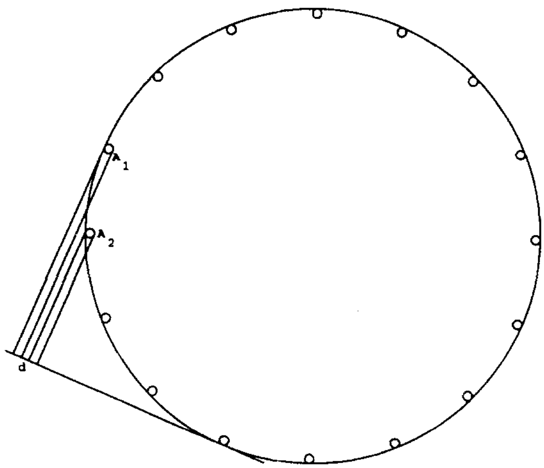

Fig. 3. E^1_{n-1}, $\theta = \frac{11}{8}\pi$.

The inequality (8.3) will be used again. The diameter of both A_1 and A_2 is d_n. Let the distance between the projections of A_1 and A_2 be d. If n is large enough,

$$\begin{aligned} d &> \frac{d_{n-1}}{2}(1 - \cos(\pi/(n+1))) - \operatorname{diam}(A_1) - \operatorname{diam}(A_2) \\ &> \frac{d_{n-1}}{2}\left(-4e^{-(n+1)!/n^2} + \frac{1}{2}\left(\frac{\pi}{n+1}\right)^2 - \frac{1}{24}\left(\frac{\pi}{n+1}\right)^4\right) \\ &> \frac{d_{n-1}}{2(n+1)^2} > 0. \end{aligned}$$

If we would have chosen other i_1 and i_2 in the above construction, then either $P_\theta(A_1)$ and $P_\theta(A_2)$ would be equal or the distance between them would be $\geq d$. Thus the proof of the lemma is complete. □

Continuation of the proof of Theorem 8.1. Fix a dyadic direction $e^{i\theta}$ and a constant N_0 as in Lemma 8.2. The lemma implies that $P_\theta(K \cap E^1_{N_0})$ is a Cantor set as in (2.2). Since a finite union of sets of capacity zero has capacity zero, cf. [Ts1, Theorem III.8], it is sufficient to show that

$$\text{(8.5)} \qquad \operatorname{cap} P_\theta(K \cap E^1_{N_0}) = 0.$$

The construction of K implies that from some step N onward, for every E^i_n there are $E^j_{n+1} \subset E^i_n$, $E^k_{n+1} \subset E^i_n$ $(j \neq k)$, such that $P_\theta(E^j_{n+1}) = P_\theta(E^k_{n+1})$ $(n \geq N)$. We may take $n \geq N > N_0$.
$P_\theta(K \cap E^1_{N_0})$ is a Cantor set with $\tilde{s}_k = s_{k+N_0}$ $(k > 1)$ and $\{\tilde{p}_k\}$ satisfying

$$\begin{aligned} &\tilde{p}_k \leq p_{k+N_0} && \text{if } k \leq N - N_0, \\ &\tilde{p}_k \leq p_{k+N_0} - 1 && \text{if } k > N - N_0. \end{aligned}$$

First assume that $N_0 = 0$. In this case Theorem 3.2 (i) gives that $P_\theta(K \cap E^1_{N_0})$ will be a Cantor set with capacity zero if $\sum_k b_k$ diverges, where

$$b_k = b_k^{(N)} = \begin{cases} \dfrac{\log s_k}{p_1 \cdots p_k} = \dfrac{\log s_k}{(k+1)!} & \text{for } k \leq N, \\ \dfrac{\log s_k}{p_1 \cdots p_N (p_{N+1} - 1) \cdots (p_k - 1)} = \dfrac{\log s_k}{(N+1)k!} & \text{for } k > N. \end{cases}$$

Substituting s_k we find that $\sum b_k^{(N)}$ diverges. If $N_0 > 0$ we get a series $\{\tilde{b}_k\}$ instead of $\{b_k\}$ which (for $k > 1$) is obtained by omitting the first N_0 terms of $\{b_k\}$ and multiplying by $p_1 \cdots p_{N_0} = (N_0 + 1)!$. Thus (8.5) is proved. □

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