Planar \(G^2\) transition with a fair Pythagorean hodograph quintic curve

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Abstract

Recently planar cubic and Pythagorean hodograph quintic transition curves that are suitable for \(G^2\) blending were developed. They are suitable for blending, e.g. rounding corners, or for smooth transition between two curves, e.g. two circular arcs. It was shown that a single cubic segment can be used as a transition curve with the guarantee that an S-shaped transition curve will have no curvature extremum, and a C-shaped transition curve will have a single curvature extremum. The results for the cubic curve are now extended to Pythagorean hodograph quintic curves. A Pythagorean hodograph curve has the attractive properties that its arclength is a polynomial of its parameter, and its offset is rational. A quintic is the lowest degree Pythagorean hodograph curve that may have an inflection point. Pythagorean hodograph curves with no curvature extrema for an S-shaped transition, and a single curvature extremum for a C-shaped transition are suitable for the design of fair curves, e.g. in highway design, or for blending in CAD applications. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Transition curves are useful in several CAD (computer aided design) applications. They may be used for blending in the plane, i.e. to round corners, or for smooth transition between two curves, usually two circular arcs. For visual smoothness it is desirable that the resulting curve have positional and curvature continuity as well as continuity of the unit tangent vector. This condition is usually referred to in the CAD and CAGD (computer aided geometric design) literature as \(G^2\) continuity. For applications such as the design of highways it is desirable that transitions be \(G^2\) and with a
small number of curvature extrema. Such curves are considered fair [1, p. 364]. The problem of finding a transition curve may be posed in a Hermite-like manner as follows.

Find a fair curve with given initial and ending radii of curvature, \( r_0 \) and \( r_1 \), and centres of curvature, \( C_0 \) and \( C_1 \), where

\[
0 < r_1 \leq r_0.
\]  

(1.1)

Observe that (1.1) can always be satisfied by labelling the given radii and centres of curvature appropriately. A positive beginning curvature is assumed; a curve with a negative beginning curvature is simply a reflection, across the line through \( C_0 \) and \( C_1 \), of a curve with a positive beginning curvature.

Parametric cubic curves are popular in CAD applications because they are the lowest degree polynomial curves that allow inflection points (where the curvature is zero), so they are suitable for the composition of \( G^2 \) curves. However, in general, a cubic segment has the following undesirable features.

- Its arc-length is the integral of the square root of a polynomial of its parameter.
- Its offset is neither polynomial, nor a rational algebraic function of its parameter.
- It may have more curvature extrema than necessary.

Pythagorean hodograph (PH) curves were introduced by Farouki and Sakkalis [2,3]. These special parametric polynomial curves do not suffer from the first two of the aforementioned undesirable features. The lowest degree PH curve that may have an inflection point, as required for an S-shaped transition curve, is quintic. The third of the aforementioned undesirable features of a general parametric cubic curve is addressed in Walton and Meek [11]; suitable restrictions were determined that allowed the use of a single cubic Bézier curve as a \( G^2 \) transition (or blending) curve between two circles. Fair curves had previously been formed using two curve segments, in particular, two clothoid spiral segments (non-polynomial) [7], two cubic spiral segments [9], and two Pythagorean hodograph quintic spiral segments [10].

The purpose of this paper is to find a single, fair PH quintic transition curve; this extends the results obtained for a single cubic transition curve. PH quintics are shape isomorphic to general cubics [4]. A PH quintic segment also has the same number of degrees of freedom as a regular cubic segment. It is thus expected that PH quintic transition segments similar to regular cubic transition segments exist. However, the curvature formulae for PH quintics and regular cubics are different, so the analyses to identify their curvature extrema are quite different.

The remainder of this paper is organized as follows. The next section gives a brief discussion of the notation and conventions used in this paper followed by a section with some theoretical background. Results for S- and C-shaped PH quintic transitions are then presented followed by illustrative examples and concluding remarks.

2. Notation and conventions

The usual Cartesian coordinate system with \( x \)- and \( y \)-axis is presumed. Angles measured counterclockwise are positive. Points, that correspond to ordered pairs in the Cartesian coordinate system, and vectors are shown as bold letters. Points and vectors may also be indicated using the ordered
pair notation, e.g. \((x, y)\). In particular, the components of a vector \(A\) may be denoted as \((A_x, A_y)\).
The vector (directed line segment) joining point \(P_i\) to point \(P_j\) is given by \(P_j - P_i\). The dot product
of two vectors, \(A\) and \(B\) is denoted as \(A \cdot B\). The norm or length of a vector \(A\) is denoted as \(\|A\|\).
The derivative of a function (scalar or vector valued), is denoted with a prime, e.g. \(Q'(t)\) or \(x'(t)\).

To aid concise writing of mathematical expressions, a scalar valued two-dimensional “cross-product”
is defined, using the symbol \(\wedge\) as in Juhász [6], by \(A \wedge B = A_xB_y - A_yB_x\). The tangent vector of a
plane parametric curve \(Q(t)\) is given by \(Q'(t)\).

3. Background

Let \(P_i, i = 0, \ldots, 5\) be six given control points. The quintic Bézier curve defined by them is [1]
\[
Q(t) = \sum_{i=0}^{5} \binom{5}{i} P_i (1-t)^{5-i} t^i, \quad 0 \leq t \leq 1
\] (3.1)
and its derivative is
\[
Q'(t) = 5 \sum_{i=0}^{4} \binom{4}{i} W_i (1-t)^{4-i} t^i, \quad 0 \leq t \leq 1,
\]
where
\[
W_i = P_{i+1} - P_i, \quad i = 0, \ldots, 4.
\]

Consider the curve \(Q(t) = x(t)T_0 + y(t)N_0\) and its derivative with respect to \(t\), \(Q'(t) = x'(t)T_0 + y'(t)N_0\) where \(x(t)\) and \(y(t)\) are polynomials in \(t\), \(T_0\) and \(N_0\) are two orthonormal vectors with \(T_0 \wedge N_0 = 1\). The curve \(Q(t)\) is said to be a PH curve if \(\{x'(t)\}^2 + \{y'(t)\}^2\) can be expressed as the
square of a polynomial in \(t\). To ensure that \(Q(t)\) is indeed a PH curve, define \(x'(t)\) and \(y'(t)\) as [2]
\[
x'(t) = u^2(t) - v^2(t)
\]
and
\[
y'(t) = 2u(t)v(t),
\]
where \(u(t)\) and \(v(t)\) are polynomials. A PH quintic Bézier form of \(Q(t)\) is obtained by defining \(u(t)\) and \(v(t)\) as [2]
\[
u(t) = u_0 (1-t)^2 + 2u_1 (1-t)t + u_2 t^2
\] (3.2)
and
\[
v(t) = v_0 (1-t)^2 + 2v_1 (1-t)t + v_2 t^2, \quad 0 \leq t \leq 1.
\] (3.3)
The Bézier control points for (3.1) are then [2]
\[
P_1 = P_0 + \frac{1}{3}(u_0^2 - v_0^2)T_0 + \frac{2}{3}u_0v_0N_0,
\] (3.4)
\[
P_2 = P_1 + \frac{1}{3}(u_0u_1 - v_0v_1)T_0 + \frac{1}{3}(u_0v_1 + u_1v_0)N_0,
\] (3.5)
Fig. 1. S- and C-shaped PH quintic control polygons.

\[ P_3 = P_2 + \frac{1}{15} (2u_1^2 - 2v_1^2 + u_0u_2 - v_0v_2)T_0 + \frac{1}{15} (4u_1v_1 + u_0v_2 + u_2v_0)N_0. \]

\[ P_4 = P_3 + \frac{1}{5} (u_1u_2 - v_1v_2)T_0 + \frac{1}{5} (u_1v_2 + u_2v_1)N_0 \]

and

\[ P_5 = P_4 + \frac{1}{5} (u_2^2 - v_2^2)T_0 + \frac{2}{5} u_2v_2N_0. \]

Eqs. (3.4)–(3.8) will be referred to as Formulation 1. There are 8 degrees of freedom: \( P_0 \) (two), \( u_0, u_1, u_2, v_0, v_1, v_2 \). This formulation is useful for algebraic manipulation but is not geometrically intuitive. It is geometrically more intuitive to formulate the 8 degrees of freedom of a planar PH quintic as 12 unknowns, \( P_0 \) (two), \( T_0, L_i = \|W_{i-1}\|, (i=1,\ldots,5), \theta_i, (i=1,\ldots,4) \) and 4 constraints, where \( \theta_i \) is the angle from \( W_i \) to \( -W_{i-1} \) \([4]\), as shown in Fig. 1. The 4 constraints are

\[ \frac{L_1}{L_5} = \left( \frac{L_2}{L_4} \right)^2, \]

\[ \theta_1 + \theta_4 = \theta_2 + \theta_3, \]

\[ 3L_1L_2L_3 \cos \theta_2 = L_1^2L_4 \cos \theta_4 + 2L_2^3 \cos \theta_1 \]

and

\[ 3L_1L_2L_3 \sin \theta_2 = L_1^2L_4 \sin \theta_4 + 2L_2^3 \sin \theta_1. \]

This will be referred to as Formulation 2.

In the work that follows, it is convenient to define the angles \( \phi_i = \pi - \theta_i \) and to impose the restrictions \( 0 < \phi_1 + \phi_2 < \pi, -\pi < \phi_3 + \phi_4 < 0 \) for an S-shaped curve, and \( 0 < \phi_i < \pi/2, (i=1,\ldots,4) \) for a C-shaped curve, to avoid degeneracy and ambiguity. To ensure fairness, further restrictions on the angles for a C-shaped transition will be imposed in Section 5. Using Formulation 1, the curvature of the PH curve \( Q(t) \) is given by \([4]\)

\[ \kappa(t) = \frac{2(u(t)v'(t) - u'(t)v(t))}{(u^2(t) + v^2(t))^2}. \]
Differentiation of (3.12) yields
\[ \kappa'(t) = \frac{2a(t)}{(u^2(t) + v^2(t))^3}, \]
where
\[ a(t) = (u''v'' - u'v')(u^2 + v^2)^2 - 4(uu'v' - u'v)(uu'' + vv') \] (3.13)
and \( u, \ u', \ v, \) and \( v' \) are functions of \( t. \)

Previously [10, 12], a PH quintic spiral, i.e. a curve with no interior curvature extrema, was obtained by introducing constraints as follows. Let \( \psi \) be the change in the angle of tangent from \( P_0 \) to \( P_5. \) Restrict this angle in magnitude to less than \( \pi. \) Let \( V \) be the intersection of \( P_1 - P_0 \) extended with \( P_4 - P_5 \) extended. So the angle from \( V - P_0 \) to \( P_5 - V \) is \( \psi \) as shown in Fig. 2.

Place the constraints
\[ P_1 = 0.5(P_0 + P_2) \]
and
\[ \|P_5 - V\| = \frac{26 + 12 \cos \psi}{63} \|V - P_0\| \]
on the control points (3.4)–(3.8). The number of degrees of freedom of \( Q(t) \) is thus reduced from the original eight \((P_0, u_0, u_1, u_2, v_0, v_1, v_2)\) to five. In a transition curve between two given circles, the endpoints are allowed to move along the circumferences, but curvatures and unit tangents must match. Blending of two circles such that \( G^2 \) continuity is maintained at the joints, while allowing the endpoints of the transition curve the freedom to move along the circumferences of the corresponding circles, imposes six scalar constraints on the transition curve.

Since five degrees of freedom of the above spiral will not generally allow matching of six constraints, the transition curve may be composed of two spirals, joined at their endpoints of zero curvature as described in [10]. In that approach, the total number of degrees of freedom is reduced from ten to seven by imposing the condition that the joint of the composite transition curve be \( G^2 \) continuous (actually \( G^1 \) since \( G^2 \) then follows automatically because both spirals have zero curvature at the joint). The remaining degree of freedom is used to set equal the magnitudes of the angles swept by the tangent vector from the beginning point to the joint, and from the joint to the ending
point; for convenience this is referred to as “the angle condition”. The angle condition simplifies the problem mathematically, resulting in a single non-linear equation in one unknown.

Later, in [11], a cubic Bézier with six degrees of freedom was developed that allowed fair \( G^2 \) blending between two circles with a single cubic Bézier curve with the angle condition placed on the control polygon. A similar approach is now used to develop a single PH quintic as a transition curve with similar constraints, namely, a fairness constraint, and the angle condition. This results in a transition curve with six degrees of freedom.

It is assumed, without loss of generality, that \( T_0 \) and \( N_0 \) are the unit tangent and normal vectors respectively of \( Q(t) \) at \( t = 0 \). Hence

\[
v_0 = 0.
\]

(3.14)

Note that since \( T_0 \) is not given, this does not reduce the number of degrees of freedom. Let \( \Omega_0 \) and \( \Omega_1 \) be circles with centres \( C_0 \) and \( C_1 \), and radii

\[
r_0 = \frac{1}{\kappa(0)}, \quad r_1 = \frac{1}{|\kappa(1)|},
\]

(3.15)

respectively, where \( r_0 \) and \( r_1 \) satisfy (1.1). Substitution of \( u(t) \) and \( v(t) \) from (3.2) and (3.3), and their derivatives, into (3.12) with subsequent evaluation at \( t = 0 \), taking (3.14) and (3.15) into account, yields

\[
r_0 = \frac{u_0^3}{4v_1}
\]

(3.16)

or

\[
v_1 = \frac{u_0^3}{4r_0}.
\]

(3.17)

To simplify subsequent algebraic expressions, let

\[
0 < \lambda = \left( \frac{r_1}{r_0} \right)^{1/3} \leq 1.
\]

(3.18)

The two cases of an S-shaped and a C-shaped transition curve are now considered separately.

4. S-Shaped transition curve

The curvature of the transition curve should change sign to allow an inflection point, so \( \kappa(1) < 0 \). Also, for an S-shaped transition, \( \Omega_0 \) and \( \Omega_1 \) should be nonenclosing and nonintersecting [10]; this is expressed mathematically as

\[
\|C_1 - C_0\| > r_0 + r_1 = r_0(1 + \lambda^3).
\]

(4.1)

The following lemma is helpful in obtaining the transition curve.

**Lemma 4.1.** Let the PH quintic defined by (3.4)–(3.8) with (3.14) satisfy the angle condition

\[
\phi_1 + \phi_2 = -(\phi_3 + \phi_4).
\]

(4.2)
Then
\[ v_2 = 0 \] (4.3)
and
\[ u_2 = \lambda u_0. \] (4.4)

**Proof.** It follows from (4.2) that \( W_0 \) and \( W_4 \) are parallel and have the same orientation which implies (4.3). Also, substitution of \( u(t) \) and \( v(t) \) from (3.2) and (3.3), and their derivatives, into (3.12) with subsequent evaluation at \( t = 1 \), using (3.15), (3.18) and (4.3), yields
\[
\lambda^3 r_0 = r_1 = -\frac{1}{\kappa(1)} = \frac{u_0^3}{4v_1}.
\] (4.5)
Substitution of \( r_0 \) from (3.16) into (4.5) yields (4.4). \( \square \)

Observe from (3.17) and Lemma 4.1 that \( u_0, u_2 \) and \( v_1 \) have the same sign which, without loss of generality, is taken as positive. The desired S-shaped transition curve is obtained by choosing the constraint \( u_1 = \frac{3}{4} u_0 \) to guarantee fairness according to the following theorem.

**Theorem 4.1.** Let
\[ u_1 = \frac{3}{4} u_0 \] (4.6)
and
\[ r_1 \leq r_0 \leq 8r_1, \] (4.7)
then, by varying \( u_0, (3.4)–(3.8) \) satisfying (3.14), (3.17) and (4.2) define a family of curves. Each value of \( u_0 \) defines a PH quintic with a single inflection point, no cusps and no curvature extrema for \( 0 \leq t \leq 1 \).

**Proof.** Observe that
\[ u(t) = u_0 \{(1 - t)^2 + \frac{3}{2}(1 - t)t + \lambda t^2\} \]
is nonzero on \([0, 1]\) and
\[ v(t) = 2v_1(1 - t)t \]
is nonzero on \((0, 1)\). Therefore the denominator in the expression of the curvature does not vanish on \([0, 1]\), so there are no cusps. Since the numerator of the curvature is quadratic, the curvature can change sign at most twice, but it has an odd number of sign changes as \( t \) varies from 0 to 1, so it has exactly one sign change. The curve thus has a single inflection point for \( 0 \leq t \leq 1 \).

Substitution of (3.14), (4.3), (4.4), (4.6) and (3.17) into (3.2) and (3.3), and the resulting expressions for \( u(t), v(t) \) and their derivatives into (3.13), followed by rearrangement and algebraic manipulation yields
\[ a(t) = -\frac{u_0^4}{r_0} f(t), \]
where 
\[ f(t) = \sum_{i=0}^{5} f_i (1-t)^{5-i}t^i \]
with
\[ f_0 = 0, \]
\[ f_1 = \frac{1}{2}u_0^2 \left( 10\lambda - 3 + \frac{u_0^3}{r_0^2} \right), \]
\[ f_2 = \frac{1}{4}u_0^2 \left( 44\lambda - 9 - \frac{u_0^3}{r_0^2} \right), \]
\[ f_3 = \frac{1}{4}u_0^2 \left( 8\lambda + 27 - \frac{u_0^3}{r_0^2} \right), \]
\[ f_4 = \frac{1}{2}u_0^2 \left( -2\lambda + 9 + \frac{u_0^3}{r_0^2} \right) \]
and
\[ f_5 = \frac{3}{2}(1-\lambda)u_0^2 \geq 0. \]

Introduce the polynomial
\[ \eta(t) = \sum_{i=0}^{3} \eta_i (1-t)^{3-i}t^i, \quad 0 \leq t < 1, \]
where \( \eta_3 = \eta_0 = \min(f_1, f_4) \) and \( \eta_2 = \eta_1 = \min(f_2, f_3) \). The polynomial \( \eta(t) \) is actually a degree-raised quadratic, namely \( \eta(t) = (3\eta_0 - \eta_1)\lambda^2 + (\eta_1 - 3\eta_0)\lambda + \eta_0 \), with endpoint values \( \eta(0) = \eta(1) = \eta_0 \), and a local interior extreme value of \( \frac{1}{4}(\eta_0 + \eta_1) \) at \( t = \frac{1}{2} \). The following four cases arise:

1. \( \eta_0 = f_1, \eta_1 = f_2 \),
2. \( \eta_0 = f_3, \eta_1 = f_2 \),
3. \( \eta_0 = f_4, \eta_1 = f_2 \), and
4. \( \eta_0 = f_4, \eta_1 = f_3 \).

From (3.18), condition (4.7) is equivalent to \( 0.5 \leq \lambda \leq 1 \) for which all four cases yield \( \eta_0 > 0 \) and \( \eta_0 + \eta_1 > 0 \); this implies that \( \eta(t) > 0 \) for \( 0 < t < 1 \). Since \( f(t) \geq t(1-t)\eta(t) \) for \( 0 \leq t \leq 1 \), it follows that \( a(t) < 0 \) for \( 0 < t < 1 \) hence the derivative of the curvature does not vanish on \([0,1]\) so there are no curvature extrema in the PH quintic as defined. □

To pick out that member of the family of S-shaped transition curves which matches the given circles, an equation for \( u_0 \) will be developed. Observe that (as illustrated in Fig. 3)
\[
\mathbf{C}_1 - \mathbf{C}_0 = \mathbf{P}_5 - \mathbf{P}_0 - (r_0 + r_1)\mathbf{N}_0.
\]
Hence, from (3.4) to (3.8) using (3.14), (3.17), (3.18), (4.3), (4.4) and (4.6),
\[
(\mathbf{C}_1 - \mathbf{C}_0) \cdot \mathbf{T}_0 = (\mathbf{P}_5 - \mathbf{P}_0) \cdot \mathbf{T}_0 = \left( 17 + 13\lambda + \lambda^2 \right) \frac{u_0^2}{5} - \frac{u_0^6}{120r_0^2}
\]
Fig. 3. An S-shaped PH quintic transition curve.

and

$$(C_1 - C_0) \cdot N_0 = (P_5 - P_0) \cdot N_0 - (1 + \lambda^3)r_0 = \frac{1}{20r_0}(2 + \lambda)u_0^4 - (1 + \lambda^3)r_0$$

(4.9)

thus

$$\|C_1 - C_0\|^2 = \{(P_5 - P_0) \cdot T_0\}^2 + \{(P_5 - P_0) \cdot N_0 - (1 + \lambda^3)r_0\}^2$$

which is the equation for $u_0$; following some algebraic manipulation it may be re-written as

$q(\rho) = 0$

where

$$q(\rho) = \frac{\rho}{576}\{51 + 26\lambda + 24\lambda^2 - 25\rho\}^2 + \left\{\frac{5}{4}(2 + \lambda)\rho - (1 + \lambda^3)\right\}^2 - \frac{\|C_1 - C_0\|^2}{r_0^2}$$

(4.10)

and

$$\rho = \frac{u_0^4}{25r_0^2}$$

(4.11)

Now, from (4.1), $q(0) = (1 + \lambda^3)^2 - \|C_1 - C_0\|^2/r_0^2 < 0$, and $\lim_{\rho \to \infty} q(\rho) > 0$, which means that $q(\rho)$ has an odd number of sign changes in the interval $(0, \infty)$. Differentiation of (4.10) yields

$$q'(\rho) = \frac{625}{192}\rho^2 + \frac{25}{144}(21 + 46\lambda - 6\lambda^2)\rho$$

$$+ \frac{1}{576}(-279 + 1212\lambda + 3124\lambda^2 - 1632\lambda^3 - 864\lambda^4)$$
which is a parabola with a turning point at $\rho < 0$. Hence $q'(\rho)$ has at most one sign change for $\rho > 0$, so $q(\rho)$ has at most two sign changes for $\rho > 0$; but $q(\rho)$ has an odd number of sign changes for $\rho > 0$ which means that it has a single sign change in $(0, \infty)$. The location of the sign change can be found using standard numerical analysis techniques such as Newton’s method or bracketing. The corresponding value for $u_0$ can be found from (4.11).

Let $\alpha$ be the angle that the vector $C_1 - C_0$ makes with the positive $x$-axis, and let $\tan \beta = (C_1 - C_0) \cdot N_0 / (C_1 - C_0) \cdot T_0$ be determined from (4.8) and (4.9). Then the angle that $T_0$ makes with the positive $x$-axis is $\alpha - \beta$, as shown in Fig. 3, from which $T_0$ and $N_0$ can be determined. Now $P_0 = C_0 - r_0 N_0$ and, as long as (4.7) holds, the S-Shaped PH quintic transition curve with no curvature extrema is given by (3.1), and (3.4)–(3.8), with $u_0$ as determined above, $v_0 = v_2 = 0$, and $v_1$, $u_1$ and $u_2$ given by (3.17), (4.6) and (4.4).

5. C-Shaped transition curve

Two cases arise for a C-shaped transition between two circles.

(1) One circle is contained in the other. In this case it may be possible to have a transition curve with no interior curvature extrema.

(2) One circle is not contained in the other. This condition is expressed mathematically as

$$\|C_1 - C_0\| > r_0 - r_1. \quad (5.1)$$

The first case is discussed in [10]; the second case is discussed here. By Kneser’s theorem [5] the transition curve cannot be a “spiral arc”, so it has at least one interior curvature extremum; here it will be required to have exactly one. It is desirable that the magnitude of the curvature be a minimum at the extremum; this lets the total curvature vary as little as possible for a more pleasing curve (see for example [8]). The transition curve should not have an inflection point, so the curvature should not change sign.

The following lemma is helpful in obtaining the transition curve.

**Lemma 5.1.** Let

$$\phi = \frac{1}{4}(\phi_1 + \phi_2 + \phi_3 + \phi_4)$$

and let the PH quintic defined by (3.4) to (3.8) with (3.14) satisfy the angle condition

$$\phi_1 + \phi_2 = \phi_3 + \phi_4. \quad (5.2)$$

Then

$$\phi_i = \phi, \quad i = 1, \ldots, 4, \quad (5.3)$$

$$\tan \phi = \frac{v_1}{u_1}, \quad (5.4)$$

$$u_1 = \frac{u_0^2}{4r_0 \tan \phi}, \quad (5.5)$$

$$u_2 = \lambda u_0 \cos 2\phi \quad (5.6)$$
and
\[ v_2 = \lambda u_0 \sin 2\phi. \] (5.7)

**Proof.** It follows from (3.9) and (5.2) that \( \phi_4 = \phi_2 \), and \( \phi_3 = \phi_1 \), so (3.10) and (3.11) now become
\[ 3L_1L_2L_3 \cos \phi_2 = L_1^2L_4 \cos \phi_2 + 2L_2^3 \cos \phi_1 \] (5.8)
and
\[ 3L_1L_2L_3 \sin \phi_2 = L_1^2L_4 \sin \phi_2 + 2L_2^3 \sin \phi_1. \] (5.9)

Division of (5.8) by \( \cos \phi_2 \), division of (5.9) by \( \sin \phi_2 \), followed by subtraction and some algebraic manipulation yields (5.3). Now (5.4) follows from (3.5) and (3.14), and (5.5) follows from (5.4) and (3.17). Substitution of \( u(t) \) and \( v(t) \) from (3.2) and (3.3), and their derivatives, into (3.12) with subsequent evaluation at \( t = 1 \) yields
\[ \frac{1}{r_1} = \kappa(1) = \frac{4(u_1v_2 - u_2v_1)}{(u_2^2 + v_2^2)^2} \]
or, using (5.4)
\[ \frac{1}{r_1} = \frac{4v_1(v_2 - u_2 \tan \phi)}{(u_2^2 + v_2^2)^2 \tan \phi}. \] (5.10)

It also follows from (5.4) that
\[ \tan 2\phi = \frac{2u_1v_1}{u_1^2 - v_1^2} \]
and from (3.6)
\[ \tan 2\phi = \frac{4u_1v_1 + u_0v_2}{2u_1^2 - 2v_1^2 + u_0u_2}. \]

Equating and re-arranging,
\[ \frac{2u_1v_1}{u_1^2 - v_1^2} = \frac{v_2}{u_2}. \] (5.11)

Now (5.10) and (5.11) are two equations in the two unknowns \( u_2 \) and \( v_2 \). Observe that (5.11) is solved for
\[ u_2 = h(1 - \tan^2 \phi), \quad v_2 = 2h \tan \phi, \] (5.12)
where \( h \) is determined, by substitution of (5.12) into (5.10), as
\[ h^3 = \frac{4r_1v_1}{(1 + \tan^2 \phi)^3} \]
or, using (3.17) and (3.18),
\[ h = \frac{\lambda u_0}{1 + \tan^2 \phi} = \lambda u_0 \cos^2 \phi. \] (5.13)

Substitution of (5.13) into (5.12) yields (5.6) and (5.7).  \( \square \)
The PH quintic with \( v_0 = 0 \) satisfying the angle condition (5.2) can thus be expressed in terms of \( u_0 \)
and \( \tan \phi \). Observe from (3.17) and (5.5) that \( u_0, u_1 \) and \( v_1 \) have the same sign which, without loss
of generality, is taken as positive. The desired C-shaped transition curve is obtained by choosing the
constraint \( u_1 = u_0 \) (which links \( u_0 \) and \( \phi \)) to guarantee fairness according to the following theorem.

**Theorem 5.1.** Let
\[
    u_1 = u_0
\]
and
\[
    u_0^2 \leq r_0 d
\]
where
\[
    d = 2\sqrt{5 + \sqrt{33}} \quad \text{(i.e. } 0 < \phi \leq 0.32\pi),
\]
then, by varying \( u_0 \), (3.4)–(3.8) satisfying (5.14) and (5.2) define a family of curves. Each value
of \( u_0 \) defines a PH quintic which is free of loops and cusps and has a single curvature extremum; the curvature magnitude is a minimum at its extremum.

**Proof.** Substitution of (5.14) into (5.4) yields
\[
    v_1 = u_0 \tan \phi.
\]
Substitution of (5.14), (5.6), (3.14), (5.16), and (5.7) into (3.2) and (3.3) yields
\[
    u(t) = u_0 \{(1 - t)^2 + 2(1 - t)t + (\lambda \cos 2\phi)t^2\}
\]
and
\[
    v(t) = u_0 \{(2 \tan \phi)(1 - t)t + (\lambda u_0 \sin 2\phi)t^2\}, \quad 0 \leq t \leq 1.
\]
Substitution of the above expressions for \( u(t), v(t) \), and their derivatives, into (3.13), followed by
re-arrangement, algebraic manipulation, and use of trigonometric identities yields
\[
    \kappa(t) = \frac{(4u_0^2 \tan \phi)\{(1 - t)^2 + 2(\lambda \cos 2\phi)(1 - t)t + \lambda t^2\}}{(u^2(t) + v^2(t))^2}
\]
and
\[
    a(t) = 4(u_0^3 \tan \phi) \sum_{i=0}^{5} a_i (1 - t)^{5-i} t^i,
\]
where
\[
    a_0 = -(1 - \lambda \cos^2 \phi),
\]
\[
    a_1 = -(8 - 5\lambda \cos^2 \phi) \tan^2 \phi,
\]
\[
    a_2 = 4(1 + \tan^2 \phi) - 2\lambda(5 - 6 \cos 2\phi) + 6\lambda^2(1 - 2 \cos^2 \phi) \cos^2 \phi,
\]
\[
    a_3 = 4\lambda(5 - \tan^2 \phi) - 2\lambda^2(1 + 15 \cos^2 \phi - 6 \cos^4 \phi),
\]
\[
    a_4 = \lambda(1 - \lambda)(8 + 7\lambda \cos^2 \phi) + \lambda(8 - 5\lambda \cos^2 \phi) \tan^2 \phi
\]
and

\[ a_5 = \lambda^2 \{ 4(1 - \lambda) + \lambda \sin^2 \phi \}. \]

It follows from (5.17) that \( \kappa(t) \) is bounded and \( \kappa(t) > 0 \) for \( 0 \leq t \leq 1 \). Furthermore the total change in the direction of the tangent vector of the curve is less than \( 2\pi \). So the transition curve is free of cusps and loops. It is clear that \( a_0, a_1 < 0 \) and \( a_4, a_5 > 0 \); the sign of \( a_2 \) is not clear. The sign of \( a_3 \) is determined below. Substitution of (5.16) into (3.17) yields

\[ \tan \phi = \frac{u_0^2}{4r_0}. \] (5.18)

By substitution of (5.18) and the corresponding relationship for \( \cos^2 \phi \), \( a_3 \) can be re-written as

\[ a_3 = \frac{\lambda u_0^2}{4r_0(16r_0^2 + u_0^4)} \begin{align*}
&\left\{ 8(1 - \lambda)(2560r_0^6 + 272r_0^4u_0^4 + r_0^2u_0^8) + r_0^4u_0^4 \left( -\frac{u_0^8}{r_0^4} + 40 \frac{u_0^4}{r_0^2} + 128 \right) \right\}.
\end{align*} \]

The coefficient of \( (1 - \lambda) \) inside the braces is positive, and the coefficient of \( r_0^4u_0^4 \) inside the braces is a downward opening quadratic in \( \mu = u_0^4/r_0^2 \) which is zero for \( \mu = 4(5 \pm \sqrt{33}) \). It thus follows from (5.15) that \( a_3 > 0 \) (i.e. \( 0 < \phi < \arctan 0.5 \sqrt{5 + \sqrt{33}} \approx 0.3256 \)).

Consider \( a_i \), \( (i = 0, \ldots, 5) \) to be the components of the control polygon of a quintic Bézier curve. It follows from the sequence of the signs of \( a_i \) that this control polygon crosses the \( t \)-axis exactly once, so by the variation diminishing property of Bézier curves [1], the curve \( a(t) \) crosses the \( t \)-axis at most once. However, \( a(0), a(1) < 0 \), so \( a(t) \) crosses the \( t \)-axis exactly once. Hence the PH quintic defined in the statement of the theorem has exactly one curvature extremum for \( 0 \leq t \leq 1 \); since \( a(0) < 0, a(1) > 0 \), and since from (5.17), \( \kappa(t) > 0 \), it follows that the magnitude of the curvature is a minimum at this extremum.

Using (5.18) the transition curve can be formulated in terms of the single unknown \( \phi \). To pick out that member of the family of S-shaped transition curves which matches the given circles, an equation for \( \phi \) will be developed, however, as is shown below, the restriction (5.15) on \( u_0 \) (which is also a restriction on \( \phi \)), causes a restriction on the data, namely

\[ \| C_1 - C_0 \| < 3.3r_0. \] (5.19)

Let the unit normal and tangent vectors to \( Q(1) \) be \( N_1 \) and \( T_1 \). To determine the transition curve an equation for the unknown \( \phi \) will be developed. Observe that (see Fig. 4)

\[ P_0 = Q(0) = C_0 - r_0N_0 \]

and

\[ P_5 = Q(1) = C_1 - r_1N_1. \]

Hence

\[ C_1 - C_0 = r_1N_1 - r_0N_0 + P_5 - P_0. \]

Let

\[ g_1(\phi) = (C_1 - C_0) \cdot T_0 = r_1N_1 \cdot T_0 + (P_5 - P_0) \cdot T_0 \]

(5.20)
and

\[ g_2(\phi) = (C_1 - C_0) \cdot N_0 = r_1 N_1 \cdot N_0 - r_0 + (P_5 - P_0) \cdot N_0. \]  \hspace{1cm} (5.21)

Observe that \( N_1 \cdot N_0 = \cos 4\phi \) and \( N_1 \cdot T_0 = \cos(4\phi + \pi/2) = -\sin 4\phi \), so using Lemma 5.1, Theorem 5.1, (3.4)–(3.8), (3.14), and (3.18), \( g_1(\phi) \) and \( g_2(\phi) \) can be re-written as

\[ g_1(\phi) = -\lambda^3 r_0 \sin 4\phi + \frac{4r_0 \tan \phi}{15} \{8 - 2 \tan^2 \phi + 4\lambda \cos 2\phi - 6\lambda \sin^2 \phi + 3\lambda^2 \cos 4\phi\} \]  \hspace{1cm} (5.22)

and

\[ g_2(\phi) = \lambda^3 r_0 \cos 4\phi - r_0 + \frac{4r_0 \tan \phi}{15} \{7 \tan \phi + 4\lambda \sin 2\phi + 3\lambda \tan \phi \cos 2\phi + 3\lambda^2 \sin 4\phi\}. \]  \hspace{1cm} (5.23)

Squaring both sides of (5.22) and (5.23), and adding gives the equation in \( \phi \) as

\[ g_1^2(\phi) + g_2^2(\phi) - \|C_1 - C_0\|^2 = 0. \]

Using algebraic and trigonometric manipulation this equation can be re-written as

\[ g(\rho) = \left(\frac{r_0}{15}\right)^2 \frac{\sigma_1(\rho) + \sigma_2(\rho)}{(1 + \rho)^4} - \|C_1 - C_0\|^2 = 0 \]  \hspace{1cm} (5.24)

where

\[ \sigma_1(\rho) = 16\rho \{8 + 4\lambda + 3\lambda^2 - 15\lambda^3 + (14 - 6\lambda - 18\lambda^2 + 15\lambda^3)\rho \\
+ (4 - 10\lambda + 3\lambda^2)\rho^2 - 2\rho^3\}^2, \]

\[ \sigma_2(\rho) = \{15(\lambda^3 - 1) + 2(-1 + 22\lambda + 24\lambda^2 - 45\lambda^3)\rho \\
+ (41 + 32\lambda - 48\lambda^2 + 15\lambda^3)\rho^2 + 4(7 - 3\lambda)\rho^3\}^2. \]
\[\rho = \tan^2 \phi. \quad (5.25)\]

To comply with Theorem 5.1 it is desirable for (5.24) to have a solution in the interval \((0, d^2/16]\). Now \(g(0) = (r_0 - r_1)^2 - \| C_1 - C_0 \|^2 < 0\). For \(g(\rho)\) to have an odd number of sign changes in \((0, d^2/16]\) it is required that \(g(d^2/16) > 0\). With the aid of the symbolic manipulator, MAPLE,

\[
g \left( \frac{d^2}{16} \right) = \left( \frac{r_0}{15} \right)^2 \left\{ 9(229 + 36\sqrt{33}) - 2(75 + 9\sqrt{33})\lambda - 2(119 + 31\sqrt{33})\lambda^2 
- 2(171 - 32\sqrt{33})\lambda^3 - 2(165 + 37\sqrt{33})\lambda^4 + 225\lambda^6 \right\} - \| C_1 - C_0 \|^2
\]

and

\[
\frac{d}{d\lambda} g \left( \frac{d^2}{16} \right) = -2 \left( \frac{r_0}{15} \right)^2 \left\{ 75 + 19\sqrt{33} + 2(119 + 31\sqrt{33})\lambda + 3(171 - 32\sqrt{33})\lambda^2 
+ 4(165 + 37\sqrt{33})\lambda^3 - 675 \right\} 
= -2 \left( \frac{r_0}{15} \right)^2 \left\{ (75 + 19\sqrt{33})(1 - \lambda)^5 + (613 + 157\sqrt{33})(1 - \lambda)^4 \lambda 
+ (2215 + 342\sqrt{33})(1 - \lambda)^3 \lambda^2 + (4377 + 422\sqrt{33})(1 - \lambda)^2 \lambda^3 
+ (4186 + 351\sqrt{33})(1 - \lambda) \lambda^4 + (811 + 133\sqrt{33})\lambda^5 \right\}
\]

from which it follows that for \(0 < \lambda < 1\), \(g(d^2/16)\) attains its smallest value at \(\lambda = 1\). This value is \((r_0/15)^224553.36 - \| C_1 - C_0 \|^2\), so \(g(d^2/16) > 0\) for (5.19). Hence \(g(\rho)\) has an odd number of sign changes in the interval \((0, d^2/16]\). Furthermore,

\[
g'(\rho) = \left( \frac{r_0}{15} \right)^2 \frac{s(\rho)}{(1 + \rho)^5},
\]

where

\[
s(\rho) = \sum_{i=0}^{7} s_i(\lambda)\rho^i = (1 + \rho)\{ \sigma_1(\rho) + \sigma_2(\rho) \} - 4\{ \sigma_1(\rho) + \sigma_2(\rho) \}.
\]

The coefficients of the polynomial \(s(\rho)\) were obtained with the aid of the symbolic manipulator MAPLE as

\[
\begin{align*}
s_0(\lambda) &= 8\{ 23 - 37\lambda - 52\lambda^2 + 123\lambda^3 - 57\lambda^4 \}, \\
s_1(\lambda) &= 8\{ 183 - 109\lambda - 56\lambda^2 + 207\lambda^3 - 225\lambda^4 \}, \\
s_2(\lambda) &= 16\{ 297 - 58\lambda - 20\lambda^2 + 36\lambda^3 - 180\lambda^4 \}, \\
s_3(\lambda) &= 16\{ 515 - 36\lambda - 86\lambda^2 + 12\lambda^3 - 150\lambda^4 \}, \\
s_4(\lambda) &= 40\{ 207 - 11\lambda - 40\lambda^2 + 9\lambda^3 - 27\lambda^4 \}, \\
s_5(\lambda) &= 8\{ 603 - 35\lambda - 76\lambda^2 + 9\lambda^3 - 27\lambda^4 \}, \\
s_6(\lambda) &= 32\{ 47 - 2\lambda - 3\lambda^2 \}
\end{align*}
\]

and

\[
s_7(\lambda) = 192.
\]
The polynomials \( s_0(\lambda) - s_6(\lambda) \) can be expressed in Bernstein basis form as

\[
\begin{align*}
\lambda_0(\lambda) &= 8\{(23(1 - \lambda)^4 + 55(1 - \lambda)^3\lambda - 25(1 - \lambda)^2\lambda^2), \\
\lambda_1(\lambda) &= 8\{183(1 - \lambda)^4 + 623(1 - \lambda)^3\lambda + 715(1 - \lambda)^2\lambda^2 + 500(1 - \lambda)^3\lambda^3 + 75\lambda^4, \\
\lambda_2(\lambda) &= 16\{297(1 - \lambda)^4 + 1130(1 - \lambda)^3\lambda + 1588(1 - \lambda)^2\lambda^2 + 1010(1 - \lambda)^3\lambda^3 + 255\lambda^4, \\
\lambda_3(\lambda) &= 16\{515(1 - \lambda)^4 + 2024(1 - \lambda)^3\lambda + 2896(1 - \lambda)^2\lambda^2 + 1792(1 - \lambda)^3\lambda^3 + 138\lambda^4, \\
\lambda_4(\lambda) &= 40\{207(1 - \lambda)^4 + 817(1 - \lambda)^3\lambda + 1169(1 - \lambda)^2\lambda^2 + 724(1 - \lambda)^3\lambda^3 + 138\lambda^4, \\
\lambda_5(\lambda) &= 8\{603(1 - \lambda)^4 + 2377(1 - \lambda)^3\lambda + 3437(1 - \lambda)^2\lambda^2 + 2164(1 - \lambda)^3\lambda^3 + 474\lambda^4, \\
\lambda_6(\lambda) &= 32\{47(1 - \lambda)^2 + 92(1 - \lambda)\lambda + 42\lambda^2\}. 
\end{align*}
\]

Now \( s_0(\lambda) \) may be negative, but \( s_i(\lambda) \geq 0 \) for \( i = 1, \ldots, 7 \), so by Descartes Rule of signs \( s(\rho) \) has at most one sign change on the interval \((0, d^2/16]\), thus \( g(\rho) \) has at most two sign changes. However \( g(\rho) \) has an odd number of sign changes on this interval, so \((5.24)\) has a unique solution for \( 0 < \lambda < 1 \) and \( \rho \) in \((0, d^2/16]\). The solution can be found using standard numerical analysis techniques such as Newton’s method or bracketing.

Once \( \rho \) is known, the corresponding values for \( \phi \) and \( u_0 \) can be found from \((5.25)\) and \((5.18)\). Values for \( u_1, u_2, v_1 \) and \( v_2 \) are available from \((5.14), (5.6), (5.4), \) and \((5.7)\). Let \( \alpha \) be the angle that the line from \( C_0 \) to \( C_1 \) makes with the positive \( x \)-axis, and let \( \tan \beta = (C_1 - C_0)^\cdot N_0/(C_1 - C_0)^\cdot T_0 \) be determined from \((5.20)\) and \((5.21)\). Then the angle that \( T_0 \) makes with the positive \( x \)-axis is \( \alpha - \beta \), as shown in Fig. 4, from which \( G, N_0 \) and \( T_1 \) can be determined. Now \( P_0 = C_0 - r_0 N_0 \) and the transition curve is given by \((3.1)\) and \((3.4) - (3.8)\).

6. Examples

Two examples from [10] are presented, but with pairs of Pythagorean hodograph (PH) quintic spirals replaced by a single fair (i.e. without extraneous curvature extrema, or inflection points) PH quintic curve.

The first example, shown in Fig. 5, represents the cross-section of a cam. It is composed of two circular arcs joined by an upper and a lower fair C-shaped PH quintic curve. This example is similar to one in [10] in which each C-shaped curve was composed of two PH quintic spiral segments. The joints where the transition curves meet the circular arcs are indicated with dots (small black-filled circles).

The second example, shown in Fig. 6, represents the profile of a vase. The side of the vase was previously represented by a pair of PH quintic spirals. It is now represented by a single S-shaped PH quintic. The base is a straight line segment joined to circular arcs by the PH quintic spirals of [10]. The circular arcs have been adjusted to accommodate the PH quintic transition curves. The endpoints of the transition curves, circular arcs, and straight line segment are indicated with dots (small black-filled circles).
7. Conclusion

Use of a fair PH quintic curve for $G^2$ blending of two circles has been demonstrated. Such blending is often desirable in CAD and CAGD applications. A particular application is in highway design where spiral segments, or compositions of spiral segments, are used for transition between two circular arcs, a circular arc and a straight line segment, or between two straight line segments. The work presented here allows for the use of a single curve segment, rather than the composition of two curve segments which are conventionally used, for transition between two circles. Use of a single spiral segment for transition between a straight line and a circle is discussed in earlier work [9,10].

To guarantee the absence of interior curvature extrema for an S-shaped transition curve, the ratio of the larger to the smaller radii of the given circular arcs is constrained to less than 8. To guarantee a single curvature extremum (at which the curvature magnitude is a minimum) for a C-shaped transition curve, the distance between the centres of the circular arcs is constrained to less than 3.3 times the larger radius. Although these are reasonable constraints in practice, from experimentation with many different pairs of circles they seem more restrictive than necessary.
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References