Existence of global bounded weak solutions to nonsymmetric systems of Keyfitz–Kranzer type

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Abstract

In this paper, we study the global $L^\infty$ solutions for the Cauchy problem of nonsymmetric system (1.1) of Keyfitz–Kranzer type. When $n=1$, (1.1) is the Aw–Rascle traffic flow model. First, we introduce a new flux approximation to obtain a lower bound $\rho^{\varepsilon, \delta} \geq \delta > 0$ for the parabolic system generated by adding “artificial viscosity” to the Aw–Rascle system. Then using the compensated compactness method with the help of $L^1$ estimate of $w^{\varepsilon, \delta}(\cdot,t)$ we prove the pointwise convergence of the viscosity solutions under the general conditions on the function $P(\rho)$, which includes prototype function $P(\rho) = \frac{1}{\gamma} \rho^{\gamma} + A$, where $\gamma \in (-1, 0) \cup (0, \infty)$, $A$ is a constant. Second, by means of BV estimates on the Riemann invariants and the compensated compactness method, we prove the global existence of bounded entropy weak solutions for the Cauchy problem of general nonsymmetric systems (1.1).

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1. Introduction

In this paper, we study the Cauchy problem for nonsymmetric system of Keyfitz–Kranzer type

$$\begin{aligned}
\rho_t + (\rho \phi(\rho, w_1, w_2, \ldots, w_n))_x &= 0, \\
(\rho w_i)_t + (\rho w_i \phi(\rho, w_1, w_2, \ldots, w_n))_x &= 0, \quad i = 1, 2, \ldots, n,
\end{aligned}$$

(1.1)
with bounded measurable initial data

\[ (\rho(x, 0), w_i(x, 0)) = (\rho_0(x), w_{i0}(x)), \quad \rho_0(x) \geq 0, \quad i = 1, 2, \ldots, n, \tag{1.2} \]

where

\[ \phi(\rho, w) = \Phi(w) - P(\rho) \tag{1.3} \]

is a nonlinear function. The more general form of (1.1) was first derived as a model for the elastic string by Keyfitz and Kranzer [13]. The symmetric system

\[ (w_i)_t + (w_i\phi(w_1, w_2, \ldots, w_n))_x = 0, \quad i = 1, 2, \ldots, n, \tag{1.4} \]

where

\[ \phi(w) = \sum_{i=1}^{n} w_i^l, \quad l > 1, \tag{1.5} \]

was well investigated in [13,14,12,6,4,20,18,11] and references cited therein.

When \( n = 1 \) and \( \Phi(w) = w \) in (1.3), system (1.1) or the nonsymmetric system of two equations

\[
\begin{align*}
(\rho)_t + (\rho(w - P(\rho)))_x &= 0, \\
(\rho w)_t + (\rho w(w - P(\rho)))_x &= 0
\end{align*}
\tag{1.6}
\]

was also introduced as a macroscopic model for traffic flow by Aw and Rascle [1], where \( \rho, w \) are the density and the velocity of cars on the roadway and the function \( P \) is smooth and strictly increasing and it satisfies

\[ 2P'(\rho) + \rho P''(\rho) > 0 \quad \text{for } \rho > 0. \tag{1.7} \]

We write the system (1.6) as

\[
\begin{align*}
(\rho)_t + (m - \rho P(\rho))_x &= 0, \\
(mw)_t + (m^2 \rho - mP(\rho))_x &= 0
\end{align*}
\tag{1.8}
\]

where \( m = \rho w \).

The eigenvalues of system (1.8) are

\[ \lambda_1 = \frac{m}{\rho} - P(\rho) - \rho P'(\rho), \quad \lambda_2 = \frac{m}{\rho} - P(\rho) \tag{1.9} \]

with corresponding right eigenvectors

\[ r_1 = \left(1, \frac{m}{\rho}\right)^T, \quad r_2 = \left(1, \frac{m}{\rho} + \rho P'(\rho)\right)^T \tag{1.10} \]
and Riemann invariants

\[ z(\rho, m) = \frac{m}{\rho} - P(\rho), \quad w(\rho, m) = \frac{m}{\rho}. \]  

(1.11)

Moreover

\[ \nabla \lambda_1 \cdot r_1 = -(2P'(\rho) + \rho P''(\rho)), \quad \nabla \lambda_2 \cdot r_2 = 0. \]  

(1.12)

Therefore system (1.6) or equivalently system (1.8) is strictly hyperbolic except for \( \rho = 0 \), where two eigenvalues coincide. The second wave family is always linearly degenerate and the first family is genuinely nonlinear except

\[ 2P'(\rho) + \rho P''(\rho) = 0, \]  

(1.13)

which is equivalent to the prototype function

\[ P(\rho) = A - \frac{B}{\rho}, \quad A, B \text{ are constants.} \]  

(1.14)

For the prototype function

\[ P(\rho) = A + \frac{1}{\gamma} \rho^{\gamma}, \]  

(1.15)

the condition in (1.7) is satisfied for \( \gamma > -1 \). If \( \gamma = 0 \) in (1.15) or \( B = 0 \) in (1.14), choose \( A = 1 \) for simplicity, then (1.6) is reduced to

\[
\begin{cases}
\rho_t + (m - \rho)x = 0, \\
m_t + \left( \frac{m^2}{\rho} - m \right)x = 0,
\end{cases}
\]  

(1.16)

or equivalently to the gas dynamics system of pressureless type

\[
\begin{cases}
\rho_t + m_1 x = 0, \\
m_{1t} + \left( \frac{m_1^2}{\rho} \right)x = 0,
\end{cases}
\]  

(1.17)

where \( m_1 = m - \rho \), which has no classical weak solution and was well studied in [2,3,9,10,26].

The Riemann problem for system (1.6) including the vacuum state (\( \rho = 0 \)) was first resolved by Aw and Rascle in [1] (see also [7] for the Riemann problem at junctions) under the conditions on \( P(\rho) \):

\[ P(0) = 0, \quad \lim_{\rho \to 0} \rho P'(\rho) = 0, \quad \text{and} \quad \rho P''(\rho) + 2P'(\rho) > 0 \quad \text{for} \quad \rho > 0, \]  

(1.18)

which is satisfied for the prototype function \( P(\rho) = \frac{1}{\gamma} \rho^{\gamma} + A \) with \( \gamma > 0, A = 0 \).
In [8], the authors studied the BV entropy solutions with the vacuum state under the conditions
\[ P'(0) = 0, \quad P'(\rho) > 0, \quad \text{for } \rho > 0 \quad \text{and} \quad |P(\rho_1) - P(\rho_2)| \leq L(\rho_1, \rho_2)|\rho_1 - \rho_2| \]
(1.19)
for some continuous function \( L \), which is satisfied for the prototype function \( P(\rho) = \frac{1}{\gamma} \rho^\gamma + A \) with \( \gamma > 1 \). The main idea in [8] is to use Serre’s argument [23,22] on systems of Temple type [25] to obtain a bound of the total variation of the Riemann invariants. However, it seems that the proof in [8] is incomplete since only second characteristic field of system (1.6) is of Temple type, and it is not obvious to be able to use Serre’s argument to prove the total variation estimates for both Riemann invariants. The existence of entropy solutions to the Cauchy problem was also studied in [7] when the road network has only one junction.

For generally nonsymmetric system (1.1), as far as we know, there is no any existence result about the Cauchy problem, except the Riemann problem [13].

In this paper, we obtain the following theorems:

**Theorem 1.** Let the initial data \((\rho_0(x), w_0(x))\) be bounded, \(\rho_0(x) \geq 0\), the total variation of the second Riemann invariant \(w_0(x)\) be bounded and \(P(\rho)\) satisfy (1.18), then the Cauchy problem (1.8)–(1.2) has a global bounded entropy solution \((\rho(x,t), w(x,t))\) and \(w_x(\cdot, t)\) is bounded in \(L^1(R)\).

**Theorem 2.** Let the initial data \((\rho_0(x), w_0(x))\) satisfy the same conditions as given in Theorem 1, but \(\rho_0(x) \geq \rho_0 > 0\) and \(c_0(x) \geq c_0 > 0\) for two constants \(\rho_0\) and \(c_0\); \(P(\rho)\) satisfy
\[ \lim_{\rho \to 0} \rho P'(\rho) = 0, \quad \lim_{\rho \to \infty} (\rho P'(\rho))^\prime \geq A, \quad \rho P''(\rho) + 2P'(\rho) > 0 \quad \text{for } \rho > 0, \]
(1.20)
where \(A\) is a constant satisfying \(A + c_0 > \sup_{x \in (-\infty, \infty)} \frac{|m_0(x)|}{\rho_0(x)}\), then the Cauchy problem (1.8)–(1.2) has a global bounded entropy solution \((\rho(x,t), w(x,t))\) and \(w_x(\cdot, t)\) is bounded in \(L^1(R)\).

**Theorem 3.** Let the initial data \((\rho_0(x), w_{i0}(x))\) be bounded, \(\rho_0(x) \geq 0\), the total variations of the invariants \(w_{i0}(x)\) be bounded; \(P(\rho)\) \(\leq 0\) satisfy
\[ P(0) = 0, \quad \lim_{\rho \to 0} \rho P'(\rho) = 0, \quad \lim_{\rho \to \infty} P(\rho) = \infty, \]
\[ \rho P''(\rho) + 2P'(\rho) < 0 \quad \text{for } \rho > 0 \]
(1.21)
and the nonlinear function \(\Phi(w)\) be nonnegative and convex, then the Cauchy problem (1.1)–(1.2) has a global bounded entropy solution \((\rho(x,t), w_i(x,t))\) and \(w_{ix}(\cdot, t)\) is bounded in \(L^1(R)\).

**Note 1.** The conditions in (1.20) are satisfied for the prototype function \(P(\rho) = \frac{1}{\gamma} \rho^\gamma + A\) with \(\gamma > -1\) and the condition \(A + c_0 > \sup_{x \in (-\infty, \infty)} \frac{|m_0(x)|}{\rho_0(x)}\) in Theorem 2 is necessary to ensure that the invariant region \(w \leq M, z \geq c_0 > 0\) is bounded when \(P(\rho) \leq 0\) or \(\gamma < 0\) for the prototype function. In fact, under this condition, the curve \(z = \frac{m}{\rho} - P(\rho) = c_0\) or \(m = c_0 \rho + \rho P(\rho)\) and the curve \(w = \frac{m}{\rho} = \sup_{x \in (-\infty, \infty)} \frac{|m_0(x)|}{\rho_0(x)} = M\) (or \(m = M \rho\)) always form a bounded region in the \((\rho, m)\) plane.
Note 2. The conditions in (1.21) are satisfied for \( P(\rho) = -\frac{1}{\gamma} \rho^\gamma + A \) with \( \gamma > 0 \).

We will prove Theorems 1–3 in the next several sections respectively. The main technique is the compensated compactness method coupled with the artificial viscosity.

When we consider the viscosity solutions of system (1.8) given by system (2.1) (or system (1.1) given by system (4.1)), a technical difficulty is to obtain the positive, lower estimate of \( \rho^\varepsilon \) since system (1.8) is singular when \( \rho = 0 \). For the isentropic gas dynamics system, DiPerna [5] first used the energy method to obtain an implicit bound \( \rho^\varepsilon \geq c(\varepsilon, t) > 0 \) for some function \( c(\varepsilon, t) \). In [15], the author obtained the upper bound estimate \( (m^2 \rho)_{xx} \leq P'(\rho^\varepsilon) \frac{\varepsilon}{\gamma} \) by using the maximum principle, and then derived an explicitly, positive, lower estimate of \( \rho^\varepsilon \). A simple proof of positive, lower bound of \( \rho^\varepsilon \) was obtained by Bereux and Sainsaulieu by using the Green function (cf. Theorem 1.0.2 in [16]). Their proof is valid for the conditions on \( P(\rho) \) given in (1.18) or (1.21) since \( P(\rho) \) is bounded in any bounded interval \( \rho \in [0, M] \). So, Theorem 1 and Theorem 3 can be proved by introducing the classical viscosity.

However, for the prototype function \( P(\rho) = \frac{1}{\gamma} \rho^\gamma + A \) with \( \gamma \in (-1, 0) \) or for the more general conditions in (1.20), all the above three methods to prove the positive, lower estimate \( \rho^\varepsilon \geq c(\varepsilon, t) > 0 \) are invalid. To overcome this difficulty, in Section 3, we construct a sequence of regular hyperbolic systems (3.1) to approximate system (1.8) by adding a small perturbation to the flux functions in system (1.8). First, we have an estimate \( \rho^\varepsilon, \delta \geq \delta > 0 \). Then repeating the process given in Section 2, we give the proof of Theorem 2. This technique was first introduced by the author in [17] to study the isentropic gas dynamics system for general pressure function. In this paper, we obtain a new application of it on system (1.8) and system (1.1).

2. Proof of Theorem 1

In this section, we prove Theorem 1.

Consider the Cauchy problem for the related parabolic system

\[
\begin{align*}
\rho_t + (m - \rho P(\rho))_x &= \varepsilon \rho_{xx}, \\
m_t + \left( \frac{m^2}{\rho} - m P(\rho) \right)_x &= \varepsilon m_{xx},
\end{align*}
\]

(2.1)

with initial data

\[
(\rho^\varepsilon(x, 0), w^\varepsilon(x, 0)) = (\rho_0(x) + \varepsilon, w_0(x)),
\]

where \((\rho_0(x), w_0(x))\) is given by (1.2).

We multiply (2.1) by \((w_\rho, w_m)\) and \((z_\rho, z_m)\) respectively, where \((w, z)\) is given by (1.11), to obtain

\[
w_t + \lambda_2 w_x = \varepsilon w_{xx} - \varepsilon \left( w_{\rho\rho} \rho_x^2 + 2w_{\rho m} \rho_x m_x + w_{mm} m_x^2 \right)
\]

\[
= \varepsilon w_{xx} - \varepsilon \left( \frac{2m}{\rho^3} \rho_x^2 - \frac{2}{\rho^2} \rho_x m_x \right)
\]

\[
= \varepsilon w_{xx} + \frac{2\varepsilon}{\rho} \rho_x w_x
\]

(2.2)
and
\[ z_t + \lambda_1 z_x = \varepsilon z_{xx} - \varepsilon \left( z_{\rho\rho} \rho_x^2 + 2z_{\rho m\rho} m_x + z_{mm} m_x^2 \right) \]
\[ = \varepsilon z_{xx} - \varepsilon \left( \frac{2m}{\rho^3} - P''(\rho) \right) \rho_x^2 - \frac{2}{\rho^2} \rho_x m_x \]
\[ = \varepsilon z_{xx} + \frac{2\varepsilon}{\rho} \rho_x z_x + \varepsilon \frac{2}{\rho} \left( 2P'(\rho) + \rho P''(\rho) \right) \rho_x^2 \]
(2.3)
or
\[ z_t + \lambda_1 z_x \geq \varepsilon z_{xx} + \frac{2\varepsilon}{\rho} \rho_x z_x. \]
(2.4)
If we consider (2.2) as an equality about the variable \( w \) and (2.4) as an inequality about \( z \), then we can get the estimates \( w(\rho^e, m^e) \leq C_1, z(\rho^e, m^e) \geq C_2 \) by applying the maximum principle to (2.2) and (2.4). Then, using the first equation in (2.1), we get \( \rho^e \geq 0 \). Therefore, the region
\[ R = \{ (\rho, m): w(\rho, m) \leq C_1, z(\rho, m) \geq C_2, \rho \geq 0 \} \]
is a bounded invariant region for two suitable constants \( C_1, C_2 \). Thus we obtain the following estimates
\[ 0 \leq \rho^e \leq M, \quad |w^e| = \left| \frac{m^e}{\rho^e} \right| \leq M \]
(2.5)
for a suitable positive constant \( M \), which depends only on the initial date, but is independent of \( \varepsilon \). Furthermore, using an argument by Bereux and Sainsaulieu (cf. Theorem 1.0.2 in [16] or [21]), we have the following positive, lower bound estimate on \( \rho^e \)
\[ \rho^e \geq c(t, \varepsilon) > 0, \quad \text{since} \quad \rho_0^e(x) \geq \varepsilon > 0, \]
(2.6)
where \( c(t, \varepsilon) \) could tend to zero as the time \( t \) tends to infinity or \( \varepsilon \) tends to zero.

Let \( \theta = w_x \). Differentiating Eq. (2.2) with respect to \( x \) and then multiplying the sequence of smooth functions \( g'(\theta, \alpha) \) to the result, we have
\[ \theta_t + (\lambda_2 \theta)_x = \varepsilon \theta_{xx} + \left( 2\varepsilon \rho^{-1} \rho_x \theta \right)_x \]
and
\[ g(\theta, \alpha)_t + \left( \lambda_2 g(\theta, \alpha) \right)_x + \left( g'(\theta, \alpha) \theta - g(\theta, \alpha) \right) \lambda_2 x \]
\[ = \varepsilon g(\theta, \alpha)_{xx} - \varepsilon g''(\theta, \alpha) \theta^2_x + \left( 2\varepsilon \rho^{-1} \rho_x g(\theta, \alpha) \right)_x \]
\[ + \left( 2\varepsilon \rho^{-1} \rho_x \right)_x \left( g'(\theta, \alpha) \theta - g(\theta, \alpha) \right). \]
(2.7)
Choosing \( g(\theta, \alpha) \) such that \( g''(\theta, \alpha) \geq 0, g'(\theta, \alpha) \rightarrow \text{sign} \theta, g(\theta, \alpha) \rightarrow |\theta| \) as \( \alpha \rightarrow 0 \), then we let \( \alpha \rightarrow 0 \) in (2.7) to get
\[ |\theta|_t + (\lambda_2 |\theta|)_x \leq \varepsilon |\theta|_{xx} + (2\varepsilon \rho^{-1} \rho_x |\theta|)_x \]  
(2.8)
in the sense of distributions. Integrating (2.8) in \( R \times [0, t] \), we have
\[ \int_{-\infty}^{\infty} |w_x|(x, t) \, dx = \int_{-\infty}^{\infty} |\theta|(x, t) \, dx \leq \int_{-\infty}^{\infty} |\theta|(x, 0) \, dx \leq M \]  
(2.9)
since \( TVw_0(x) \) is bounded.

**Lemma 4.**
\[ g(\rho \varepsilon)_t + \left( \int_0^{\rho \varepsilon} g'(s)f'(s) \, ds + g(\rho \varepsilon)w^\varepsilon \right)_x, \]  
(2.10)
\[ (\rho \varepsilon w^\varepsilon)_t + (\rho \varepsilon w^\varepsilon (w^\varepsilon - P(\rho \varepsilon)))_x, \]  
(2.11)
are compact in \( H^{-1}_{loc}(R \times R^+) \), where \( f(\rho) = -\rho P'(\rho) \) and \( g(\rho) \) is an arbitrary smooth function. Particularly, if \( g(\rho) = \rho \),
\[ \rho \varepsilon_t + (\rho \varepsilon w - \rho \varepsilon P(\rho \varepsilon))_x \]  
are compact in \( H^{-1}_{loc}(R \times R^+) \).

**Proof of Lemma 4.** We multiply the first equation in (2.1) by \( g'(\rho) \), to obtain
\[ g(\rho)_t - g'(\rho)(\rho P(\rho) - \rho w)_x = \varepsilon g(\rho)_{xx} - \varepsilon g''(\rho)\rho_x^2 \]  
(2.13)
or
\[ g(\rho)_t + \left( \int_0^{\rho} g'(s)f'(s) \, ds + g(\rho)w \right)_x \\
= \varepsilon g(\rho)_{xx} - \varepsilon g''(\rho)\rho_x^2 + (g(\rho) - \rho g'(\rho))w_x. \]  
(2.14)
Since the estimate in (2.9), the last term on the right-hand side of (2.14) is bounded in \( L^1_{loc}(R \times R^+) \), so we may choose a strictly convex function \( g(\rho) \) to obtain that
\[ \varepsilon(\rho_x^2)^2 \]  
are bounded in \( L^1_{loc}(R \times R^+) \)
(2.15)
with the help of (2.14), and then to use (2.14) again to get the proof of (2.10) for any smooth function \( g(\rho) \) by using Murat’s Theorem [19,24].

To prove (2.11), we multiply (2.1) by \( (\eta^*(\rho, m)_\rho, \eta^*(\rho, m)_m) \), where \( \eta^*(\rho, m) = \rho F'(\frac{m}{\rho}) \) is an entropy of (1.6) (cf. [1]), then
\[ \eta^*(\rho, m)_x + q^*(\rho, m)_x = \varepsilon \eta^*(\rho, m)_{xx} - \varepsilon F''(w) \left( \frac{m^2}{\rho^3} \rho_x^2 - \frac{2m}{\rho^2} \rho_x \rho_x + \frac{1}{\rho} m^2 \right) \]
\[ = \varepsilon \eta^*(\rho, m)_{xx} - \frac{F''(w)}{\rho} \left( m_x - \frac{m}{\rho} \rho_x \right)^2 \]
\[ = \varepsilon \eta^*(\rho, m)_{xx} - \varepsilon F''(w) \rho w_x^2 \quad (2.16) \]

where

\[ q^*(\rho, m) = \left( m - \rho P(\rho) \right) F \left( \frac{m}{\rho} \right) \]

is the entropy flux corresponding to \( \eta^*(\rho, m) \).

Thus we may choose a strictly convex function \( F(w) \) in (2.16) to obtain that

\[ \varepsilon \rho^\varepsilon \left( w_x^\varepsilon \right)^2 \text{ are bounded in } L^1_{\text{loc}}(R \times R^+) \quad (2.17) \]

Then we may rewrite the right-hand side of the second equation in (2.1) as \( \varepsilon (\rho w_x + \rho \rho_x)_x \), which is clearly compact in \( H^{-1}_{\text{loc}}(R \times R^+) \) since the estimates in (2.15) and (2.17). So we get the proof of (2.11).

Using the Curl–Div Theorem in the compensated compactness theory [16] to the function pairs given in (2.10) and (2.12), we have for \( g(\rho) = f(\rho) = -\rho P(\rho) \)

\[ \rho^\varepsilon \int_k f'^2(s) \, ds - f^2(\rho^\varepsilon) = \bar{\rho}^\varepsilon \int_k f'^2(s) \, ds - \left( f(\rho^\varepsilon) \right)^2 + \rho^\varepsilon f(\rho^\varepsilon) w^\varepsilon - f(\rho^\varepsilon) \bar{\rho}^\varepsilon w^\varepsilon, \]

where \( k \) is a constant and \( \bar{\eta}(u^\varepsilon) \) denotes the weak-star limit of \( \eta(u^\varepsilon) \).

Let \( \bar{\rho}^\varepsilon = \rho \). Then by simple calculations, we have from (2.18) that

\[ (\rho^\varepsilon - \rho) \int_\rho^{\rho^\varepsilon} f'^2(s) \, ds - \left( f(\rho^\varepsilon) - f(\rho) \right)^2 + \left( f(\rho^\varepsilon) - f(\rho) \right)^2 = \bar{\rho}^\varepsilon f(\rho^\varepsilon) w^\varepsilon - f(\rho^\varepsilon) \bar{\rho}^\varepsilon w^\varepsilon. \]

(2.19)

Using the Curl–Div Theorem again to the function pairs in (2.11)–(2.12), we have

\[ \bar{\rho}^\varepsilon \rho^\varepsilon \left( w^\varepsilon \right)^2 + f(\rho^\varepsilon) w^\varepsilon - \bar{\rho}^\varepsilon w^\varepsilon \bar{\rho}^\varepsilon w^\varepsilon + f(\rho^\varepsilon) = 0 \quad (2.20) \]

or

\[ \bar{\rho}^\varepsilon f(\rho^\varepsilon) w^\varepsilon - f(\rho^\varepsilon) \bar{\rho}^\varepsilon w^\varepsilon = \left( \bar{\rho}^\varepsilon w^\varepsilon \right)^2 - \bar{\rho}^\varepsilon \bar{\rho}^\varepsilon \left( w^\varepsilon \right)^2. \]

(2.21)
Combining (2.19) and (2.21), we have

\[
(\rho^\varepsilon - \rho) \int_\rho f'(s) \, ds - (f(\rho^\varepsilon) - f(\rho))^2 + \left( \frac{f(\rho^\varepsilon) - f(\rho)}{\rho^\varepsilon} \right)^2 = (\rho^\varepsilon w^\varepsilon)^2 - \rho^\varepsilon \rho^\varepsilon (w^\varepsilon)^2.
\]

(2.22)

Since

\[
(f(\rho^\varepsilon) - f(\rho))^2 = \left( \int_\rho f'(s) \, ds \right)^2 \leq (\rho^\varepsilon - \rho) \int_\rho (f'(s))^2 \, ds
\]

and

\[
\left( \int_\Omega \rho^\varepsilon w^\varepsilon \, dx \, dt \right)^2 \leq \int_\Omega \rho^\varepsilon \, dx \, dt \int_\Omega \rho^\varepsilon (w^\varepsilon)^2 \, dx \, dt,
\]

then the left-hand side of (2.22) is nonnegative and the right-hand side is nonpositive, and so that both sides of (2.22) must be zero. From

\[
(\rho^\varepsilon - \rho) \int_\rho f'(s) \, ds - (f(\rho^\varepsilon) - f(\rho))^2 + \left( \frac{f(\rho^\varepsilon) - f(\rho)}{\rho^\varepsilon} \right)^2 = 0,
\]

(2.23)

we get the pointwise convergence of \( \rho^\varepsilon \); from

\[
(\rho^\varepsilon w^\varepsilon)^2 - \rho^\varepsilon \rho^\varepsilon (w^\varepsilon)^2 = 0
\]

(2.24)

we get the pointwise convergence of \( w^\varepsilon \) in the region of \( \rho > 0 \). Therefore, we get the proof of Theorem 1.

3. Proof of Theorem 2

In this section, we prove Theorem 2.

Consider the following approximated hyperbolic systems of system (1.8)

\[
\begin{aligned}
\rho_t + \left( \frac{\rho - \delta}{\rho} m - (\rho - \delta) P(\rho) \right)_x &= 0, \\
m_t + \left( \frac{(\rho - \delta)m^2}{\rho^2} - \frac{\rho - \delta}{\rho} m P(\rho) \right)_x &= 0,
\end{aligned}
\]

(3.1)

where \( \delta \) is a positive, small perturbation constant. When \( \delta = 0 \), system (3.1) is reduced to the original system (1.8).
The Jacobian matrix of the flux functions in (3.1) is
\[
\begin{pmatrix}
\frac{\delta}{\rho^2} m - P(\rho) - (\rho - \delta) P'(\rho) \\
\left( -\frac{1}{\rho^2} + \frac{2\delta}{\rho^3} \right) m^2 - \frac{\delta}{\rho^2} P(\rho) m - \frac{\rho - \delta}{\rho} P'(\rho) m
\end{pmatrix},
\]
and its characteristic equation is
\[
\lambda^2 - \left[ \left( \frac{2}{\rho} - \frac{\delta}{\rho^2} \right) m - (\rho - \delta) P'(\rho) - \frac{2\rho - \delta}{\rho} P(\rho) \right] \lambda
+ \left( \frac{\delta}{\rho^2} m - P(\rho) - (\rho - \delta) P'(\rho) \right) \left( \frac{2(\rho - \delta)}{\rho^2} m - \frac{\rho - \delta}{\rho} P(\rho) \right)
- \frac{\rho - \delta}{\rho} \left( \left( -\frac{1}{\rho^2} + \frac{2\delta}{\rho^3} \right) m^2 - \frac{\delta}{\rho^2} P(\rho) m - \frac{\rho - \delta}{\rho} P'(\rho) m \right) = 0
\] (3.2)
or
\[
\lambda^2 - \left[ \left( \frac{2}{\rho} - \frac{\delta}{\rho^2} \right) m - (\rho - \delta) P'(\rho) - \frac{2\rho - \delta}{\rho} P(\rho) \right] \lambda
+ \left( \frac{\rho - \delta}{\rho} P'(\rho) m + \frac{\rho - \delta}{\rho} \left( P(\rho) + (\rho - \delta) P'(\rho) \right) P(\rho) \right) = 0.
\] (3.3)

Since
\[
\left[ \left( \frac{2}{\rho} - \frac{\delta}{\rho^2} \right) m - (\rho - \delta) P'(\rho) - \frac{2\rho - \delta}{\rho} P(\rho) \right]^2
- 4 \left[ \frac{\rho - \delta}{\rho^3} m^2 - 2\frac{\rho - \delta}{\rho^2} P(\rho) m \right.
\left. - \left( \frac{\rho - \delta}{\rho} \right)^2 P'(\rho) m + \frac{\rho - \delta}{\rho} \left( P(\rho) + (\rho - \delta) P'(\rho) \right) P(\rho) \right]
= \frac{4\rho^2 - 4\delta\rho + \delta^2}{\rho^4} m^2 + (\rho - \delta)^2 P'(\rho) + \left( \frac{2\rho - \delta}{\rho} \right)^2 P^2(\rho) - \frac{4\rho - 2\delta}{\rho^2} (\rho - \delta) P'(\rho) m
- \frac{4(\rho - \delta)(\rho - \delta)}{\rho^3} P'(\rho) P(\rho) - 4\frac{\rho - \delta}{\rho^3} m^2 + 8\frac{\rho - \delta}{\rho^2} P(\rho) m
\left. + 4 \left( \frac{\rho - \delta}{\rho} \right)^2 P'(\rho) m - 4 \left( P(\rho) + (\rho - \delta) P'(\rho) \right) \frac{\rho - \delta}{\rho} P(\rho) \right]
= \frac{\delta^2}{\rho^3} m^2 - \frac{2\delta^2}{\rho^3} P(\rho) m + \frac{2\delta^2 - 2\delta\rho}{\rho^2} P'(\rho) m
\left. + \frac{\delta^2}{\rho^2} P^2(\rho) - \frac{2\delta^2 - 2\delta\rho}{\rho} P'(\rho) P(\rho) + (\rho - \delta)^2 P'(\rho) \right]
= \frac{\delta^2}{\rho^2} \left( \frac{m}{\rho} - P(\rho) \right)^2 + \frac{2\delta^2 - 2\delta\rho}{\rho} P'(\rho) \left( \frac{m}{\rho} - P(\rho) \right) + (\rho - \delta)^2 P'(\rho) \right)
= \left( \frac{\delta}{\rho} \left( \frac{m}{\rho} - P(\rho) \right) - (\rho - \delta) P'(\rho) \right)^2.
\] (3.4)
then two eigenvalues of system (3.1) are
\[ \lambda_1^\delta = \frac{m}{\rho} - P(\rho) - (\rho - \delta)P'(\rho), \quad \lambda_2^\delta = \frac{\rho - \delta}{\rho} \left( \frac{m}{\rho} - P(\rho) \right) \] (3.5)
and corresponding Riemann invariants \((z^\delta, w^\delta)\) are
\[ z^\delta = \lambda_2^\delta = \frac{\rho - \delta}{\rho} \left( \frac{m}{\rho} - P(\rho) \right) = \frac{\rho - \delta}{\rho} z, \quad w^\delta = w = \frac{m}{\rho}. \] (3.6)

By simple calculations,
\[ z^\rho = -\frac{\rho - 2\delta}{\rho^3} m - \frac{\rho - \delta}{\rho} P'(\rho) - \frac{\delta}{\rho^2} P(\rho), \quad z^m = \frac{\rho - \delta}{\rho^2}, \quad z_{mm} = 0, \] (3.7)
\[ z^\rho = \frac{2\rho - 6\delta}{\rho^4} m - \frac{\rho - \delta}{\rho} P''(\rho) - \frac{2\delta}{\rho^2} P'(\rho) + \frac{2\delta}{\rho^3} P(\rho), \quad z^m = -\frac{\rho - 2\delta}{\rho^3} \] (3.8)
and
\[ z^x = \left( -\frac{\rho - 2\delta}{\rho^3} m - \frac{\rho - \delta}{\rho} P'(\rho) - \frac{\delta}{\rho^2} P(\rho) \right) \rho_x + \frac{\rho - \delta}{\rho^2} m_x. \] (3.9)

Now consider the Cauchy problem for the related parabolic system
\[
\begin{align*}
\rho_t + \left( \frac{\rho - \delta}{\rho} m - (\rho - \delta)P(\rho) \right)_x &= \varepsilon w_{xx}, \\
m_t + \left( (\rho - \delta)m^2 - \frac{\rho - \delta}{\rho} mP(\rho) \right)_x &= \varepsilon m_{xx},
\end{align*}
\] (3.10)
with the initial data (1.2). Since \(\rho_0(x) \geq \rho_0 > 0, z_0(x) \geq c_0 > 0\) as given in Theorem 2, then \(z_0^\delta(x) \geq \frac{1}{2} c_0 > 0\) for small \(\delta\).

We multiply (3.10) by \((w^\rho, w^m)\) and \((Z^\rho, Z^m)\) respectively, where \(Z = z^\delta\), to obtain
\[ w_t + \lambda_2^\delta w_x = \varepsilon w_{xx} - \varepsilon \left( w^\rho \rho_x^2 + 2w^m \rho_x m_x + w^m m_x^2 \right) \]
\[ = \varepsilon w_{xx} - \varepsilon \left( \frac{2m}{\rho^3} \rho_x^2 - \frac{2}{\rho^2} \rho_x m_x \right) \]
\[ = \varepsilon w_{xx} + \frac{2\varepsilon}{\rho} \rho_x w_x \] (3.11)
and
\[ Z_t + \lambda_1^\delta Z_x = \varepsilon Z_{xx} - \varepsilon \left( Z^\rho \rho_x^2 + 2Z^m \rho_x m_x + Z^m m_x^2 \right) \]
\[ = \varepsilon Z_{xx} - \varepsilon \left( \left( \frac{2}{\rho^3} - \frac{6\delta}{\rho^4} \right) m - \frac{\rho - \delta}{\rho} P''(\rho) - \frac{2\delta}{\rho^2} P'(\rho) + \frac{2\delta}{\rho^3} P(\rho) \right) \rho_x^2 \]
\[ - \frac{2(\rho - 2\delta)}{\rho^3} \rho_x m_x \] (3.12)
By simple calculations,

\[
\begin{align*}
\left( \frac{2}{\rho^3} - \frac{6\delta}{\rho^3} \right) m - \frac{\rho - \delta}{\rho} P''(\rho) - \frac{2\delta}{\rho^2} P'(\rho) + \frac{2\delta}{\rho^3} P(\rho) \right) \rho_x^2 - \frac{2(\rho - 2\delta)}{\rho^3} \rho_x m_x \\
= -\frac{2}{\rho} \left( \left( \frac{\rho - 2\delta}{\rho^3} m - \frac{\rho - \delta}{\rho} P'(\rho) - \frac{\delta}{\rho^2} P(\rho) \right) \rho_x + \frac{\rho - \delta}{\rho^2} m_x \right) \rho_x \\
+ \frac{2\delta}{(\rho - \delta)\rho} \left( \left( \frac{\rho - 2\delta}{\rho^3} m - \frac{\rho - \delta}{\rho} P'(\rho) - \frac{\delta}{\rho^2} P(\rho) \right) \rho_x + \frac{\rho - \delta}{\rho^2} m_x \right) \rho_x \\
+ \left( \frac{2\delta^2}{(\rho - \delta)\rho^3} m - \frac{\rho - \delta}{\rho} P''(\rho) - \frac{2\rho - \delta}{\rho^2} P'(\rho) + \frac{2\delta^2}{(\rho - \delta)\rho^3} P(\rho) \right) \rho_x^2 \\
= \left( -\frac{2}{\rho} + \frac{2\delta}{(\rho - \delta)\rho} \right) Z_x \rho_x + \left( -\frac{2\delta^2}{(\rho - \delta)^2}\rho_x \right) Z - \frac{\rho - \delta}{\rho^2} \left( \rho P''(\rho) + 2P'(\rho) \right) \rho_x^2.
\end{align*}
\]

So we obtain from (3.13) and (3.12) that

\[
Z_t + \lambda_1^2 Z_x + \varepsilon \left( -\frac{2}{\rho} + \frac{2\delta}{(\rho - \delta)\rho} \right) Z_x \rho_x - \varepsilon \frac{2\delta^2}{(\rho - \delta)^2}\rho_x Z \\
= \varepsilon Z_{xx} + \varepsilon \frac{\rho - \delta}{\rho^2} \left( \rho P''(\rho) + 2P'(\rho) \right) \rho_x^2 \geq \varepsilon Z_{xx}.
\]

(3.14)

If we consider (3.11) as an equality about the variable \(w\) and (3.14) as an inequality about \(Z\), then again we can get the estimates \(w(\rho, m, \varepsilon, \delta) \leq C_1, Z(\rho, m, \varepsilon, \delta) \geq \frac{1}{2} \varepsilon_0 > 0\) by applying the maximum principle to (3.11) and (3.14). Then

\[
z(\rho, m, \varepsilon, \delta) = \frac{\rho \varepsilon \delta}{\rho - \delta} Z(\rho, m, \varepsilon, \delta) \geq \frac{1}{2} \varepsilon_0 > 0.
\]

Using the first equation in (3.10), we get \(\rho \varepsilon \delta \geq \delta\). Therefore, the region

\[
R = \{ (\rho, m) : w(\rho, m) \leq C_1, z(\rho, m) \geq \frac{1}{2} \varepsilon_0, \rho \geq \delta \}
\]

is a bounded invariant region for a suitable constant \(C_1\).

Furthermore, applying Theorem 1.0.2 in [16] again to the first equation in (3.10), we have the following positive, lower bound estimate on \(\rho \varepsilon \delta\)

\[
\rho \varepsilon \delta \geq c(t, \varepsilon, \delta) > \delta > 0, \quad \text{since} \quad \rho_0 \varepsilon \delta(x) \geq 2\delta,
\]

(3.15)

where \(c(t, \varepsilon, \delta)\) could tend to \(\delta\) as the time \(t\) tends to infinity or \(\varepsilon\) tends to zero.

Thus we obtain the following estimates

\[
0 < \delta < c(t, \varepsilon, \delta) \leq \rho \varepsilon \delta \leq M, \quad |w \varepsilon \delta| = \left| \frac{m \varepsilon \delta}{\rho \varepsilon \delta} \right| \leq M
\]

(3.16)
for a suitable positive constant $M$, which depends only on the initial date, but is independent of $\varepsilon, \delta$.

Using the same technique as given in Section 2, we have the estimate in (2.9)

$$
\int_{-\infty}^{\infty} |w_{x}^{\varepsilon, \delta}(x, t)| \, dx \leq \int_{-\infty}^{\infty} \left| \frac{d}{dx} w_{0}(x) \right| \, dx \leq M
$$

(3.17)

and the following lemma.

**Lemma 5.**

$$
g\left(\rho_{\varepsilon, \delta}\right)_{t} + \left(\int g'(s)f'(s) \, ds + g\left(\rho_{\varepsilon, \delta}\right)w_{\varepsilon, \delta}\right)_{x}
$$

(3.18)

and

$$
\left( g\left(\rho_{\varepsilon, \delta}\right)w_{\varepsilon, \delta}\right)_{t} + \left( g\left(\rho_{\varepsilon, \delta}\right)\left(w_{\varepsilon, \delta}\right)^{2} + \int g'(s)f'(s) \, ds \, w_{\varepsilon, \delta}\right)_{x}
$$

(3.19)

are compact in $H_{\text{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^{+})$ as $\varepsilon, \delta$ go to zero, where $f(\rho) = -\rho P(\rho)$ and $g(\rho) \geq 0$ is a strictly increasing, smooth function satisfying

$$
\lim_{\rho \to 0} g(\rho)P(\rho) = 0
$$

(3.20)

such that $\int^{\rho} g'(s)f'(s) \, ds$ is a regular function at $\rho = 0$.

**Note.** For the prototype function $P(\rho) = \frac{1}{\gamma}\rho^{\gamma} + A$ with $\gamma \in (-1, 0)$, we choose $g(\rho) = \rho^\alpha$ with $\alpha > -\gamma$ or for the convenience in the proof of Lemma 4, $\alpha > 2$.

**Proof of Lemma 5.** We multiply the first equation in (3.10) by $g'(\rho)$, to obtain

$$
g(\rho)_{t} - g'(\rho)(\rho P(\rho) - \rho w)_{x} - \delta g'(\rho)w_{x} + \delta g'(\rho)P(\rho)_{x}
$$

$$
= \varepsilon g(\rho)_{xx} - \varepsilon g''(\rho)\rho_{x}^{2}
$$

(3.21)

or

$$
g(\rho)_{t} + \left(\int^{\rho} g'(s)f'(s) \, ds + g(\rho)w\right)_{x}
$$

$$
= \varepsilon g(\rho)_{xx} - \varepsilon g''(\rho)\rho_{x}^{2} - \delta \left(\int^{\rho} g'(s)P'(s) \, ds\right)_{x} + (g(\rho) - \rho g'(\rho) + \delta g'(\rho))w_{x}.
$$

(3.22)
Due to the estimate in (3.17), the last term on the right-hand side of (3.22) is bounded in $L^1_{loc}(R \times R^+)$, so we may choose a strictly convex function $g(\rho)$ to obtain that

$$\varepsilon(\rho^{\varepsilon,\delta})^2$$

are bounded in $L^1_{loc}(R \times R^+)$ (3.23) and then with the help of (3.22) to get the $H^{-1}_{loc}$ compactness of (3.18) as $\varepsilon, \delta$ go to zero.

To prove the compactness of (3.19), we first use the same technique given in Section 2 to obtain that

$$\varepsilon\rho^{\varepsilon,\delta}(w^{\varepsilon,\delta}_x)^2$$

are bounded in $L^1_{loc}(R \times R^+)$ (3.24).

Multiplying (3.11) by $g(\rho)$, and (3.22) by $w$, then adding the result, we have

$$
(g(\rho)w)_t + \left( \int g'(s)f'(s)ds w + g(\rho)w^2 \right)_x
$$

$$
= \varepsilon(g(\rho)w)_{xx} - \delta \left( \int g'(s)P'(s)ds w \right)_x - \varepsilon g''(\rho)\rho \rho_x^2 + \delta \left( \int g'(s)P'(s)ds \right) w_x
$$

$$
+ \left( g(\rho) - \rho g'(\rho) + \delta g'(\rho) \right) w w_x + \left( \int g'(s)f'(s)ds + g(\rho)w \right) w_x
$$

$$
- \frac{\rho - \delta}{\rho} \left( \frac{m}{\rho} - P(\rho) \right) g(\rho)w_x + \frac{2\varepsilon}{\rho} \left( g(\rho) - \rho g'(\rho) \right) \rho_x w_x.
$$

(3.25)

All the terms on the right-hand side of (3.25) are bounded in $L^1_{loc}(R \times R^+)$ except the first two terms, which are compact in $H^{-1}_{loc}(R \times R^+)$ as $\varepsilon, \delta$ go to zero since the estimates given in (3.17), (3.20), (3.23) and (3.24). The left-hand side of (3.25) is clearly bounded in $W^{-1,\infty}_{loc}(R \times R^+)$. So, Murat’s Theorem [19,24] gives the compactness of (3.19). \(\square\)

To complete the proof of Theorem 2, we choose two pairs of functions given by (3.18)

$$
\left( g(\rho^{\varepsilon,\delta}), \int_{\rho^{\varepsilon,\delta}} g'(s)f'(s)ds + g(\rho^{\varepsilon,\delta})w^{\varepsilon,\delta}_x \right)
$$

(3.26)

and

$$
\left( \int_{\rho^{\varepsilon,\delta}} g'(s)f'(s)ds, \int_{\rho^{\varepsilon,\delta}} g'(s)f''(s)ds + \int_{\rho^{\varepsilon,\delta}} g'(s)f'(s)ds w^{\varepsilon,\delta}_x \right).
$$

(3.27)

Let $g(\rho^{\varepsilon,\delta}) = v^{\varepsilon,\delta}$, $\int_{\rho^{\varepsilon,\delta}} g'(s)f'(s)ds = F(v^{\varepsilon,\delta})$, then clearly $\int_{\rho^{\varepsilon,\delta}} g'(s)f''(s)ds = \int_{v^{\varepsilon,\delta}} F''(s)ds$. Using the Curl–Div Theorem to (3.26) and (3.27), we have
\[ v^\varepsilon,\delta \int_k F'(s) \, ds - F^2(v^\varepsilon,\delta) = v^\varepsilon,\delta \int_k F'(s) \, ds - (F(v^\varepsilon,\delta))^2 + g(\rho^\varepsilon,\delta) \int g'(s) f'(s) \, ds w^\varepsilon,\delta - \int g'(s) f'(s) \, ds g(\rho^\varepsilon,\delta) w^\varepsilon,\delta, \quad (3.28) \]

where \( k \) is a constant. Let \( v^\varepsilon,\delta = v \). By simple calculations, we have from (3.28) that

\[ (v^\varepsilon,\delta - v) \int F'(s) \, ds - (F(v^\varepsilon,\delta) - F(v))^2 + (F(v^\varepsilon,\delta) - F(v))^2 \]
\[ = g(\rho^\varepsilon,\delta) \int g'(s) f'(s) \, ds w^\varepsilon,\delta - \int g'(s) f'(s) \, ds g(\rho^\varepsilon,\delta) w^\varepsilon,\delta. \quad (3.29) \]

Using the Curl–Div Theorem again to the pairs of functions given in (3.18) and (3.19), we have

\[ g(\rho^\varepsilon,\delta) \int g'(s) f'(s) \, ds w^\varepsilon,\delta - \int g'(s) f'(s) \, ds g(\rho^\varepsilon,\delta) w^\varepsilon,\delta \]
\[ = (g(\rho^\varepsilon,\delta) w^\varepsilon,\delta)^2 - g(\rho^\varepsilon,\delta) \cdot g(\rho^\varepsilon,\delta)(w^\varepsilon,\delta)^2. \quad (3.30) \]

Combining (3.29) and (3.30), we have

\[ (v^\varepsilon,\delta - v) \int F'(s) \, ds - (F(v^\varepsilon,\delta) - F(v))^2 + (F(v^\varepsilon,\delta) - F(v))^2 \]
\[ = (g(\rho^\varepsilon,\delta) w^\varepsilon,\delta)^2 - g(\rho^\varepsilon,\delta) \cdot g(\rho^\varepsilon,\delta)(w^\varepsilon,\delta)^2. \quad (3.31) \]

Since the left-hand side of (3.31) is nonnegative, and the right-hand side is nonpositive, we know that both sides of (3.31) must be zero. From

\[ (v^\varepsilon,\delta - v) \int F'(s) \, ds - (F(v^\varepsilon,\delta) - F(v))^2 + (F(v^\varepsilon,\delta) - F(v))^2 = 0 \quad (3.32) \]

we get the pointwise convergence of \( v^\varepsilon,\delta \), and so the convergence of \( \rho^\varepsilon,\delta \) since \( g(\rho) \) is a strictly increasing function.

From

\[ (g(\rho^\varepsilon,\delta) w^\varepsilon,\delta)^2 - g(\rho^\varepsilon,\delta) \cdot g(\rho^\varepsilon,\delta)(w^\varepsilon,\delta)^2 = 0, \quad (3.33) \]
we have the pointwise convergence of $w^{\varepsilon, \delta}$ in the region of $\rho > 0$. Therefore, we get the proof of Theorem 2.

4. Proof of Theorem 3

In this section, we prove Theorem 3.
Consider the Cauchy problem for the related parabolic system

\[
\begin{align*}
\rho_t + \left( \rho \phi(\rho, w_1, w_2, \ldots, w_n) \right)_x &= \varepsilon \rho_{xx}, \\
(\rho w_i)_t + \left( \rho w_i \phi(\rho, w_1, w_2, \ldots, w_n) \right)_x &= \varepsilon (\rho w_i)_{xx}, \quad i = 1, 2, \ldots, n,
\end{align*}
\]  

(4.1)

with initial data

\[
\left( \rho^\varepsilon(x, 0), w^\varepsilon_i(x, 0) \right) = \left( \rho_0(x) + \varepsilon, w_{i0}(x) \right),
\]  

(4.2)

where $(\rho_0(x), w_{i0}(x))$ is given by (1.2).

Substituting the first equation in (4.1) into the second, we have

\[
w_{i\varepsilon} + \phi(\rho, w_1, w_2, \ldots, w_n)w_{ix} = \varepsilon w_{ixx} + 2\varepsilon \frac{\rho_x}{\rho} w_{ix},
\]  

(4.3)

which implies

\[
\Phi_t + \phi(\rho, w_1, w_2, \ldots, w_n)\Phi_x = \varepsilon \Phi_{xx} - \varepsilon \sum_{i=1}^n \Phi_{w_iw_i} w_{ix}^2 + 2\varepsilon \frac{\rho_x}{\rho} \Phi_x.
\]  

(4.4)

Using the first equation in (4.1), we have

\[
P(\rho)_t + \phi(\rho, w_1, w_2, \ldots, w_n)P(\rho)_x + \rho P'(\rho)\phi_x = \varepsilon P(\rho)_{xx} - \varepsilon P''(\rho)\rho_x^2.
\]  

(4.5)

(4.4) and (4.5) give us the following inequality

\[
\phi_t + (\phi - \rho P'(\rho))\phi_x = \varepsilon \phi_{xx} + 2\varepsilon \frac{\rho_x}{\rho} \phi_x - \varepsilon \sum_{i=1}^n \Phi_{w_iw_i} w_{ix}^2 + \frac{\varepsilon}{\rho} (2P'(\rho) + \rho P''(\rho))\rho_x^2 \leq \varepsilon \phi_{xx} + 2\varepsilon \frac{\rho_x}{\rho} \phi_x
\]  

(4.6)

due to the conditions in Theorem 3. Applying the maximum principle to (4.6), we have the upper bound of $\phi \leq M$ and so the upper bound of $\rho^\varepsilon$

\[
\rho^\varepsilon \leq M
\]  

(4.7)

due to the conditions in Theorem 3 again. Applying the maximum principle again to (4.3), we have the boundedness of $w^\varepsilon_i$

\[
|w^\varepsilon_i| \leq M,
\]  

(4.8)
where $M$ in (4.7) and (4.8) is a positive constant independent of $\varepsilon$. Furthermore, applying Theorem 1.0.2 in [16] again to the first equation in (4.1), we have the following positive, lower bound estimate on $\rho^\varepsilon$

$$\rho^\varepsilon \geq c(t, \varepsilon) > 0,$$

(4.9)

where $c(t, \varepsilon)$ could tend to 0 as the time $t$ tends to infinity or $\varepsilon$ tends to zero.

Similarly to the proof of (2.9), we have

$$\int_{-\infty}^{\infty} |w_{ix}(x, t)| dx \leq \int_{-\infty}^{\infty} |w_{ix}(x, 0)| dx \leq M$$

(4.10)

since $TV w_{i0}(x)$ is bounded.

To complete the proof of Theorem 3, we need the following

Lemma 6.

$$g(\rho^\varepsilon)_t + \left( \int g'(s)f'(s) ds + g(\rho^\varepsilon)\Phi(w^\varepsilon) \right)_x,$$

(4.11)

$$\left( \rho^\varepsilon \Phi(w^\varepsilon) \right)_t + \left( \rho^\varepsilon \Phi^2(w^\varepsilon) + f(\rho^\varepsilon)\Phi(w^\varepsilon) \right)_x,$$

(4.12)

are compact in $H^{-1}_{loc}(R \times R^+)$, where $f(\rho) = -\rho P(\rho)$ and $g(\rho)$ is an arbitrary smooth function. Particularly, if $g(\rho) = \rho$,

$$\rho^\varepsilon_t + \left( \rho^\varepsilon \Phi(w^\varepsilon) - \rho^\varepsilon P(\rho^\varepsilon) \right)_x$$

are compact in $H^{-1}_{loc}(R \times R^+)$. (4.13)

Proof of Lemma 6. The proof is very similar to that of Lemma 4. In fact, we multiply the first equation in (4.1) by $g'(\rho)$, to obtain

$$g(\rho)_t + \left( \int_{\rho}^{\rho^\varepsilon} g'(s)f'(s) ds + g(\rho)\Phi(w) \right)_x$$

$$= \varepsilon g(\rho)_{xx} - \varepsilon g''(\rho)\rho^2_x + (g(\rho) - \rho g'(\rho))\Phi(w)_x.$$

(4.14)

Since the estimate in (4.10), the last term on the right-hand side of (4.14) is bounded in $L^1_{loc}(R \times R^+)$, so we may choose a strictly convex function $g(\rho)$ to obtain that $\varepsilon(\rho^\varepsilon)_x$ are bounded in $L^1_{loc}(R \times R^+)$ with the help of (4.14), and then to use (4.14) again to get the proof of (4.11).

Second, we multiply (4.1) by $(\eta^*(\rho, m_i)_\rho, (\eta^*(\rho, m_i))_{m_i})$, where $m_i = \rho w_i$ and $\eta^*(\rho, m_i) = \rho F(m_{\rho})$, $F$ strictly convex, is an entropy of (1.1) with corresponding entropy flux $q^*(\rho, m_i) = \rho F(m_{\rho})(\Phi(w) - P(\rho))$, then we have that

$$\varepsilon \rho^\varepsilon (w^\varepsilon_{ix})^2, \ i = 1, 2, \ldots, n,$$

are bounded in $L^1_{loc}(R \times R^+)$ (4.15)

and obtain the proof of (4.12). \qed
Since (4.11) and (4.12) are very similar to (2.10) and (2.11), we can use the same way given in the proof of Theorem 1 to obtain Eqs. (2.23) and (2.24), and so the pointwise convergence of \( \rho^\varepsilon \) and \( \Phi(w^\varepsilon_1, w^\varepsilon_2, \ldots, w^\varepsilon_n) \).

To prove the pointwise convergence of \( w^\varepsilon_i \), we first have that both \( w^\varepsilon_{ix} \) and \( ((w^\varepsilon_i)^2)_x \) are compact in \( H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}^+) \) since they are bounded both in \( L^1_{loc}(\mathbb{R} \times \mathbb{R}^+) \) and in \( W^{-1,\infty}(\mathbb{R} \times \mathbb{R}^+) \). Thus we may apply the Curl–Div Theorem to the pairs of functions

\[
(\rho^\varepsilon, \rho^\varepsilon(\Phi(w^\varepsilon) - P(\rho^\varepsilon)) + w^\varepsilon_i)
\]

and

\[
(\rho^\varepsilon w^\varepsilon_i, \rho^\varepsilon w^\varepsilon_i(\Phi(w^\varepsilon) - P(\rho^\varepsilon)) + (w^\varepsilon_i)^2)
\]

to obtain

\[
\rho^\varepsilon((\rho^\varepsilon w^\varepsilon_i(\Phi(w^\varepsilon) - P(\rho^\varepsilon)) + (w^\varepsilon_i)^2) - (\rho^\varepsilon w^\varepsilon_i)(\rho^\varepsilon(\Phi(w^\varepsilon) - P(\rho^\varepsilon)) + w^\varepsilon_i) = 0.
\]

Since the strong convergence of \( \rho^\varepsilon \) and \( \Phi(w^\varepsilon) \), we have

\[
\rho((\overline{w^\varepsilon_i})^2 - (\overline{w^\varepsilon_i})^2) = 0
\]

which includes the pointwise convergence of \( w^\varepsilon_i \) on the region \( \rho > 0 \). Thus, we complete the proof of Theorem 3.

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