Reduced critical branching processes in random environment

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Abstract

Let \( Z(n), n = 0, 1, 2, \ldots \) be a critical branching process in random environment and \( Z(m, n) \), \( m \leq n \), the corresponding reduced process. We consider the case when the offspring generating functions are fractional linear and show that for any fixed \( m \) the conditional distribution of \( Z(m, n) \) given \( Z(n) > 0 \) converges to a non-trivial limit as \( n \to \infty \). We also prove the convergence of the conditional distribution of the process \( \{ n^{-1/2} \log Z([m], n), 0 \leq t \leq 1 \} \) given \( Z(n) > 0 \) to the law of a transformation of the Brownian meander. Some applications of the above results to random walks in random environment are indicated. © 1997 Elsevier Science B.V.

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1. Introduction and main results

Branching processes in random environment (BPRE's) form a class of well-motivated and relatively simple probabilistic models, of which the mathematical study proved to be very difficult. So far only a few facts related to the limiting behaviour of the processes have been established (for references see e.g. Section 10.2 in the survey paper Vatutin and Zubkov, 1993). The first BPRE model was suggested in Smith and Wilkinson (1969). In this discrete-time process, the offspring law of the \( n \)th generation particles is given by a common random distribution

\[
\pi_n = \pi_n(\omega) = (\pi_n^{(0)}, \pi_n^{(1)}, \pi_n^{(2)}, \ldots), \quad \pi_n^{(i)} \geq 0, \quad \sum_{i=0}^{\infty} \pi_n^{(i)} = 1,
\]

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the random elements \( \pi_n, n \geq 0 \), being independent and identically distributed. Denote by

\[
f_n(s) = \pi_n^{(0)} + \pi_n^{(1)} s + \pi_n^{(2)} s^2 + \cdots
\]

the (random) generating function of the law \( \pi_n \). According to the model, given the whole environment sequence \( \{\pi_n\}_{n \geq 0} \), the reproduction of particles occurs in the same way as in an ordinary non-homogeneous Galton–Watson branching process. That is, given the size \( Z(n) \) of the \( n \)th generation, the distribution of the number of particles in the next generation is specified by the relation

\[
E \left[ s^{Z(n+1)} | Z(0), Z(1), \ldots, Z(n); \pi_0, \pi_1, \pi_2, \ldots \right] = (f_n(s))^{Z(n)}.
\]

We assume in what follows that the initial value \( Z(0) = 1 \). The assertions of our Theorems 1 and 2 below remain valid for the case \( Z(0) > 1 \) as well (the proofs will need only minor modifications). We shall also restrict our attention to critical BPRE’s, i.e. to the case when \( E(\log f'_0(1)) = 0 \). Recall that critical and subcritical (with \( E(\log f'_0(1)) < 0 \)) BPRE’s become extinct with probability 1 as \( n \to \infty \). Therefore, to obtain meaningful limit distributional results one has to consider conditional distributions given the non-extinction of the process.

Let \( Z(m,n) \) denote the number of particles at time \( m \leq n \) which have non-empty offspring at time \( n \). The process \( \{Z(m,n), 0 \leq m \leq n\} \) is called the reduced process for \( Z(n) \). Note that \( Z(m,n) \) is non-decreasing in \( m \), \( Z(n,n) = Z(n) \), and \( Z(0,n) = 1 \) if \( Z(n) \geq 1 \). The study of reduced processes for ordinary branching processes has by now a relatively long history. They were introduced by Fleischmann and Prehn (1974), who investigated the structure of reduced subcritical Galton–Watson processes. Then Zubkov (1975) and Fleischmann and Siegmund–Schultze (1977) considered the critical case. For more general models of branching processes, the reduced processes were analysed later by Fleischmann and Prehn (1975, 1978), Vatutin (1979) and Sagitov (1995). As far as we know, the present paper is the first one dealing with reduced processes for BPRE’s.

The purpose of the present paper is to study the limiting distributions of the reduced processes for BPRE’s with fractional linear generating functions. This means that

\[
1 - f_n(s) = \frac{\alpha(n)}{1 - \beta(n)} = \frac{\alpha(n)s}{1 - \beta(n)s} \quad \text{for all } n = 0, 1, 2, \ldots,
\]

where \( \{(\alpha(n), \beta(n))\} \) is an i.i.d. sequence of random vectors with \( \alpha(n), \beta(n) \in (0, 1) \) and \( \alpha(n) + \beta(n) < 1 \). Even in this special case, when one can get readily explicit formulae for conditional generating functions of \( Z(n) \) given the environment, studying the limiting properties of the reduced process is a rather difficult task.

**Remark 1.** Note that there exists an interesting relationship between the BPRE’s from this particular class and stopped simple random walks in random environment (RWRE’s). Roughly speaking, \( Z(n) \) corresponds to the number of ‘down crossings’ of the level \( n \) in such a walk with the starting point at 1 until it first hits 0 (for more detail see Kesten et al., 1974).
Let \( X_i = \log f'_i(1) \), \( i \geq 1 \), and \( S_0 = 0 \), \( S_n = X_1 + \cdots + X_n \), \( n \geq 1 \). As it was shown in Kozlov (1976), the limiting behaviour of BPRE’s with fractional linear generating functions is closely related to the properties of \( \{S_n\} \). We shall exploit this relationship to prove two limit results for conditional distributions of the reduced BPRE given the non-extinction of the BPRE itself. The first one shows the character of the behaviour of \( Z(m,n) \) for finite \( m \) as \( n \to \infty \), while the second one is a functional limit theorem for the whole trajectory of the reduced BPRE.

Set \( \eta_i = f''_i(1)/2(f'_i(1))^2 \), \( i \geq 1 \).

**Theorem 1.** Let \( \{S_n\} \) be a non-lattice random walk, condition (1) hold, and

\[
E(X_i) = 0, \quad \sigma^2 = \text{Var}(X_i) \in (0, \infty), \quad E(\eta_i) < \infty, \quad E(|X_i|) < \infty.
\]

Then for all \( m = 0, 1, 2, \ldots \) and \( k = 1, 2, \ldots \) there exist the limits

\[
\lim_{n \to \infty} p_k(m) = \lim_{n \to \infty} P(Z(m,n) = k \mid Z(n) > 0) > 0, \quad \sum_{k=1}^{\infty} p_k(m) = 1.
\]

On the other hand, for any fixed \( k = 1, 2, \ldots \)

\[
\lim_{m \to \infty} p_k(m) = 0.
\]

**Remark 2.** Let \( \theta = \max\{m < n: Z(m,n) = 1\} \). The difference \( n - \theta \) is called the distance to the closest mutual ancestor and the respective particle is called the closest mutual ancestor for the particles of the \( n \)th generation. For critical Galton–Watson processes, \( \theta \) is asymptotically uniformly distributed over \([0, n]\), while for supercritical ones, the distribution of \( \theta \) converges to a proper law, so that the closest mutual ancestor belongs in that case to one of the first generations in the process (see Zubkov, 1975). Our Theorem 1 shows that the latter holds true for critical BPRE’s as well. Therefore, in this aspect, critical BPRE’s display behaviour similar to that of supercritical Galton–Watson processes.

In terms of related RWRE’s (see Remark 1), Theorem 1 asserts that, given the maximum of the stopped (on hitting the point 0) random walk exceeds \( n \), the walk is rather likely to return several times to the vicinity of 0 before the stopping time. For an ordinary simple random walk with no drift, such a behaviour is unlikely.

To state our next result, recall the definition of the Brownian meander process \( W^+ \) which can be defined as

\[
W^+(t) = |(1 - \tau)^{-1/3} W(\tau + (1 - \tau)t)|, \quad t \in [0, 1],
\]

with \( W \) being the Brownian motion process and \( \tau = \sup\{t \in [0, 1]: W(t) = 0\} \). Brownian meander appears as the limiting process in the conditional invariance principle (when the random walk is conditioned to stay positive), see Iglehart (1974) and Bolthausen (1976).

To simplify notation, we shall write in what follows \( nt, nu, \) etc. instead of their integer parts \( \lfloor nt \rfloor, \lfloor nu \rfloor \) and so on. Symbols \( c, c_1, c_2 \) and so on will be used to denote
positive constants which can be different in different formulae. This will lead to no confusion.

It was proved in Afanasev (1993) that, given \( Z(n) > 0 \), conditional finite-dimensional distributions of the process

\[ L_n^0(t) = \sigma^{-1} n^{-1/2} \log Z(nt), \quad t \in [0,1], \tag{5} \]

converge to the corresponding distributions of the Brownian meander process \( \{W^+(t), t \in [0,1]\} \) as \( n \to \infty \). Later Kozlov (1995) proved (without assuming (1)) that this convergence actually takes place for distributions of the processes in the Skorokhod space \( D[0,1] \). This means that, in a sense, the conditional logarithmic behaviour of the BPRE given its non-extinction at the terminal time \( n \) is the same as that of the process \( \{\exp(S_n), t \in [0,1]\} \) of conditional mean values (when the environment is fixed) given the latter “stays above 1”.

It turns out that a similar assertion holds for the reduced BPRE’s as well: the value of the reduced BPRE gives the size of the “future bottleneck” (minimum value till the terminal time \( n \)) in the mean values process. Note that the following theorem does not follow from the above-mentioned result for ordinary BPRE’s.

**Theorem 2.** Under the assumptions of Theorem 1, the conditional distribution of the process

\[ L_n(t) = \sigma^{-1} n^{-1/2} \log Z(nt,n), \quad t \in [0,1], \]

in the Skorokhod space \( D[0,1] \) given \( Z(n) > 0 \) converges weakly as \( n \to \infty \) to the law of the process

\[ M(t) = \min_{t \leq u \leq 1} W^+(u), \quad t \in [0,1], \]

where \( W^+(t) \) is the Brownian meander.

### 2. Proof of Theorem 1

To prove Theorem 1, we make use of condition (1) to find an explicit representation for the probabilities of interest. Then we develop the approach suggested by Kozlov (1976) to estimate the contributions to the probabilities from the related random walks on the segments between successive lower ladder epochs for the walk \( \{S_n\} \). Unfortunately, this technique cannot be used in its present form in the case of the general BPRE’s.

It will be convenient to introduce symbols \( E_\pi \) and \( P_\pi \) to denote conditional expectation and probability, respectively, given the whole environment \( \pi = (\pi_0, \pi_1, \pi_2, \ldots) \). For \( r < n \), we set

\[ a_{r,n} = e^{-S_n+S_r} = f'_r(1) \cdots f'_{n-1}(1) = E_\pi(Z(n) | Z(r) = 1), \]
\[ a_n = a_{0,n} = e^{-S_0}, \quad a_{r,r} = a_0 = 1, \quad \text{and put} \]
\[ b_{r,n} = \sum_{i=r+1}^{n} \eta_i a_{i,i-1}, \quad b_n = b_{0,n}, \quad n > 0, \quad b_{r,r} = b_0 = 0. \]

Condition (1) enables one to find compact representations for conditional generating functions (given the environment)
\[ F_{r,n}(s) = f_r(f_{r+1}(\cdots f_{n-1}(s) \cdots)), \quad F_{r,n}(s) = s, \quad F_n(s) = F_0,n(s). \]
of \( Z(n) \) for the BPRE starting at time \( r \) with one particle. They are given by the following assertion from Agresti (1975).

**Lemma 1.** If condition (1) holds, then for \( m \leq n \)
\[ 1 - F_{m,n}(s) = \left(\frac{a_{m,n}}{1 - s} + b_{m,n}\right)^{-1}. \tag{6} \]
In particular, \( 1 - F_n(s) = \left(\frac{a_n}{1 - s} + b_n\right)^{-1}. \)

The basic relation which is to be used throughout the paper and can easily be verified by direct computation using Lemma 1 is that for any \( m < n \)
\[ a_n + b_n = b_m + a_m \left(a_{m,n} + b_{m,n}\right). \tag{7} \]

Put
\[ R_{m,n} = F_{m,n}(0) = P_Z(Z(n) = 0 \mid Z(m) = 1), \]
\[ Q(n) = 1/P(Z(n) > 0) = 1/(E(1 - F_n(0))). \]

It also follows from Lemma 1 that
\[ (a_{m,n} + b_{m,n})^{-1} = 1 - R_{m,n} = P_Z(Z(n) > 0 \mid Z(m) = 1) \leq 1 \tag{8} \]
and, in particular,
\[ (a_n + b_n)^{-1} = 1 - F_n(0) = P_Z(Z(n) > 0) \leq 1. \tag{9} \]

Relations (8) and (9) clearly imply that
\[ a_{m,n} + b_{m,n} \leq a_n + b_n, \quad (x a_{m,n} + b_{m,n})^{-1} \leq 1, \quad x \geq 1. \tag{10} \]

Now introduce the quantities
\[ w_{m,n} = a_m (1 - R_{m,n})^{-1} = a_m b_{m,n} + a_n. \]

It is easy to see that \( w_{m,n} = a_m b_{m,q} + w_{q,n} \) for \( m \leq q < n \). Set also
\[ Y_n(m,k) = w_{m,n} b_m^{k-1} (a_n + b_n)^{-k-1}, \quad T_n(m,k) = b_m^{k-1} (a_n + b_n)^{-k}. \]
Lemma 2. If condition (1) holds, then for \( m \leq n \)

\[
P(Z(m, n) = k \mid Z(n) > 0) = Q(n)E(Y_n(m, k)),
\]

\[
P(Z(m, n) \geq k \mid Z(n) > 0) = Q(n)E(T_n(m, k)).
\]

Proof. Clearly,

\[
P(Z(m, n) = k \mid Z(n) > 0) = Q(n)P(Z(m, n) = k; Z(n) > 0).
\]

Since \( Z(m, n) \) given \( Z(m) \) and the environment is a binomial random variable with the parameters \( Z(m) \) and \( 1 - R_m,n \), we have, using the total probability formula and notation \( x^{[k]} = x(x - 1) \cdots (x - k + 1) \), that

\[
P(Z(m, n) = k; Z(n) > 0) = \frac{1}{k!} E \left[ (1 - R_{m,n})^k R_{m,n}^{Z(m) - k} \right]
\]

\[
= \frac{1}{k!} E \left[ (1 - R_{m,n})^k E_{m} \left( Z(m)^{[k]} R_{m,n}^{Z(m) - k} \right) \right]
\]

\[
= \frac{1}{k!} E \left[ (1 - R_{m,n})^k F_m^{(k)}(R_{m,n}) \right].
\]

Here, the last equality follows from the fact that, for a random variable \( X \) with a generating function \( g \), one has \( E \left( X^{[k]} x^{-k} \right) = g^{(k)}(x) \), \(|x| < 1\).

Now, we find the \( k \)th derivative of \( F_m \) from Lemma 1 to conclude that the last probability is equal to

\[
E \left[ (1 - R_{m,n})^k a_m b_m^{k-1} (a_m + b_m(1 - R_{m,n}))^{-k-1} \right]
\]

\[
E \left[ (1 - R_{m,n})^{-1} a_m b_m^{k-1} (1 - F_n(0))^{k+1} \right] = E \left[ w_m b_m^{k-1} (a_m + b_m)^{-k-1} \right],
\]

which proves the first part of Lemma 2. The second part follows from the first one, formula for the sum of a geometric series, and the relation \( b_n - b_m = a_m b_{m,n} \). \( \square \)

As it was mentioned in Section 1, for a critical BPRE with \( P(\tau_n^{(1)} = 1) < 1 \), the non-extinction probability tends to zero as \( n \to \infty \). We shall need the following result on the asymptotic behaviour of this probability from Kozlov (1976).

Lemma 3. Under the conditions of Theorem 1, there exists a positive constant \( \gamma \) such that

\[
1/Q(n) = P(Z(n) > 0) \sim \gamma n^{-1/2} \quad \text{as} \quad n \to \infty.
\]

For the random walk \( S_n \) we introduce the strict lower ladder epochs \( \tau_j \) by putting

\[
\tau_0 = 0, \quad \tau_{j+1} = \min\{n > \tau_j : S_n < S_{\tau_j}\}, \quad j = 0, 1, 2, \ldots.
\]

Clearly, \( \tau_1 \) is the first hitting time of the set \((-\infty, 0)\) by the walk \( \{S_n\} \). In what follows, we shall use the symbols \( E_{j,n} \) and \( P_{j,n} \) to denote the conditional expectation and probability given the event \( \{\tau_{j-1} \leq n < \tau_j\} \), respectively, \( j = 1, 2, \ldots \).
The next lemma is one of the key steps in the proof of Theorem 1. This result was stated as Lemma 1 in Kozlov (1976), but its proof there had a gap which we fill here.

**Lemma 4.** Under the conditions of Theorem 1

\[
\lim_{q \to \infty} \lim_{n \to \infty} E_1, a(w_{q,n}) = 0. \tag{12}
\]

**Proof.** For any fixed \( \delta \in (0, 1) \),

\[
w_{q,n} = a_q b_{q,n(1-\delta)} + w_{n(1-\delta),n}. \tag{13}
\]

It was proved in Kozlov (1976) that

\[
\lim_{q \to \infty} \lim_{n \to \infty} E_1, a(a_q b_{q,n(1-\delta)}) = 0. \tag{14}
\]

On the other hand, relation (35) in that paper implies that

\[
E_1, a[w_{n(1-\delta),n}] \leq cn^{1/2} \int_0^\infty e^{-x} ((x+1)E(\eta_1) + E(|\eta_1 X_1|))dx \left( \sum_{i=0}^{n} G_i(x) \right),
\]

where

\[
G_i(x) = P(S_j \leq S_i, 0 \leq j \leq i; S_i \leq x).
\]

It is proved in Eppel (1979) that, under the conditions of Theorem 1, there exists an absolute constant \( c_1 < \infty \) such that for all \( x \in [0, \infty) \) and \( i = 1, 2, \ldots \)

\[
G_i(x + 1) - G_i(x) \leq c_1(x + 1)i^{-3/2}.
\]

Therefore for each \( i = 1, 2, \ldots \)

\[
\int_0^\infty e^{-x} ((x+1)E(\eta_1) + E(|\eta_1 X_1|))dG_i(x)
\]

\[
\leq \sum_{k=0}^{\infty} e^{-k} ((k+2)E(\eta_1) + E(|\eta_1 X_1|))(G_i(k+1) - G_i(k))
\]

\[
\leq c_1 i^{-3/2} \sum_{k=0}^{\infty} e^{-k}(k+1)((k+2)E(\eta_1) + E(|\eta_1 X_1|)) \leq c_2 i^{-3/2}
\]

and, hence,

\[
E_1, a(w_{n(1-\delta),n}) \leq c_3 n^{1/2} \sum_{i=n(1-\delta)}^{n} i^{-3/2} \leq c_4 \delta.
\]

Since \( \delta > 0 \) is arbitrary, combining the last estimate with (13) and (14) completes the proof of Lemma 4.

**Proof of Theorem 1.** According to Lemma 2 to prove (3) we have to show that \( Q(n)E[Y_{n}(m,k)] \) converges to a positive limit as \( n \to \infty \). Fixing an integer \( J > 0, \)
we have by the total probability formula

$$Q(n) E[Y_n(m,k)] = Q(n) \sum_{j=1}^{\infty} E[Y_n(m,k) I(\tau_{j-1} \leq n < \tau_j)]$$

$$= \sum_{j=1}^{J} E_{j,n}(Y_n(m,k)) Q(n) P(\tau_{j-1} \leq n < \tau_j)$$

$$+ Q(n) E[Y_n(m,k) I(\tau_J \leq n)],$$

(15)

where $I(A)$ is the indicator function of the event $A$. First, we consider the last term. Clearly,

$$Y_n(m,k) \leq w_{m,n}(a_n + b_n)^{-2} \leq (a_n + b_n)^{-1} = 1 - F_n(0).$$

(16)

Now, it follows from relations (46)-(49) in Kozlov (1976) that

$$\limsup_n \limsup_{n \to \infty} Q(n) E(1 - F_n(0)) I(\tau_J \leq n)) = 0.$$  (17)

Turning back to the sum in (15), we note that since

$$P(\tau_{j-1} \leq n < \tau_j) \sim c_j n^{-1/2}$$

as $n \to \infty$ (cf. Kozlov 1976, p.794), it follows from Lemma 3 that, for any $j$, the quantity

$$Q(n) P(\tau_{j-1} \leq n < \tau_j)$$

converges to a positive limit. This means that it remains to show that for each fixed $j$ the finite limit

$$A_j = \lim_{n \to \infty} E_{j,n}(Y_n(m,k))$$

(19)

exists and, for at least one $j$, is positive.

First, we consider the case $j = 1$. For $n > q > m$ we have

$$|Y_n(m,k) - a_m b_{m,q} b_{m}^{k-1} (a_q + b_q)^{-k-1}|$$

$$= |w_{m,n} b_{m}^{k-1} (a_n + b_n)^{-k-1} - a_m b_{m,q} b_{m}^{k-1} (a_q + b_q)^{-k-1}|$$

$$\leq |(w_{m,n} - a_m b_{m,q}) b_{m}^{k-1} (a_n + b_n)^{-k-1}|$$

$$+ a_m b_{m,q} b_{m}^{k-1} |(a_q + b_q)^{-k-1} - (a_n + b_n)^{-k-1}| =: \hat{Y}.$$  (20)

Since $w_{m,n} - a_m b_{m,q} = w_{q,n}$ and

$$b_{m}^{k-1} (a_n + b_n)^{-k-1} \leq b_{m}^{k-1} (a_n + b_n)^{-k-1} \leq (a_n + b_n)^{-2} \leq 1$$

by (9), the first term in the expression for $\hat{Y}$ from (20) does not exceed $w_{q,n}$. For the second one we have

$$|(a_q + b_q)^{-k-1} - (a_n + b_n)^{-k-1}|$$

$$\leq \frac{(k + 1)(a_n + b_n)^k}{(a_n + b_n)^{k+1}} (a_n + b_n - a_q - b_q)$$

$$\leq (a_q + b_q)^{-k}(k + 1) w_{q,n}$$
in view of (9) and
\[ a_n + b_n - a_q - b_q \leq \frac{a_n + b_n - b_q}{n}. \]

Now, similar to (21)
\[ \frac{a_m b_m q b_m^{k-1} (a_q + b_q)^{-k}}{k} \leq 1, \]
so that \( Y \leq (k + 2)w_{q,n} \). Therefore, by Lemma 4
\[ \lim \sup \lim \sup E_1, n \left[ Y_n(m,k) - \frac{a_m b_m q b_m^{k-1} (a_q + b_q)^{-k}}{k} \right] \leq \lim \sup \lim \sup (k + 2)E_1, n(w_{q,n}) = 0. \]  
(22)

Now note that for each fixed \( q > m \) there exists the limit
\[ \lim_{k \to \infty} \frac{a_m b_m q b_m^{k-1} (a_q + b_q)^{-k}}{k} \in (0, 1]. \]  
(23)

This is an immediate consequence of Theorem B from Kozlov (1976) which asserts that the joint conditional distribution of the random variables \( S(1), \eta_1, S(2), \eta_2, \ldots, S(q), \eta_q \), given the event \( \{\tau_1 > n\} \) converges to a proper non-degenerate distribution as \( n \to \infty \). The last relation in (23) follows from the fact that the expression under the expectation sign is positive and does not exceed \( Y_q(m,k) \), and estimate (16). Combining (22) and (23), we establish that the limit in (19) exists for \( j = 1 \).

The next step is to show that \( A_1 > 0 \). Put
\[ \tau_m(x) = \min\{n > m: S_n < S_m - x\}, \quad x \geq 0. \]

By an argument similar to that from the proof of Lemma 2 in Kozlov (1976) one can show that for any \( x \geq 0 \) there exists the limit
\[ H(x) = \lim_{n \to \infty} E \left[ (1 + a_n + b_n)^{-1} \mid \tau_0(x) > n \right] \]  
(24)
and for each fixed \( R > 0 \) there is a constant \( c_1(R) > 0 \) such that \( H(x) \geq c_1(R) \) for all \( x \in [0, R) \). It is clear from the definition of \( Y_q(m,k) \) and (7) that
\[ E_1, n(Y_q(m,k)) \geq E_1, n \left[ a_m \eta_{m+1} b_m^{k-1} (b_m + a_m)^{-k} \right] \]
\[ = E_1, n \left[ a_m \eta_{m+1} b_m^{k-1} (b_{m+1} + a_{m+1} + b_{m+1})^{-k} \right] \]
\[ \geq E_1, n \left[ K_n N_{m,n}^{k+1} \right] =: M_{m,n} \]  
(25)
with
\[ K_m = a_m \eta_{m+1} b_m^{k-1} (b_{m+1} + a_{m+1})^{-k}, \quad N_{m,n} = (1 + a_{m+1} + b_{m+1})^{-1}. \]

As is well known, for a zero-drift random walk with finite variance,
\[ P(\tau_1 > n) \sim cn^{-1/2}. \]  
(26)
Therefore, since $I(\tau_1 > n) = I(\tau_1 > m + 1)I(\tau_{m+1}(S_{m+1}) > n)$,

$$M_{m, n} = (P(\tau_1 > n))^{-1} E[K_mN_{m,n}^{k+1}I(\tau_1 > n)]$$

$$\geq cn^{1/2}E[I(\tau_1 > m + 1)K_mE(N_{m,n}^{k+1}I(\tau_{m+1}(S_{m+1}) > n)|\tilde{\pi}_m)]$$

$$= cn^{1/2}E[I(\tau_1 > m + 1)K_mE(N_{m,n}^{k+1}I(\tau_{m+1}(S_{m+1}) > n)|S_{m+1})],$$

(27)

where

$$\tilde{\pi}_m = (\pi_0, \ldots, \pi_m)$$

denotes the initial segment of the random environment. The inner conditional expectation in the last line of (27) is

$$P(\tau_{m+1}(S_{m+1}) > n|S_{m+1})E(N_{m,n}^{k+1}I(\tau_{m+1}(S_{m+1}) > n, S_{m+1})$$

on the event $\{S_{m+1} > 0\} \supset \{\tau_1 > m + 1\}$. By Jensen’s inequality for conditional expectations

$$E(N_{m,n}^{k+1}I(\tau_{m+1}(S_{m+1}) > n, S_{m+1})$$

$$\geq \left(E(N_{m,n} I(\tau_{m+1}(S_{m+1}) > n, S_{m+1})\right)^{k+1}.$$

Now, it follows from (24) that for any $x > 0$

$$\lim \inf_{n \to \infty} E(N_{m,n}^{k+1}I(\tau_{m+1}(S_{m+1}) > n, S_{m+1} = x) \geq H^{k+1}(x).$$

Therefore, by Fatou lemma, one has from (25)–(28) that

$$A_1 \geq \lim \inf_{n \to \infty} M_{m, n} \geq c E[I(\tau_1 > m + 1)K_mH^{k+1}(S_{m+1})] > 0.$$

(29)

The last inequality holds due to the fact that the expression under the expectation sign here is always non-negative and is positive on the event

$$\{\tau_1 > m + 1, \eta_{m+1}b_m > 0, S_{m+1} < R\}$$

(recall that $H(x) \geq c_1(R) > 0, x \in [0, R]$), of which the probability is clearly positive for some $R > 0$.

Now, consider the case $j > 1$. For a fixed $T, j \leq T < n/2$, we have from (18) that

$$E_{j,n}[Y_n(m, k)] \sim cn^{1/2}\{E[Y_n(m, k)I(T < \tau_{j-1} \leq n < \tau_j)]$$

$$+E[Y_n(m, k)I(\tau_{j-1} \leq T, \tau_j > n)]\}.$$

(30)
By (16)

\[ E \left[ Y_n(m,k) I(T < \tau_{j-1} \leq n < \tau_j) \right] \leq P(T < \tau_{j-1} \leq n < \tau_j) \]

\[ = \sum_{p=T+1}^{n} P(\tau_{j-1} = p, \tau_j > n) = \sum_{p=T+1}^{n} P(\tau_{j-1} = p) P(\tau_1 > n - p) \]

\[ \leq P(\tau_1 > n/2) \sum_{p=T+1}^{n/2} P(\tau_{j-1} = p) + \max_{p > n/2} P(\tau_{j-1} = p) \sum_{r=0}^{n/2} P(\tau_{j-1} > r). \]

It follows from the estimates (26) and \( P(\tau_{j-1} = n) \leq c n^{-3/2} \) (Theorem D from Eppel, 1979) that the last expression does not exceed

\[ c(n^{-1/2} T^{-1/2} + n^{-3/2} n^{1/2}) \]

so that

\[ \limsup_{T \to \infty} \limsup_{n \to \infty} n^{1/2} E \left[ Y_n(m,k) I(T < \tau_{j-1} \leq n < \tau_j) \right] = 0. \]  

(31)

Further

\[ E \left[ Y_n(m,k) I(\tau_{j-1} \leq T, \tau_j > n) \right] = \sum_{p=j-1}^{T} E \left[ Y_n(m,k) I(\tau_{j-1} = p, \tau_j > n) \right]. \]  

(32)

For a fixed \( p > m \) we have

\[ E \left[ Y_n(m,k) I(\tau_{j-1} = p, \tau_j > n) \right] \]

\[ = P(\tau_{j-1} = p, \tau_j - \tau_{j-1} > n - p) E \left[ Y_n(m,k) I(\tau_{j-1} = p, \tau_j - \tau_{j-1} > n - p) \right] \]

\[ = P(\tau_{j-1} = p) P(\tau_1 > n - p) E \left[ E \left[ Y_n(m,k) | \pi_{p-1}; \tau_{j-1} = p, \tau'_1 > n - p \right] \right] \]

\[ \sim cn^{-1/2} E \left[ E \left[ Y_n(m,k) | \pi_{p-1}; \tau_{j-1} = p, \tau'_1 > n - p \right] \right]. \]  

(33)

by (26), where \( \tau'_1 = \tau_j - \tau_{j-1} \) is independent of \( \pi_{p-1} = (\pi_0, \ldots, \pi_{p-1}) \) and has the same distribution as \( \tau_1 \). From the definition of \( w_{m,n} \) and (7) it follows that

\[ Y_n(m,k) = w_{m,n} b_m^{-k-1} (a_n + b_n)^{-k-1} = (a_m b_{m,p} + a_p b_{p,n} + a_p a_{p,n}) b_m^{-k-1} (a_p a_{p,n} + b_p + b_{p,n}) b_{p,n}^{-k+1}. \]

Therefore, for fixed \( \pi_{p-1} \) we can argue in the same way as in the case \( j - 1 \) and prove that the limit

\[ \lim_{n \to \infty} E \left[ Y_n(m,k) | \pi_{p-1}; \tau_{j-1} = p, \tau'_1 > n - p \right] \]

exists. The case \( p \leq m \) can be treated in a similar way.

Since the expression under the expectation sign on the right-hand side of (33) is bounded by a positive constant with probability 1, we can pass to the limit under this expectation and make use of (32) and (33) to show that

\[ n^{1/2} E \left[ Y_n(m,k) I(\tau_{j-1} \leq T, \tau_j > n) \right] \]

also converges to a finite limit as \( n \to \infty \). This together with (30) and (31) complete the proof of (19) for \( j > 1 \). Combination of (15) and (17) with the fact \( A_1 > 0 \) proves the first part of (3).

Now we shall prove the second part of (3) which means that the limiting distribution is proper. It follows from what we have just established that

\[
P\left(Z(m,n) \geq \frac{k}{n}, Z(n) > 0\right) \to 1 - \sum_{j=1}^{k-1} p_j(m) \quad \text{as} \quad n \to \infty.
\]

Further, since by (10)

\[
b_m^{-1}(a_n + b_n)^{-k} = \frac{1}{a_n + b_n} \left( \frac{b_m}{a_n + b_n} \right)^{k-1} \leq \frac{1}{a_m,n + b_m,n} \left( \frac{b_m}{a_n + b_n} \right)^{k-1} = (a_{m,n} + b_{m,n})^{-1} \zeta_{m}^{k-1}
\]

and the random variables \( a_{m,n} + b_{m,n} \) and

\[
\zeta_m = b_m/(a_m + b_m) < 1
\]

are independent, we have from Lemma 2 that the left-hand side of (34) equals

\[
Q(n) E \left[T_{n}(m,k)\right] \leq Q(n) E \left[(a_{m,n} + b_{m,n})^{-1}\right] E \left[\zeta_{m}^{-k}\right]
\]

as \( n \to \infty \), and hence

\[
1 - \sum_{j=1}^{k-1} p_j(m) \leq E \left[\zeta_{m}^{-k}\right].
\]

Letting \( k \to \infty \), we obtain the required assertion from (35).

Now, we shall prove convergence (4). Since for \( k > 1 \) one has from Lemma 2 that

\[
p_k(m) = \lim_{n \to \infty} Q(n) E \left(Y_n(m,k)\right)
\]

\[
= \lim_{n \to \infty} Q(n) E \left[ \frac{w_{m,n}}{(a_n + b_n)^2} \left( \frac{b_m}{a_n + b_n} \right)^{k-1} \right]
\]

\[
\leq \lim_{n \to \infty} Q(n) E \left[ \frac{w_{m,n}}{(a_n + b_n)^2} \right] = \lim_{n \to \infty} Q(n) E \left(Y_n(m,1)\right) = p_1(m),
\]

it suffices to show that \( p_1(m) \to 0 \) as \( m \to \infty \). To this end, in view of (15), (17), (31), and (32) it remains to establish that for each fixed \( p \geq j - 1 \)

\[
\lim sup_{m \to \infty} \lim sup_{n \to \infty} Q(n) E \left[Y_n(m,1) I(\tau_j = p, \tau_j > n)\right] = 0.
\]

Observe that for each \( p < m \) we have from (9) and (7) that

\[
Y_n(m,1) = \frac{w_{m,n}}{(a_n + b_n)^2} \leq \frac{w_{m,n}}{a_p + b_p} \leq \frac{w_{m,n}}{a_{m,n} + b_{m,n}} = \frac{a_{p,m}}{a_{m,n} + b_{m,n}}.
\]
so that

\[
E \left[ Y_n(m, 1) I(\tau_{j-1} = p, \tau_j - \tau_{j-1} > n - p) \right]
\]

\[
\leq E \left[ \frac{a_{p,m}}{a_{m,n} + b_{m,n}} I(\tau_{j-1} = p, \tau_j - \tau_{j-1} > n - p) \right]
\]

\[
= P(\tau_{j-1} = p) E \left[ \frac{a_{p,m}}{a_{m,n} + b_{m,n}} I(\tau_j - \tau_{j-1} > n - p) \right]
\]

\[= P(\tau_1 > n - p) E_{1,n-p} W_{m-p,n-p}. \tag{38}\]

Now, (37) follows immediately from (11), (12), (26), and (38), which completes the proof of Theorem 1.

3. Proof of Theorem 2

First, note that \( L_n(1) = L_0(n) = 1 \) and hence, the conditional distribution of \( L_n(1) \) given \( Z(n) > 0 \) converges to the distribution of \( W^+(1) \) (cf. (5)). Since the sets \( \{ f \in D[0,1] : f \text{ is non-decreasing, } f(1) \leq c \}, c < \infty \), are compacts in \( D[0,1] \), this means that the sequence of conditional distributions of \( \{ L_n(t), 0 \leq t \leq 1 \} \) given \( Z(n) > 0 \) is tight in the space \( D[0,1] \). Therefore, to prove Theorem 2 it remains to establish convergence of finite-dimensional distributions.

We shall need several auxiliary results.

Lemma 5. If condition (2) holds, then for any fixed \( j \geq 1 \) the conditional distribution of the process \( \{ \sigma^{-1} n^{-1/2} S_n, t \in [0,1] \} \) in the space \( D[0,1] \) given \( \tau_{j-1} \leq n < \tau_j \), converges weakly as \( n \to \infty \) to the distribution of the Brownian meander \( \{ W^+(t), t \in [0,1] \} \).

Since the same limiting relation takes place for conditional distributions of the same process given \( \{ \tau_1 > n \} \) (i.e. the walk stays positive till time \( n \), Bolthausen, 1975), this lemma means that, given the \( j \)th time interval between two successive lower ladder epoches for the walk \( \{ S_m \} \) “covers” the point \( n \), the \( (j - 1) \)th lower ladder epoch \( \tau_{j-1} \) is negligible in comparison with the length of this covering interval.

Proof. Convergence of finite-dimensional distributions was proved in Afanasev (1993). The tightness can be established by a standard argument (cf. Iglehart, 1974).

Introduce the processes

\[
M_n(t) = \sigma^{-1} n^{-1/2} \min_{t \leq n \leq 1} S_n.
\]

Since the minimum is a continuous functional in \( D[0,1] \), Lemma 5 implies the following statement.
Corollary 1. Under the assumptions of Lemma 5, the conditional distribution of the process \( \{M_n(t), t \in [0, 1]\} \) given \( \tau_{j-1} \leq n < \tau_j \) converges weakly as \( n \to \infty \) to the distribution of \( \{M(t), t \in [0, 1]\} \).

For a fixed vector \( x = (x_1, \ldots, x_N) \) with positive components set

\[
V_n(x) = \bigcap_{i=1}^{N} \{M_n(t_i) \geq x\}, \quad V(x) = \bigcap_{i=1}^{N} \{M(t_i) \geq x\}.
\]

(39)

Lemma 6. Under the conditions of Theorem 2, for any fixed \( x > 0 \) and \( j \geq 1 \)

\[
E \left[ I(V_n(x)) P_\pi(Z(n) > 0) I(\tau_{j-1} \leq n < \tau_j) \right] \leq (1 + o(1)) P(V(x)) E \left[ I(V_n(x)) P_\pi(Z(n) > 0) I(\tau_{j-1} \leq n < \tau_j) \right].
\]

Proof. By Lemma 3 from Afanasev (1993) the process \( \{\sigma^{-1}n^{-1/2}S_m\} \) and the vector \((a_n, b_n)\) are asymptotically (as \( n \to \infty \)) conditionally independent given the \( j \)-th interval between successive lower ladder epochs covers the point \( n \). The desired assertion follows now immediately from this fact, Corollary 1, and representation (9).

Corollary 2. Under the conditions of Theorem 2, for any fixed \( x > 0 \)

\[
\lim_{n \to \infty} Q(n) E \left[ I(V_n(x)) P_\pi(Z(n) > 0) \right] = P(V(x)).
\]

Proof. By Lemma 6

\[
Q(n) E \left[ I(V_n(x)) P_\pi(Z(n) > 0) \right] \\
= Q(n) \sum_{j=1}^{J} E \left[ I(V_n(x)) P_\pi(Z(n) > 0) I(\tau_{j-1} \leq n < \tau_j) \right] \\
+ Q(n) E \left[ I(V_n(x)) P_\pi(Z(n) > 0) I(\tau_j \leq n) \right] \\
= (1 + o(1)) P(V(x)) Q(n) E \left[ P_\pi(Z(n) > 0) I(\tau_{j-1} \leq n < \tau_j) \right] \\
+ Q(n) E \left[ I(V_n(x)) P_\pi(Z(n) > 0) I(\tau_j \leq n) \right] \\
= (1 + o(1)) P(V(x)) - O(Q(n) E \left[ P_\pi(Z(n) > 0) I(\tau_j \leq n) \right]).
\]

Applying the relation (17) completes the proof of Corollary 2.

Introduce the events \( A_n(t, x) = \{L_n(t) \geq x\}, \ x > 0, \) and let, for fixed arbitrary \( 0 < t_1 < t_2 < \cdots < t_N \leq 1 \) and \( x = (x_1, \ldots, x_N) > 0, \)

\[
U_n(x) = \bigcap_{i=1}^{N} A_n(t_i, x_i).
\]

Lemma 7. Under the conditions of Theorem 2, for any \( \varepsilon > 0 \) and \( x + \varepsilon = (x_1 + \varepsilon, \ldots, x_N + \varepsilon) \),

\[
\lim_{n \to \infty} P(U_n(x)|Z(n) > 0) \geq P(V(x + \varepsilon)).
\]
Proof. We have

\[
P(U_n(x)|Z(n) > 0) = Q(n) P(U_n(x) \cap \{Z(n) > 0\})
\]

\[
\geq Q(n) E[P_n(U_n(x) \cap \{Z(n) > 0\}) I(V_n(x + \varepsilon))]
\]

\[
= Q(n) E[P_n(Z(n) > 0) I(V_n(x + \varepsilon))]
\]

\[
- Q(n) E[P_n(U_n^c(x) \cap \{Z(n) > 0\}) I(V_n(x + \varepsilon))]
\]

\[
\geq Q(n) E[P_n(Z(n) > 0) I(V_n(x + \varepsilon))]
\]

\[
- Q(n) \sum_{i=1}^N E[P_n(A_i^n(t_i,x_i) \cap \{Z(n) > 0\}) I(M_n(t_i) \geq x_i + \varepsilon)]. \quad (40)
\]

By Corollary 2

\[
Q(n) E[P_n(Z(n) > 0) I(V_n(x + \varepsilon))] \to P(V(x + \varepsilon)) \quad (41)
\]

as \(n \to \infty\).

On the other hand, following the argument from the proof of Lemma 2, it is not hard to verify that

\[
P_n(Z(m,n) < k|Z(n) > 0) = 1 - \left(\frac{b_m}{a_n + b_n}\right)^k
\]

\[
\leq k(a_n + b_n - b_m)/(a_n + b_n) \leq k(a_n + b_n - b_m)
\]

by (9). Therefore, on the event

\[
\{M_n(t_i) \geq x_i + \varepsilon\} = \left\{ e^{-s_i} \leq e^{-an^l z(t_i + \varepsilon)}, j = |nt_i|, \ldots, n \right\},
\]

we have

\[
P_n(A_i^n(t_i,x_i)|Z(n) > 0) = P_n(Z(nt_i,n) < e^{an^l z(t_i + \varepsilon)}|Z(n) > 0)
\]

\[
\leq e^{an^l z(t_i)} \left( e^{-s_i} + \sum_{j=nt_i+1}^n \eta_j e^{-s_{j-1}} \right) \leq e^{an^l z(t_i)} \left( 1 + \sum_{j=nt_i+1}^n \eta_j \right).
\]

Hence,

\[
\sum_{i=1}^N E[P_n(A_i^n(t_i,x_i) \cap \{Z(n) > 0\}) I(M_n(t_i) \geq x_i + \varepsilon)]
\]

\[
\leq \sum_{i=1}^N E[P_n(Z(n) > 0) e^{-an^l z(t_i)} \left( 1 + \sum_{j=nt_i+1}^n \eta_j \right)] \leq N e^{-an^l z} E(\eta_1),
\]

so that the last term in (40) tends to zero as \(n \to \infty\) which, together with (41), completes the proof of Lemma 7.

Lemma 8. Under the conditions of Theorem 2, for any \(x > 0\),

\[
\limsup_{n \to \infty} P(U_n(x)|Z(n) > 0) \leq P(V(x)).
\]
Proof. Since \( Z(nt, n) \leq \min_{s \leq 1} Z(ns) \), one has
\[
U_n(x) \subseteq \bigcap_{i=1}^{N} \{L_n^0(t_i) \geq x_i\}.
\]

The assertion of the lemma follows now immediately from the fact that the conditional process \( \{L_n^0(t), 0 \leq t \leq 1\} \) given \( \{Z(n) > 0\} \) converges to the process \( \{W^+(t)\} \) (see Section 1), for the mapping \( W^+ \mapsto M \) is continuous and the corresponding set has a null boundary with respect to the distribution of \( M \).

Convergence of finite-dimensional distributions follows now from Lemmas 7 and 8. Theorem 2 is proved.

References