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Average of the Navier's Law on the Rapidly Oscillating Boundary

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We study the flow of Newtonian fluid in a domain with periodically wrinkled boundary with slip (Navier's) boundary condition. The goal of this paper is to replace a microscopic boundary condition, posed on the rough boundary, by some macroscopic boundary condition, posed on the middle surface of the oscillating boundary. Depending on the shape of wrinkles and the friction coefficient we get four different effective models. © 2001 Academic Press

Key Words: homogenization; rapidly oscillating boundary; Navier's law; Γ-convergence.

1. INTRODUCTION

The most usual boundary condition imposed on the solid wall for a viscous fluid is the no-slip condition. However, another approach is sometimes suggested, supposing that there is a stagnant layer of fluid close to the wall allowing a fluid to slip. In such a situation Navier's law (see [20]), saying that the slip velocity is proportional to the shear stress while the normal velocity remains zero, is commonly used. Such boundary conditions can be induced by the effects of a rough boundary, as in [1, 2, 15, 24], or a perforated boundary (it is then called Beavers and Joseph's law) as for instance in [15]. Such boundary behaviour can also be result of an exterior electric field, as in [9] and [10].

Our intention is to consider the flow of Newtonian fluid in domains with one part of the boundary being a rough surface. On that rough part of the boundary we impose the slip boundary condition. Solving numerically a problem in such a domain that has small cogs is almost impossible. Therefore the idea is to simplify the model with a minimal loss of accuracy by averaging the microeffects caused by small wrinkles of the surface.

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More precisely, our goal is to study the behaviour of the solution in the vicinity of the boundary in order to smooth out the ruffles, i.e., to replace that boundary condition posed on the wrinkled surface by some other condition posed on its middle surface that leads to a fairly good approximation of the original problem and is more convenient for numerical resolution. To do so we use the *method of homogenization.*

To describe a rough boundary (see Fig. 1) we denote by $\Omega \subset \mathbf{R}^n$, $n = 2, 3$ a smooth domain with outer unit normal $\mathbf{n}$. Let $\Gamma_2 \subset \partial\Omega$ be parameterized by $\mathbf{x}(z)$. By h we denote a smooth periodic function with period $Y =]0, 1[^{n-1}$. In vicinity of Γ_2, we define the curvilinear coordinates (z, t), where t is normal and z tangential to Γ_2. We then define the rough boundary Γ_2^ε as a manifold parameterized by $\mathbf{y}_\varepsilon(z) = \mathbf{x}(z) + \varepsilon^m h(z/\varepsilon)\mathbf{n}$, where ε^m is the height and ε is the length of each cog for a small parameter $\varepsilon \ll 1$.

It should be noticed that the cogs of that rough boundary are almost periodic. Almost, because of a little distortion due to the curvilinear coordinates. This structure describes the most common laboratory model of a rough boundary.

To find the effective boundary condition on a smoothed boundary we need to study the limit as ε tends to 0. Depending on the power m and the boundary data we expect to get different models.

On the rough part of the boundary we pose the condition

$$(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mu \frac{\partial u^\varepsilon}{\partial \mathbf{n}} = \varepsilon^k (g^\varepsilon - \kappa u^\varepsilon).$$

We suppose that $k + m \geq 1$. Depending on k and m we get four different effective boundary conditions that can be written in the form

$$(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mu \frac{\partial u}{\partial \mathbf{n}} = \lambda(g - \kappa u).$$

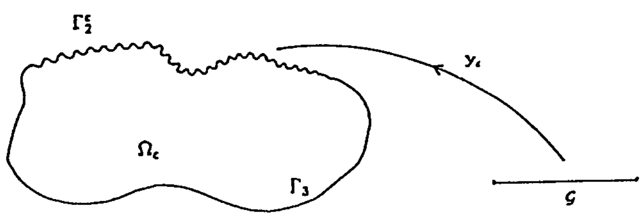

FIG. 1. Domain with oscillating boundary.

1. If $k + m = 1$ then:

   (a) If $k = 0$ then $\lambda = \int_Y \sqrt{1 + |G|}$, where G is defined by (7). If our local curvilinear system on Γ_2 is conformal (i.e., if the metric tensor is scalar, see Remark 2) then λ is the surface of one ruffle.

   (b) If $k > 0$ then $\lambda = \int_Y |G|^{1/2}$.

2. If $k + m > 1$ then:

   (a) If $k = 0$ then $\lambda = 1$.

   (b) If $k > 0$ then $\lambda = 0$.

That problem can be placed in class of so-called singular perturbation problems. Singular, because a small change of the parameter ε can cause an important change of solution, particularly in some vicinity of the boundary.

The problem of wrinkled boundary has been successfully solved by Sanchez-Palencia [22], Belyaev [5], and Checkin, Friedman, and Piatnitski [8] for a heat conduction equation with a Neumann and Robin boundary condition and it justifies the radiator effect. In fact they proved that the heat flux on a wrinkled boundary is proportional to the area of the ruffle.

Some results about the eigenvalue problem for an elliptic operator in a domain with oscillating boundary were stated in the short paper [5]. An interesting study of the rough boundaries in elasticity was done by Kohn and Vogelius in [16–18]. We refer particularly to applications of the Γ-convergence by Buttazzo and Kohn [7].

Navier's law for the viscous flow near the impermeable rough wall was studied in [1, 2]. A rigorous justification of the Navier's law with the friction coefficient of order ε was obtained in [15], starting from the no-slip condition. Some numerical computations for a turbulent flow past a wavy boundary have been presented in Valentin and Le Tallec [25] and for a non-Newtonian flow along a vertical sinusoidal surface in Kumari, Pop, and Takhar [19]. Effects of a rough wall on hydrodynamic drag were studied in [3].

2. THE GEOMETRY

We generalize the ideas from [22] and [5] to describe the rough boundary. We denote by $\Omega \subset \mathbf{R}^n$ a bounded C^2 domain placed only on one side of its boundary Γ. We suppose that Γ can be separated in two disjoint parts Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and that Γ_2 can be parameterized by a single parameterization. This is a technical assumption that simplifies the presentation. Let $U \subset \mathbf{R}^{n-1}$ and $\mathbf{x} : U \to \mathbf{R}^n$ be the parameterization of

Γ_2. We suppose that $\mathbf{x}: U \to \mathscr{O} = \mathbf{x}(U)$ is bijective, that $\mathbf{x}$ and $\mathbf{x}^{-1}$ are of class C^2, and that $\nabla \mathbf{x}(z)$ has rank $n-1$ at every point $z \in U$. In order to have the smooth junction between the rough boundary and the rest of the boundary we suppose that $h \in C_0^2(Y, \mathbf{R}_+)$, with $Y =]0,1[^{n-1}$, is the shape function for cogs. Let $0 < \varepsilon \ll 1$ be a small parameter (we can think of it as the period of ruffles). Let $\mathbf{y}: U \times \mathbf{R} \to \mathbf{R}^n$ be the mapping defined by

$$\mathbf{y}(z,t) = \mathbf{x}(z) + t\mathbf{n}(z),$$

where $\mathbf{n}$ is the exterior unit normal on Γ_2. For sufficiently small ε_0 there exists $\mathscr{G} \subseteq U$, such that the mapping $\mathbf{y}: \mathscr{G} \times]-\varepsilon_0, \varepsilon_0[\to \mathbf{y}(\mathscr{G} \times]-\varepsilon_0, \varepsilon_0[)$ is a diffeomorphism (see, e.g., [13, 21]). For the sake of simplicity, we suppose that $\mathscr{G} = U$ (that is not a loss of generality since we could have replaced, from the beginning, U by, possibly, a smaller set $\mathscr{G}$). Given $\varepsilon \in]0, \varepsilon_0^{1/m}[$ we define, by periodic repetition of the canonical cell εY, the rectangular set $\mathscr{Z}_\varepsilon = \varepsilon \bigcup_{i \in \mathscr{J}} (i + Y)$, where $\mathscr{J} = \{i \in \mathbf{Z}^n;\ \varepsilon(i+Y) \subset \mathscr{G}\}$. We now extend h by Y-periodicity in $\mathscr{Z}_\varepsilon$ and by zero to the rest of $\mathscr{G}$. By construction, such h is obviously in $C_0^2(\mathscr{G})$. We note that the measure of the small set $\mathscr{G} \setminus \mathscr{Z}_\varepsilon$ is of order ε. The parameterization of the wrinkled boundary Γ_2^ε, $\mathbf{y}_\varepsilon : \mathscr{G} \to \mathbf{R}^n$, for $\varepsilon^m < \varepsilon_0$, is now given by

$$\mathbf{y}_\varepsilon(z) = \mathbf{y}(z, \varepsilon^m h(z/\varepsilon)).$$

We now define

$$\Omega_\varepsilon = \Omega \cup \{\mathbf{y}(z,t) \in \mathbf{R}^n;\ z \in \mathscr{G}, 0 \le t < \varepsilon^m h(z/\varepsilon)\}$$

and we pose

$$\Gamma_2^\varepsilon = \{\mathbf{y}_\varepsilon(z) \in \mathbf{R}^n;\ z \in \mathscr{G}\} \text{ so that } \Gamma_1 = \partial\Omega_\varepsilon \setminus \Gamma_2^\varepsilon.$$

Remark 1. We note that the definition of Γ_2^ε depends on the choice of parameterization of Γ_2, i.e. on $\mathbf{x}$. In case $n = 2$ it would be natural to parameterize Γ_2 by its arc length. In that case Γ_2^ε is a periodic union of cogs. In general Γ_2^ε is not actually periodic due to the distortion coming from the choice of parameterization.

3. THE EQUATIONS

A stationary flow of an incompressible, viscous, Newtonian fluid is governed by the Navier–Stokes system. As described before, we impose a no-slip condition on the part of the boundary denoted by Γ_1 and a nonhomogeneous slip condition on the rough part of the boundary denoted Γ_2^ε. Our system can be written as

$$(u^\varepsilon \nabla) u^\varepsilon - \mu \Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon, \operatorname{div} u^\varepsilon = 0 \text{ in } \Omega_\varepsilon \tag{1}$$

$$u^\varepsilon = 0 \text{ on } \Gamma_1, u^\varepsilon \cdot \mathbf{n} = 0 \text{ on } \Gamma_2^\varepsilon \tag{2}$$

$$(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \mu \frac{\partial u^\varepsilon}{\partial \mathbf{n}} + \varepsilon^k \kappa u^\varepsilon = \varepsilon^k g^\varepsilon \text{ on } \Gamma_2^\varepsilon. \tag{3}$$

To place our problem in an appropriate functional framework we define the functional spaces

$$V_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon)^n, \operatorname{div} v = 0, v = 0 \text{ on } \Gamma_1, v \cdot \mathbf{n} = 0 \text{ on } \Gamma_2^\varepsilon \right\}$$

$$H_\varepsilon = \left\{ v \in L^2(\Omega_\varepsilon)^n, \operatorname{div} v = 0, v \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_\varepsilon \right\},$$

equipped by the norms

$$\|v\|_{V_\varepsilon} = \|\nabla v\|_{L^2(\Omega_\varepsilon)^{n^2}}, \|v\|_{H_\varepsilon} = \|v\|_{L^2(\Omega_\varepsilon)^n}.$$

In order to have a well-posed problem we add the following assumptions on f^ε, g^ε, and κ:

(i) $f^\varepsilon \in V_\varepsilon'$.

(ii) $g^\varepsilon \in H^{1/2}(\Gamma_2^\varepsilon)^n$.

(iii) $\kappa \in C(\overline{\mathscr{G}})$, $0 < \kappa_0 \le \kappa \le \kappa_1$.

To write the variational formulation of (1)–(3) it is important to notice that, due to the choice of V_ε, for any $\phi \in V_\varepsilon$

$$\begin{aligned} &{}_{V_\varepsilon'}\langle -\mu \Delta u^\varepsilon + \nabla p^\varepsilon \mid \phi \rangle_{V_\varepsilon} \\ &= \mu \int_{\Omega_\varepsilon} \nabla u^\varepsilon \nabla \phi + {}_{H^{-1/2}(\Gamma_2^\varepsilon)}\left\langle -\mu \frac{\partial u^\varepsilon}{\partial \mathbf{n}} + p^\varepsilon \mathbf{n} \,\middle|\, \phi \right\rangle_{H^{1/2}(\Gamma_2^\varepsilon)} \\ &= \mu \int_{\Omega_\varepsilon} \nabla u^\varepsilon \nabla \phi + {}_{H^{-1/2}(\Gamma_2^\varepsilon)}\left\langle -(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \mu \frac{\partial u^\varepsilon}{\partial \mathbf{n}} \,\middle|\, \phi \right\rangle_{H^{1/2}(\Gamma_2^\varepsilon)} \\ &= \mu \int_{\Omega_\varepsilon} \nabla u^\varepsilon \nabla \phi + \varepsilon^k \int_{\Gamma_2^\varepsilon} (\kappa u^\varepsilon - g^\varepsilon) \phi. \end{aligned}$$

The variational formulation of (1)–(3) now reads as follows:

Find $u^\varepsilon \in V_\varepsilon$ such that

$$\mu \int_{\Omega_\varepsilon} \nabla u^\varepsilon \nabla \phi + \int_{\Omega_\varepsilon} (u^\varepsilon \nabla) u^\varepsilon \phi + \varepsilon^k \int_{\Gamma_2^\varepsilon} \kappa u^\varepsilon \phi$$

$$= {}_{V_\varepsilon'}\langle f^\varepsilon \mid \phi \rangle_{V_\varepsilon} + \varepsilon^k \int_{\Gamma_2^\varepsilon} g^\varepsilon \phi, \tag{4}$$

for any $\phi \in V_\varepsilon$.

4. SOME TECHNICAL RESULTS

We denote by $\mathbf{a}_i(z)$ the basis of the tangent space on Γ_2 at point $z \in \mathscr{G}$ such that

$$(\mathbf{a}_1(z), \ldots, \mathbf{a}_{n-1}(z)) = \nabla \mathbf{x}(z) \tag{5}$$

and by $\mathbf{a}_i^t(z)$, $|t| < \varepsilon_0$, $i = 1, \ldots, n-1$ the vectors chosen such that $(\mathbf{a}_1^t(z), \ldots, \mathbf{a}_{n-1}^t(z)) = \nabla_z \mathbf{y}(z,t)$. Now the surface element on $\Gamma_2^t = \mathbf{y}(\mathscr{G}, t)$ can be expressed by (see, e.g., [21])

$$dS_t = \sqrt{\det\left[\mathbf{a}_i^t \cdot \mathbf{a}_j^t\right]}\, dz = \sqrt{\det[\mathbf{a}_i \cdot \mathbf{a}_j] + O(t)}\, dz. \tag{6}$$

For the ruffled boundary we can now see by a direct computation that (see also [22] for the 2D case or [8] for the case of plain Γ_2):

LEMMA 1. *Let dS_ε be the surface element on Γ_2^ε with $\varepsilon^m < \varepsilon_0$ and let dS be the surface element on Γ_2. Then*

$$dS_\varepsilon = \sqrt{1 + \varepsilon^{2(m-1)} G(z, z/\varepsilon)}\,(1 + o(\varepsilon))\, dS,$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to $z \in \mathscr{G}$ and

$$G(z, \xi) = \frac{h'(\xi)^2}{|\mathbf{a}_1(z)|^2} \quad \text{if } n = 2$$

$$G(z,\xi) = \frac{|\mathbf{a}_1(z)|^2\left(\dfrac{\partial h(\xi)}{\partial \xi_2}\right)^2 + |\mathbf{a}_2(z)|^2\left(\dfrac{\partial h(\xi)}{\partial \xi_1}\right)^2 - \mathbf{a}_1(z)\cdot\mathbf{a}_2(z)\dfrac{\partial h(\xi)}{\partial \xi_1}\dfrac{\partial h(\xi)}{\partial \xi_2}}{\det\left[\mathbf{a}_i(z)\cdot\mathbf{a}_j(z)\right]} \quad \text{if } n = 3.$$

Proof. In case $n = 2$ we have

$$dS_\varepsilon = |\mathbf{y}_\varepsilon'(z)| = \left(|\mathbf{x}'(z)|^2 + \varepsilon^{2(m-1)} h'(z/\varepsilon)^2 + O(\varepsilon^m)\right)^{1/2} dz.$$

For $n = 3$ we obtain

$$\begin{aligned} dS_\varepsilon &= \sqrt{\det\left[\frac{\partial \mathbf{y}_\varepsilon}{\partial z_i} \cdot \frac{\partial \mathbf{y}_\varepsilon}{\partial z_j}\right]}\, dz \\ &= \left\{\det[\mathbf{a}_i \cdot \mathbf{a}_j] + \varepsilon^{2(m-1)}\left[\left(\frac{\partial h}{\partial \xi_1}(z/\varepsilon)\right)^2 |\mathbf{a}_2|^2 + \left(\frac{\partial h}{\partial \xi_2}(z/\varepsilon)\right)^2 |\mathbf{a}_1|^2 \right.\right. \\ &\qquad \left.\left. - \mathbf{a}_1 \cdot \mathbf{a}_2 \frac{\partial h}{\partial \xi_1}(z/\varepsilon)\frac{\partial h}{\partial \xi_2}(z/\varepsilon)\right] + O(\varepsilon^{3m-2})\right\}^{1/2} dz. \end{aligned}$$

∎

Remark 2. When the basis $\mathbf{a}_i$ is orthonormal (as for instance in [22] where the natural parameterization was used or in [8], where Γ_2 is plain, parameterized by identity $\mathbf{x}(z) = z$), function $G(z, \xi) = G(\xi) = |\nabla h(\xi)|^2$. In case $n = 2$ we can choose such natural parameterization (i.e., parameterization by arc length) for any smooth Γ_2 to have $|\mathbf{x}'| = 1$ giving $G = |\nabla h|^2$. In case $n = 3$, locally, parameterization can be chosen such that the metric tensor $[\mathbf{a}_i \cdot \mathbf{a}_j]$ is scalar; i.e., it has the form $g(z)\,\mathbf{I}$ (see, e.g., [13], Théorème 1, p. 116). With such choice we also get $G = |\nabla h|^2$.

Following [8] (Lemma 1), with a slight modification due to the curved boundary, we get the following:

LEMMA 2. *There exists a constant $C_1 > 0$ and $\varepsilon_1 > 0$ such that*

$$\big(\forall \varepsilon \in]0, \varepsilon_1[\big)\big(\forall \varphi \in H^1(\Omega_\varepsilon)\big)\Big(\|\varphi\|_{L^2(\Omega_\varepsilon \setminus \Omega)} \leq C_1 \varepsilon^{m/2} \|\varphi\|_{H^1(\Omega_\varepsilon)}\Big). \tag{8}$$

LEMMA 3 (Poincaré's inequality). *There exists a constant $C_2 > 0$ and $\varepsilon_1 > 0$ such that*

$$\big(\forall \varepsilon \in]0, \varepsilon_1[\big)\big(\forall \varphi \in V_\varepsilon\big)\big(\|\varphi\|_{H_\varepsilon} \leq C_2 \|\varphi\|_{V_\varepsilon}\big). \tag{9}$$

Proof. Since $\varphi = 0$ on Γ_1, for the restriction of φ on Ω we have

$$\|\varphi\|_{L^2(\Omega)^n} \leq C(\Omega) \|\nabla \varphi\|_{L^2(\Omega)^{n^2}} \leq C(\Omega) \|\nabla \varphi\|_{L^2(\Omega_\varepsilon)^{n^2}}. \tag{10}$$

On $\Omega_\varepsilon \setminus \Omega$ we have, using Lemma 2,

$$\begin{aligned} \|\varphi\|^2_{L^2(\Omega_\varepsilon)^n} &= \|\varphi\|^2_{L^2(\Omega)^n} + \|\varphi\|^2_{L^2(\Omega_\varepsilon \setminus \Omega)} \\ &\leq C(\Omega)^2 \|\nabla \varphi\|^2_{L^2(\Omega)^n} + C_1^2 \varepsilon^m \|\varphi\|^2_{H^1(\Omega_\varepsilon)^n} \\ &\leq \big(C(\Omega)^2 + C_1^2 \varepsilon^m\big) \|\nabla \varphi\|^2_{L^2(\Omega_\varepsilon)^{n^2}} + C_1^2 \varepsilon^m \|\varphi\|^2_{L^2(\Omega_\varepsilon)^n}. \end{aligned} \tag{11}$$

Now (11) implies

$$\|\varphi\|^2_{L^2(\Omega_\varepsilon)^n} \leq \frac{C(\Omega)^2 + C_1^2 \varepsilon^m}{1 - C_1^2 \varepsilon^m} \|\nabla \varphi\|^2_{L^2(\Omega_\varepsilon)^{n^2}}$$

which proves the claim with $C_2^2 = 2(\Omega)^2 + 1$ and $\varepsilon_1 = (2C_1^2)^{-1/m}$. ∎

5. EXISTENCE, UNIQUENESS, AND REGULARITY

In this section we state some existence, uniqueness, and regularity results for Problems (1)–(3). Since the proofs follow exactly the same lines as for the similar results proved in Temam [23] for the case of the Dirichlet's boundary conditions we leave them as an easy exercise to the reader.

THEOREM 1. *Suppose that f^ε and g^ε satisfy assumptions* (i) *and* (ii). *Then Problems* (1)–(3) *admit a solution $u^\varepsilon \in V_\varepsilon$, $p^\varepsilon \in L^2(\Omega_\varepsilon)/\mathbf{R}$. In addition, there exists a constant $M > 0$ (independent on ε but depending on κ_0 and μ) such that, if*

$$\|f^\varepsilon\|_{V'_\varepsilon} + \|g^\varepsilon\|_{L^2(\Gamma_2^\varepsilon)} < M, \tag{12}$$

then the solution is unique.

6. A PRIORI ESTIMATES

Supposing that Assumptions (i) and (ii), including Assumption (12), hold. Using the results from the previous section we derive the a priori estimates that will play the essential role in passing to the limit as $\varepsilon \to 0$ in our system (1)–(3). In this section, C denotes a positive constant not depending on ε and, possibly, taking different values in different places.

LEMMA 4.

$$\|u^\varepsilon\|_{V_\varepsilon}^2 \le \mu^{-2}\|f^\varepsilon\|_{V'_\varepsilon}^2 + \varepsilon^k(\kappa_0\,\mu)^{-1}\|g^\varepsilon\|_{L^2(\Gamma_2^\varepsilon)}^2 \tag{13}$$

$$\varepsilon^k\|u^\varepsilon\|_{L^2(\Gamma_\varepsilon^2)^n}^2 \le (\kappa_0\,\mu)^{-1}\|f^\varepsilon\|_{V'_\varepsilon}^2 + \varepsilon^k\kappa_0^{-2}\|g^\varepsilon\|_{L^2(\Gamma_2^\varepsilon)}^2. \tag{14}$$

Proof. We choose u^ε as test function in (4) and we obtain

$$\mu\|u^\varepsilon\|_{V_\varepsilon}^2 + \varepsilon^k\int_{\Gamma_2^\varepsilon}\kappa|u^\varepsilon|^2 = {}_{V'_\varepsilon}\langle f^\varepsilon \mid u^\varepsilon\rangle_{V_\varepsilon} + \varepsilon^k\int_{\Gamma_2^\varepsilon} g^\varepsilon u^\varepsilon$$

implying the claim. ∎

Our next step is to estimate the restriction of the pressure on Ω.

LEMMA 5.

$$\|p^\varepsilon\|_{L^2(\Omega)/\mathbf{R}} \le C, \tag{15}$$

with C depending on $\|f^\varepsilon\|_{V'_\varepsilon}$, $\|g^\varepsilon\|_{L^2(\Gamma_2^\varepsilon)^n}$.

Proof. We define ϕ as a solution of the problem

$$\begin{cases} \operatorname{div}\phi = \tilde{p}^\varepsilon \text{ in } \Omega \\ \phi = 0 \text{ on } \partial\Omega, \end{cases}$$

such that

$$\|\nabla\phi\|_{L^2(\Omega)^n} \le C\|\tilde{p}^\varepsilon\|_{L^2(\Omega)},$$

where $\tilde{p}^\varepsilon = p^\varepsilon - \frac{1}{|\Omega|}\int_\Omega p^\varepsilon\,dx$. Using ϕ as a test function in our Problem (1) we get

$$\mu\int_{\Omega_\varepsilon}\nabla u^\varepsilon\,\nabla\phi + \int_{\Omega_\varepsilon}(u^\varepsilon\nabla)u^\varepsilon\phi = {}_{V'_\varepsilon}\langle f^\varepsilon \mid \phi\rangle_{V_\varepsilon} + \|\tilde{p}^\varepsilon\|^2_{L^2(\Omega)}.$$

The above estimate leads to

$$\|\tilde{p}^\varepsilon\|_{L^2(\Omega))} \le C\Big\{\mu\|u^\varepsilon\|_{V_\varepsilon} + \|f^\varepsilon\|_{V'_\varepsilon} + \|\nabla u^\varepsilon\|^2_{L^2(\Omega)}\Big\}\|\tilde{p}^\varepsilon\|_{L^2(\Omega)},$$

giving (15). ∎

7. THE CONVERGENCE

In order to have a uniform estimate (with respect to ε) for $(u^\varepsilon, p^\varepsilon)$ we suppose that

$$\|f^\varepsilon\|_{V'_\varepsilon} \le C, \tag{16}$$

$$\varepsilon^k\|g^\varepsilon\|_{L^2(\Gamma_2^\varepsilon)^n} \le C. \tag{17}$$

This implies that the constants in estimates (13)–(15) are independent from ε, i.e., that the restrictions of $(u^\varepsilon, p^\varepsilon)$ on Ω satisfy

$$\|u^\varepsilon\|_{H^1(\Omega)^n}, \|p^\varepsilon\|_{L^2(\Omega)/\mathbf{R}} \le C, \tag{18}$$

with C independent from ε. Thus there exists a subsequence $\{u^\varepsilon, p^\varepsilon\}_{\varepsilon>0}$ and $(u \in H^1(\Omega)^n,\ p \in L^2(\Omega)/\mathbf{R})$ such that, as $\varepsilon \to 0$,

$$\begin{cases} u^\varepsilon \rightharpoonup u \text{ weakly in } H^1(\Omega)^n \\ p^\varepsilon \rightharpoonup p \text{ weakly in } L^2(\Omega)/\mathbf{R}. \end{cases} \tag{19}$$

In analogy with Section 2 we introduce the functional spaces

$$V_0 = \Big\{v \in H^1(\Omega)^n, \operatorname{div} v = 0, v = 0 \text{ on } \Gamma_1, v\cdot\mathbf{n} = 0 \text{ on } \Gamma_2\Big\},$$

$$\Gamma_2 = \partial\Omega \setminus \Gamma_1$$

$$H_0 = \Big\{v \in L^2(\Omega)^n, \operatorname{div} v = 0, v\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\Big\},$$

$$V_{\Gamma_1} = \Big\{v \in H^1(\Omega)^n, \operatorname{div} v = 0, v = 0 \text{ on } \Gamma_1\Big\} \subset V_0$$

equipped by the norms

$$\|v\|_{V_0} = \|v\|_{V_{\Gamma_1}} = \|\nabla v\|_{L^2(\Omega)^{n^2}}, \|v\|_{H_0} = \|v\|_{L^2(\Omega)^n}.$$

In order to pass to the limit, as $\varepsilon \to 0$, in (4), we assume (in addition) that the restrictions of f^ε on Ω and g^ε satisfy

$$f^\varepsilon \to f^0 \text{ strongly in } V'_{\Gamma_1} \tag{20}$$

$$\bar{g}^\varepsilon \to g^0 \text{ strongly in } L^2(\Gamma_2)^n, \tag{21}$$

where $\bar{g}^\varepsilon[\mathbf{y}(z,0)] = g^\varepsilon\{\mathbf{y}[z, \varepsilon^m h^\varepsilon(z)]\}$ for $z \in \mathscr{G}$, i.e., for $\mathbf{x} = \mathbf{y}(z,0) \in \Gamma_2$ and $\mathbf{y}_\varepsilon = \mathbf{y}[z, \varepsilon^m h(z/\varepsilon)] \in \Gamma_2^\varepsilon$. As $\operatorname{div} u^\varepsilon \rightharpoonup \operatorname{div} u$ and for any $\phi \in C^1(\mathbf{R}^n)$ one has

$$0 = \int_{\Omega_\varepsilon} u^\varepsilon \nabla\phi \to \int_\Omega u\nabla\phi = \int_{\Gamma_2} (u \cdot \mathbf{n})\,\phi$$

obviously $u \in V_0$.

We can now state our main result:

THEOREM 2. *The functions (u, p) are the solution to the problem*

$$(u\nabla)u - \mu\Delta u + \nabla p = f^0, \operatorname{div} u = 0 \text{ in } \Omega \tag{22}$$

$$u = 0 \text{ on } \Gamma_1, u \cdot \mathbf{n} = 0 \text{ on } \Gamma_2 \tag{23}$$

$$(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\,\mu\frac{\partial u}{\partial \mathbf{n}} + \kappa\lambda u = \lambda g^0 \text{ on } \Gamma_2, \tag{24}$$

where

1. $\lambda = \int_Y \sqrt{1 + |G(z,\xi)|}\, d\xi$ *if* $m = 1$, $k = 0$, (*note that* $\lambda > 1$ *and that in cases described in Remark* 2, λ *is, in fact, equal to the area of one cog*).
2. $\lambda = \int_Y |G(z,\xi)|^{1/2}\, d\xi$ *if* $0 < m < 1$, $k = 1 - m$ (*note that* $\lambda \geq h_0 > 0$, *where* $h_0 = |\mathbf{a}_1|^{-1} \int_Y |h'|$, *in case* $n = 2$ *and* $h_0 = (2 \det[\mathbf{a}_i \cdot \mathbf{a}_j])^{-1/2} \int_Y (|\mathbf{a}_2|^2 |\partial_1 h|^2 + |\mathbf{a}_1|^2 |\partial_2 h|^2)^{1/2}$, *for* $n = 3$).
3. $\lambda = 1$ *if* $k = 0$, $m > 1$,
4. $\lambda = 0$, *if* $k > 0$, $k + m > 1$.

Moreover, if Assumptions (*i*) *and* (*ii*) *and* (12) *with* f^0, λg^0, $\lambda\kappa$ *in place of* f^ε, g^ε, κ *hold, then* (22)–(24) *have a unique solution and the whole sequence* $\{(u^\varepsilon, p^\varepsilon)\}_{\varepsilon > 0}$ *converges to* (u, p).

We first prove the easy part, that the limit (u, p) satisfies the system (22) and the boundary condition on Γ_1.

LEMMA 6. *There exists a subsequence, denoted again by* $\{u^\varepsilon, p^\varepsilon\}_{\varepsilon > 0}$, *converging in the sense of* (19) *to the limit* (u, p) *that satisfies the system* (22) *and the boundary conditions* (23).

Proof. (i) We first verify the equations. For $\varphi \in C_0^\infty(\Omega)^n$ we get from (1)

$$\int_\Omega (u^\varepsilon \nabla) u^\varepsilon \varphi + \mu \int_\Omega \nabla u^\varepsilon \, \nabla \varphi = {}_{V'_\varepsilon}\langle f^\varepsilon \mid \varphi \rangle_{V_\varepsilon} + \int_\Omega p^\varepsilon \operatorname{div} \varphi .$$

Using (19)–(21) and, if necessary, extracting the subsequence, we pass to a limit as $\varepsilon \to 0$ we find that (u, p) satisfy

$$(u\nabla)u - \mu\Delta u + \nabla p = f^0, \operatorname{div} u = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma_1, \; u \cdot \mathbf{n} = 0 \text{ on } \Gamma_2 .$$

∎

It only remains to prove that u satisfies the boundary condition (24) on Γ_2. The trouble is that, if we want to use the energy method, we need a test function that has a bounded gradient and has a normal component equal to 0 on Γ_2 as well as on Γ_2^ε. Such construction is complicated and it seems to be more convenient to use the Γ convergence introduced by De Giorgi and Franzoni in [12] (see also [11] as a general reference on Γ-convergence). In fact we use its modification to variable domains given by Anzellotti, Baldo, and Percivale in [4]. We take Definition 2.1 from [4], with $X_\varepsilon = L^2(\Omega_\varepsilon)^n$, $Y = L^2(\Omega)^n$ with strong topology and $q_\varepsilon : X_\varepsilon \to Y$ defined as a simple restriction $q_\varepsilon(v) = v|_\Omega$. In that case their definition reads as follows:

DEFINITION 1. We say that the sequence $\{F_\varepsilon\} : L^2(\Omega_\varepsilon)^n \to \mathbf{R}$ is $\Gamma(L^2)$ —converging to a functional $F_0 : L^2(\Omega)^n \to \mathbf{R}$ at point $\varphi \in L^2(\Omega)^n$, and we write

$$\Gamma(L^2) - \lim_{\varepsilon \to 0} F_\varepsilon = F_0 \; in \; \varphi$$

if:

(a) For each sequence $\{\psi_\varepsilon\}$, $\psi_\varepsilon \in L^2(\Omega_\varepsilon)^n$, such that $\psi_\varepsilon|_\Omega \to \varphi$ in $L^2(\Omega)^n$, we have

$$\liminf_{\varepsilon \to 0} F_\varepsilon(\psi_\varepsilon) \geq F_0(\psi). \tag{25}$$

(b) There exists a sequence $\{\psi_\varepsilon\}$, $\psi_\varepsilon \in L^2(\Omega_\varepsilon)^n$, such that $\psi_\varepsilon|_\Omega \to \varphi$ in $L^2(\Omega)^n$ and

$$\limsup_{\varepsilon \to 0} F_\varepsilon(\psi_\varepsilon) \leq F_0(\varphi). \tag{26}$$

We say that

$$\Gamma(L^2) - \lim_{\varepsilon \to 0} F_\varepsilon = F_0 \text{ on } L^2(\Omega)^n$$

if

$$\Gamma(L^2) - \lim_{\varepsilon\to 0} F_\varepsilon = F_0 \text{ in } \varphi$$

for every $\varphi \in L^2(\Omega)^n$.

The main property of such convergence is that the $\Gamma(L^2)$-convergence of functionals implies the convergence of their minima and their minimisers. We recall the following proposition (see, e.g., [4, Proposition 2.4]):

PROPOSITION 1. *Let*

$$\Gamma(L^2) - \lim_{\varepsilon\to 0} F_\varepsilon = F_0 \text{ on } L^2(\Omega)^n.$$

Let there exist a compact set $K \subset L^2(\Omega)^n$ such that

$$\inf_{K_\varepsilon} F_\varepsilon = \inf_{L^2(\Omega_\varepsilon)^n} F_\varepsilon, \tag{27}$$

where $K_\varepsilon = \{\varphi \in L^2(\Omega_\varepsilon)^n;\ \varphi|_\Omega \in K\}$ (so called L^2-equicoercivity). Then we have

$$\min_{L^2(\Omega)^n} F_0 = \lim_{\varepsilon\to 0}\left(\inf_{L^2(\Omega_\varepsilon)^n} F_\varepsilon\right).$$

Furthermore, if $\{\varphi_\varepsilon\}$ are the minimisers of F_ε and $\varphi_\varepsilon|_\Omega \to \varphi$ in $L^2(\Omega)^n$, then φ is a minimiser of F_0 and

$$F_\varepsilon(\varphi_\varepsilon) \to F_0(\varphi).$$

Proof of Theorem 2. Let $\{u^\varepsilon\}_{\varepsilon>0}$ be the subsequence from Lemma 6 and u its limit. We define four functionals

$$\Phi^\varepsilon(\varphi) = \frac{\mu}{2}\int_{\Omega_\varepsilon} |\nabla\varphi|^2 + \frac{\varepsilon^k}{2}\int_{\Gamma_2^\varepsilon} \kappa\varphi^2,\ \varphi \in V_\varepsilon$$

$$\Psi^\varepsilon(\varphi) = -\int_{\Omega_\varepsilon} (u^\varepsilon\nabla)u^\varepsilon\varphi + {}_{V'_\varepsilon}\langle f^\varepsilon \mid \varphi\rangle_{V_\varepsilon} + \varepsilon^k\int_{\Gamma_2^\varepsilon} g^\varepsilon\varphi,\ \varphi \in V_\varepsilon$$

$$\Phi^0(\varphi) = \frac{\mu}{2}\int_{\Omega} |\nabla\varphi|^2 + \frac{1}{2}\int_{\Gamma_2} \kappa\lambda\varphi^2,\ \varphi \in V_0$$

$$\Psi^0(\varphi) = -\int_{\Omega} (u\nabla)u\varphi + {}_{V'_{\Gamma_1}}\langle f^0 \mid \varphi\rangle_{V_{\Gamma_1}} + \int_{\Gamma_2} \lambda g^0\varphi,\ \varphi \in V_0.$$

Functions u^ε and u can now be seen as the solutions of the variational problems

$$\inf_{v \in V_\varepsilon} \left[\Phi^\varepsilon(v) - \Psi^\varepsilon(v) \right] \tag{28}$$

$$\inf_{w \in V_0} \left[\Phi^0(w) - \Psi^0(w) \right], \tag{29}$$

respectively. It is important to notice that u^ε and u in the definition of Ψ^ε and Ψ^0 are fixed (u^ε chosen as the converging subsequence from Lemma 6 and u as its limit) and the above infima are taken for those fixed Ψ^ε and Ψ^0. For any V_ε-bounded sequence $\varphi^\varepsilon \in V_\varepsilon$ such that $\varphi^\varepsilon|_\Omega \rightharpoonup \varphi$ weakly in $H^1(\Omega)$ we have

$$\Psi^\varepsilon(\varphi^\varepsilon) \to \Psi^0(\varphi), \text{ as } \varepsilon \to 0. \tag{30}$$

Indeed, we have proved the convergence for the first three integrals in Ψ^ε in the proof of Lemma 6. It only remains to prove the convergence for the last one. Due to (21) and the compactness of the trace from $H^1(\Omega)$ to $L^2(\Gamma_2)^n$, we obtain

$$\begin{aligned}
\varepsilon^k \int_{\Gamma_2^\varepsilon} g^\varepsilon \varphi^\varepsilon &= \varepsilon^k \int_{\mathscr{G}} g^\varepsilon [\mathbf{y}(z, \varepsilon^m h(z/\varepsilon))] \varphi^\varepsilon [\mathbf{y}(z, \varepsilon^m h(z/\varepsilon))] \\
&\qquad \times \sqrt{1 + \varepsilon^{2(m-1)} |G(z, z/\varepsilon)|}\,(1 + o(\varepsilon))\, dS \\
&= \int_{\mathscr{G}} \bar{g}^\varepsilon \varphi^\varepsilon [\mathbf{y}(z,0)]\, \lambda_\varepsilon\, dS \\
&\qquad + \int_{\mathscr{G}} \bar{g}^\varepsilon \int_0^{\varepsilon^m h(z/\varepsilon)} \frac{\partial}{\partial t} \varphi^\varepsilon [\mathbf{y}(z,t)]\, dt \lambda_\varepsilon\, dS + o(\varepsilon) \\
&= I^1 + I^2 + o(\varepsilon),
\end{aligned} \tag{31}$$

where $\lambda_\varepsilon(z) = \varepsilon^k \sqrt{1 + \varepsilon^{2(m-1)} |G(z, z/\varepsilon)|}$,

$$I^1 = \int_{\mathscr{G}} \bar{g}^\varepsilon \varphi^\varepsilon [\mathbf{y}(z,0)]\, \lambda_\varepsilon\, dS$$

and

$$I^2 = \int_{\mathscr{G}} \bar{g}^\varepsilon \int_0^{\varepsilon^m h(z/\varepsilon)} \frac{\partial}{\partial t} \varphi^\varepsilon [\mathbf{y}(z,t)]\, dt \lambda_\varepsilon\, dS.$$

Since the Jacobian $J^t = \det([\mathbf{a}_i \cdot \mathbf{a}_j]_{i,j=1,\dots,n})$, $\mathbf{a}_n = \mathbf{n}$, satisfies the uniform (with respect to t) estimate $c_0 \le |J^t| \le c_1$, we have

$$|I^2| \le C \int_{\mathscr{G}} \int_0^{\varepsilon^m h(z/\varepsilon)} |\nabla \varphi^\varepsilon|\, |J^t|\, dt\, dS = C \int_{\Omega_\varepsilon \setminus \Omega} |\nabla \varphi^\varepsilon| \to 0.$$

On the other hand, the trace operator $\gamma^1 : H^1(\Omega)^n \to L^2(\Gamma_2)^n$, $\gamma^1(\varphi) = \varphi|_{\Gamma_2}$ is compact and therefore $\gamma^1(\varphi^\varepsilon) \to \gamma^1(\varphi)$ strongly in $L^2(\Gamma_2)^n$. The periodicity lemma (see, for example, Bensoussan, Lions, and Papanicolaou [6]) gives

$$\lambda_\varepsilon \rightharpoonup \lambda \text{ weak* in } L^\infty(\mathscr{G}) \text{ if } m + k = 1.$$

On the other hand, obviously

$$\lambda_\varepsilon \to \lambda \text{ strongly in } L^\infty(\mathscr{G}) \text{ if } m + k > 1,$$

where

$$\lambda(z) = \begin{cases} \int_Y (1 + G(z,\xi))^{1/2}\, d\xi & \text{if } m = 1, k = 0 \\ \int_Y |G(z,\xi)|^{1/2}\, d\xi & \text{if } m + k = 1, k > 0 \\ 1 & \text{if } m > 1, k = 0 \\ 0 & \text{if } m + k > 1, k > 0. \end{cases}$$

Thus, omitting the notation γ^1, we have

$$I^1 = \int_{\Gamma_2} \varphi^\varepsilon \bar{g}^\varepsilon \lambda_\varepsilon \to \int_{\Gamma_2} \lambda g \varphi.$$

Our next step is to prove that the functional defined by

$$F^\varepsilon(v) = \begin{cases} \Phi^\varepsilon(v) - \Psi^\varepsilon(v) & \text{for } v \in V_\varepsilon \\ +\infty & \text{otherwise} \end{cases}$$

converges in the $\Gamma(L^2)$ sense to the functional

$$F^0(v) = \begin{cases} \Phi^0(v) - \Psi^0(v) & \text{for } v \in V_0 \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\{v^\varepsilon\}_{\varepsilon > 0}$ be a sequence in $L^2(\Omega_\varepsilon)^n$ with restrictions on Ω converging to some v^0 in $L^2(\Omega)^n$. If

$$\liminf_{\varepsilon \to 0} F^\varepsilon(v^\varepsilon) = +\infty$$

then obviously

$$\liminf_{\varepsilon \to 0} F^\varepsilon(v^\varepsilon) \geq F^0(v^0).$$

Otherwise, since

$$F^\varepsilon(v^\varepsilon) \geq \frac{\mu}{2}|v^\varepsilon|^2_{V_\varepsilon} + \frac{\kappa_0}{2}|v^\varepsilon|^2_{L^2(\Gamma_2^\varepsilon)} - C|v^\varepsilon|_{V_\varepsilon}$$

there exists a V_ε-bounded subsequence, denoted again by $\{v^\varepsilon\}$, such that

$$v^\varepsilon|_\Omega \rightharpoonup v^0 \text{ weakly in } V_0 \text{ and } v^\varepsilon \to v^0 \text{ strongly in } L^2(\Gamma_2)^n.$$

As in (31) we prove that

$$\varepsilon^k \int_{\Gamma_2^\varepsilon} \kappa |v^\varepsilon|^2 \to \int_{\Gamma_2} \kappa\lambda |v^0|^2.$$

The convexity of V_ε norm implies that

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 \geq \int_\Omega |\nabla v^0|^2$$

so that (30) implies

$$\liminf_{\varepsilon\to 0} F^\varepsilon(v^\varepsilon) \geq F^0(v^0).$$

We need to verify condition (b) in the definition of Γ-convergence. If $v^0 \notin V_0$ the construction is obvious. Let $v^0 \in V_0$ and let $\{v^\varepsilon\}$ be a V_ε-bounded sequence having restrictions on Ω that converge strongly to v^0 in V_0. Then

$$\varepsilon^k \int_{\Gamma_2^\varepsilon} \kappa |v^\varepsilon|^2 \to \int_{\Gamma_2} \kappa\lambda |v^0|^2$$

and

$$\int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 \to \int_\Omega |\nabla v^0|^2.$$

But then

$$F^\varepsilon(v^\varepsilon) \to F^0(v^0).$$

We have proved that

$$\Gamma(L^2) - \lim_{\varepsilon\to 0} F^\varepsilon = F^0 \text{ in } L^2(\Omega)^n.$$

The a priori estimate (13) implies that the infimum of F^ε is attained in a ball in $H^1(\Omega_\varepsilon)^n$ with some radius R, i.e., in $K_\varepsilon = \{u \in L^2(\Omega_\varepsilon)^n;\ u|_\Omega \in K\}$, with $K_\varepsilon = \{u \in H^1(\Omega)^n;\ \|u\|_{H^1(\Omega)} \leq R\}$. As $H^1(\Omega)^n$ is compactly embedded in $L^2(\Omega)^n$ the conditions of Proposition 1 are fulfilled. On the other hand we know that $u^\varepsilon|_\Omega \to u$ so that u is the (unique) minimiser of F^0.

The corresponding Euler's equation for (29) is

$$\mu\int_{\Omega}\nabla u\,\nabla\varphi+\int_{\Gamma_2}\lambda\kappa u\varphi+\int_{\Omega}(u\nabla)u\varphi={}_{V'_{\Gamma_1}}\langle f^0\mid\varphi\rangle_{V_{\Gamma_1}}+\int_{\Gamma_2}\lambda g^0\varphi,\ \forall\varphi\in V_0,$$

which proves our claim. ∎

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