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Bounds on the Number of Affine, Symmetric, and Hadamard Designs and Matrices

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Lower bouds on the number of non-isomorphic embeddings of a symmetric net into affine designs with classical parameters, of an affine design into symmetric designs with classical parameters, and of a symmetric Hadamard design of order n into ones of order $2n$ are obtained. The bound of Jungnickel on the number of affine 2-$(q^d, q^{d-1}, (q^{d-1}-1)/(q-1))$ designs $(d \geqslant 3)$ that contain the classical (q, q^{d-2})-net is improved by a factor of $q^{3+4+\cdots+d}(q-1)^{d-2}$. Similarly, the bound of Jungnickel for the number of symmetric 2-$((q^{d+1}-1)/(q-1), (q^d-1)/(q-1), (q^{d-1}-1)/(q-1))$ designs $(d \geqslant 3)$ that contain the the classical affine design $AG(d, q)$ as a residual design is improved to match that of Kantor. Furthermore, for d large and by starting with rigid symmetric and affine designs, the lower bound for the number of non-isomorphic symmetric 2-$((q^{d+1}-1)/(q-1), (q^d-1)/(q-1), (q^{d-1}-1)/(q-1))$ designs is improved to $(q^{d-1}+\cdots+q)!$. By using the Paley design of order $n=(q+1)/4$, $q \equiv 3 \pmod 4$ a prime power, a lower bound for the number of Hadamard designs of order $q+1$ is also obtained. In particular, by choosing a non-classical net and non-classical affine design as the starting point, the bound on the number of symmetric 2-(40, 13, 4) designs is improved from 389 to 1, 108, 800, and the bound on the number of affine 2-(64, 16, 5) designs is improved from 157 to 10, 810, 800. A similar method also improves the number of non-isomorphic Hadamard 2-(31, 15, 7) designs from 1, 266, 891 to 11, 727, 788 and the number of non-isomorphic Hadamard 2-(39, 19, 9) designs from 38 to 5.87×10^{14}. The number of inequivalent Hadamard matrices of order 40 is at least 3.66×10^{11}.
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1. A GENERAL BOUND ON THE NUMBER OF COMPLETIONS

Let $D=(V, \mathscr{B})$ be a block design with point set V and collection of blocks $\mathscr{B}$. Let $\mathrm{Sym}(V)$ denote the symmetric group acting on the point set V. Two designs $D_1=(V, \mathscr{B}_1)$, $D_2=(V, \mathscr{B}_2)$ are isomorphic if there is an element in $\mathrm{Sym}(V)$ that maps $\mathscr{B}_1$ to $\mathscr{B}_2$. An *automorphism* of a design $D=(V, \mathscr{B})$ is any permutation from $\mathrm{Sym}(V)$ that preserves $\mathscr{B}$.

An incidence structure $S=(V', \mathscr{B}')$ is a substructure (or a subdesign) of $D=(V, \mathscr{B})$ if $V' \subseteq V$ and $\mathscr{B}' \subseteq \{B \cap V' \mid B \in \mathscr{B}\}$. The design D is then called a *completion* (or embedding) of S. Two designs D_1 and D_2 are *distinct* completions of S if they are both defined over the same point set V, both contain S as a substructure, but their collections of blocks are distinct. Note that distinct completions may or may not be isomorphic as designs.

Let D be a design with a subdesign S, and let $\mathrm{Aut}(D)$ denote the full automorphism group of D. We define $\mathrm{Aut}(S)$ as the subgroup of $\mathrm{Sym}(V)$ that stabilizes S. Note that $\mathrm{Aut}(S)$ depends only on S and is generally independent of the design D that S is obtained from.

Given an incidence structure S, we consider the set $\mathscr{C}$ of all distinct completions of S to a design with given parameters. We would like to obtain bounds for the the number of non-isomorphic completions amongst the $|\mathscr{C}|$ distinct completions.

The $|\mathscr{C}|$ distinct completions are partitioned into orbits under the action of $\mathrm{Aut}(S)$. Let D be the orbit representative of one such orbit and let $\mathrm{Aut}(S)_D$ be the subgroup of $\mathrm{Aut}(S)$ stabilizing D. The size of the orbit containing D is then $|\mathrm{Aut}(S)|/|\mathrm{Aut}(S)_D|$. Since the sum of the sizes of all the orbits is $|\mathscr{C}|$, we have

$$|\mathscr{C}| = |\mathrm{Aut}(S)| \sum_{D \in I} \frac{1}{|\mathrm{Aut}(S)_D|}, \tag{1}$$

where the summation is over all orbit representatives D of $\mathscr{C}$ under the action of $\mathrm{Aut}(S)$.

Since $\mathrm{Aut}(S)$ is only a subgroup of $\mathrm{Sym}(V)$, two completions D_1 and D_2 from different orbits under $\mathrm{Aut}(S)$ may still be isomorphic. Suppose there exists $\gamma \in \mathrm{Sym}(V)$ such that $D_1^{\gamma}=D$. Then S^{γ} is a substructure in D, which implies that D contains both S and S^{γ}. If there exists also a $\sigma \in \mathrm{Aut}(D)$ such that $S^{\gamma\sigma}=S$, then $\gamma\sigma$ takes D_1 to D and fixes S. That is, D_1 and D are in the same orbit under the action of $\mathrm{Aut}(S)$. Thus, let $D(S)$ be the subset of the orbit of S under $\mathrm{Sym}(V)$ that is contained in D. The number of orbits of $D(S)$ under $\mathrm{Aut}(D)$ is equal to the number of times an isomorphic image of D appears in distinct orbits of $\mathscr{C}$ under $\mathrm{Aut}(S)$. In particular,

$\mathrm{Aut}(D)$ acts transitively on $D(S)$ if and only if all isomorphic images of D appear in the same orbit of $\mathscr{C}$ under $\mathrm{Aut}(S)$. For later calculations, it is useful to observe that $\mathrm{Aut}(S)_D = \mathrm{Aut}(D)_S$ and that $|\mathrm{Aut}(D)| = |\mathrm{Aut}(S)_D| \times |S^{\mathrm{Aut}(D)}|$, where $S^{\mathrm{Aut}(D)}$ is the orbit of S under the action of $\mathrm{Aut}(D)$.

Equation (1) is the basis for testing the consistency of computer generated enumeration results. For example, see [7, 8]. However, it can also be used to derive a lower bound on the number of non-isomorphic completions of S.

THEOREM 1.1. *Let S be an incidence structure with automorphism group* $\mathrm{Aut}(S)$. *Let G be the largest subgroup which is contained in* $\mathrm{Aut}(S)_D$ *for all completions $D \in \mathscr{C}$. Denote by N the maximum number of orbits of $D(S)$ under* $\mathrm{Aut}(D)$ *over all possible completions D of S. Then there exist at least*

$$\frac{|\mathscr{C}| \cdot |G|}{|\mathrm{Aut}(S)| \cdot N} \tag{2}$$

non-isomorphic completions of S contained in $\mathscr{C}$.

Proof. It follows from (1) that $\mathscr{C}$ is partitioned into at least $|\mathscr{C}| \times |G| / |\mathrm{Aut}(S)|$ orbits. Since isomorphic copies of a completion D can occur in at most N of these orbits, the theorem follows.

2. AFFINE DESIGNS FROM NETS

The classical affine design $AG(d, q)$ is the 2-$(q^d, q^{d-1}, (q^{d-1}-1)/(q-1))$ design having as points and blocks the points and hyperplanes in the d-dimensional affine space over the field of order q (q a prime power). A symmetric (q, q^{d-2})-net is an incidence structure with q^d points and q^d blocks such that: (i) the blocks are partitioned into q^{d-1} parallel classes of size q so that any two distinct blocks from the same class are disjoint, while any two blocks from different parallel classes meet in exactly q points; (ii) the points are partitioned into q^{d-1} point classes (or lines) of size q so that any two distinct points from the same class do not occur together in any block, while any two points from different classes occur together in q blocks. Let D be an affine design with the parameters of an $\mathrm{AG}(d, q)$, $(d \geqslant 3)$, and let S be a (q, q^{d-2})-net contained in D. S can be completed to D by adjoining $q(q^{d-2} + \cdots + 1)$ blocks which are obtained from an affine design T with the parameters of $\mathrm{AG}(d-1, q)$, by associating a point of T with a point class in S, and by expanding a block in T into a block in D by replacing every point v' in T by the points in the point class that v' is associated with.

Different mappings of points in T to the point classes of S give potentially distinct completions. Following the notation in [4], we define $c(d, q)$ as the number of distinct completions of a given (q, q^{d-2})-net and $A(d, q)$ as the number of isomorphism classes of affine designs with the parameters of $\mathrm{AG}(d, q)$. In [4], Jungnickel proved that

$$c(d, q) = (q^{d-1})! \sum_T \frac{1}{|\mathrm{Aut}\, T|},$$

where T runs over a complete system of representatives of the isomorphism classes of affine designs with the parameters of $\mathrm{AG}(d-1, q)$. By using the fact that $\mathrm{Aut}(\mathrm{AG}(d-1, q)) = \mathrm{A}\Gamma\mathrm{L}(d-1, q)$, Jungnickel obtained

$$c(d, q) \geqslant \frac{(q^{d-1})!}{|\mathrm{A}\Gamma\mathrm{L}(d-1, q)|}. \tag{3}$$

In case that $\mathrm{AG}(d-1, q)$ is the unique (up to isomorphism) design with these parameters, then the above is an equality.

In [4], Jungnickel used the classical symmetric (q, q^{d-2}) net $(d \geqslant 3)$ as the substructure S. Its automorphism group $\mathrm{Aut}(S)$ is of order $|\mathrm{A}\Gamma\mathrm{L}(d, q)|(q-1)/(q^d-1)$ and N, the maximum number of times that S occurs in $AG(d, q)$, is $(q^d-1)/(q-1)$. Jungnickel obtained the following lower bound for the number $A(d, q)$ of isomorphism classes of affine designs with the parameters of $AG(d, q)$:

THEOREM 2.1 (Jungnickel [4]).

$$A(d, q) \geqslant \frac{(q^{d-1})!}{|\mathrm{A}\Gamma\mathrm{L}(d-1, q)|\;|\mathrm{A}\Gamma\mathrm{L}(d, q)|}. \tag{4}$$

Note that this bound is the same as the bound obtained from Theorem 1.1 by assuming that G is trivial. However, it was noted in [6] that, when S is the classical symmetric net, $\mathrm{Aut}(D)$ is non-trivial for all completions. In fact, the subgroup G of $\mathrm{Aut}(S)$ fixing all the point classes automatically fixes all the completions of S. Consider $D = AG(d, q)$ with $\mathrm{Aut}(D) = \mathrm{A}\Gamma\mathrm{L}(d, q)$, $(d \geqslant 3)$. Since $\mathrm{Aut}(D)$ acts transitively on the $(q^d-1)/(q-1)$ copies of S contained in D, $|\mathrm{Aut}(S)| = |\mathrm{A}\Gamma\mathrm{L}(d, q)|\,(q-1)/(q^d-1)$. The design T used in the construction of the completion D is $\mathrm{AG}(d-1, q)$. Let V' be the point set of T. Since $\mathrm{Aut}(S)$ fixes D, it also fixes T. Since there is a one-to-one association of the points in V' with the point classes of S, we can define an homomorphism ϕ from $\mathrm{Aut}(S)$ onto $Sym(V')$. The image $\phi(\mathrm{Aut}(S))$ is the automorphism group of $\mathrm{AG}(d-1, q)$, which is

$\mathrm{A}\Gamma\mathrm{L}(d-1, q)$. The kernel of ϕ is the subgroup G of $\mathrm{Aut}(S)$ fixing all point classes. Hence,

$$|G| = \frac{|\mathrm{A}\Gamma\mathrm{L}(d, q)|\,(q-1)}{(q^d-1)|\mathrm{A}\Gamma\mathrm{L}(d-1, q)|},$$

which simplifies to $q^d(q-1)$. Hence, by using (2), the Jungnickel bound (4) can be improved to the following

THEOREM 2.2.

$$A(d, q) \geqslant \frac{(q^{d-1})!\, q^d(q-1)}{|\mathrm{A}\Gamma\mathrm{L}(d-1, q)|\;|\mathrm{A}\Gamma\mathrm{L}(d, q)|}. \tag{5}$$

The additional factor $q^d(q-1)$ can also be used, as in [4, Th. 2.6], to recursively improve the lower bound.

THEOREM 2.3. *Define a function $a(d, q)$ recursively as follows. Put*

$$a(3, q) = \frac{q^3(q-1)(q^2)!}{f^2 q^9 (q-1)^2\,(q^2-1)^2\,(q^3-1)},$$

and, for $d \geqslant 4$,

$$a(d, q) = \frac{a(d-1, q)\, q^d(q-1)(q^{d-1})!}{f^2 q^{d^2}(q-1)^2\,(q^{d-1}-1)^2\,(q^d-1)},$$

where $q = p^f$ for a prime p. Then $A(d, q) \geqslant a(d, q)$.

Thus, when compared to [4, Th. 2.6], the bound for $A(d, q)$ is improved by a factor of $q^{3+4+\cdots+d}(q-1)^{d-2}$.

EXAMPLE 2.4. If $d=3, q=3$ both bounds (4), (5) give $A(3,3) \geqslant 1$. Actually, there are 68 non-isomorphic affine 2-(27,9,4) designs [10, 11], and 58 of these designs do contain (3,3)-nets [13].

EXAMPLE 2.5. Let $d=3$, $q=4$. The Jungnickel bound (4) is $A(3,4) \geqslant 157$. The improved bound from (5) is 30,030. In fact, an estimate for the number of non-isomorphic completions of the classical $(4,4)$-net to 2-$(64, 16, 5)$ designs is $30030 \cdot 21 \approx 6.3 \times 10^5$, by assuming that, for most completions, $|\mathrm{Aut}(S)_D| = q^d(q-1)$, the minimum possible, and that $D(S)$ is mostly trivial, and when it is non-trivial, $\mathrm{Aut}(D)$ acts transitively on it.

EXAMPLE 2.6. One can get a better lower bound on $A(3,4)$ by using a different symmetric net. The $(4,4)$-net listed in the Appendix has an automorphism group Aut(S) of order 64, and whose point classes are fixed by the cyclic group of order 4. From (3), $c(3,4)=3632428800$. Then (2) gives the lower bound

$$A(3,4) \geqslant (3632428800 \cdot 4)/(64 \cdot 21) = 10,810,800.$$

3. SYMMETRIC DESIGNS FROM AFFINE ONES

The classical symmetric design $PG(d,q)$ with parameters 2-$((q^{d+1}-1)/(q-1), (q^d-1)/(q-1), (q^{d-1}-1)/(q-1))$ has as blocks the hyperplanes in the d-dimensional projective space over the field of order q. Let D be a symmetric design with the parameters of $\mathrm{PG}(d,q)$ ($d \geqslant 3$), and let S be a residual affine design from D with the parameters of $\mathrm{AG}(d,q)$. S can be completed to D by enlarging the blocks of S by blocks of a symmetric design T with the parameters of $\mathrm{PG}(d-1,q)$. More specifically, points in T are associated with points of D not in S, and blocks in T are associated with parallel classes in S. All blocks of S in a parallel class are enlarged by including the points in the associated block of T.

Again, we let $c_s(d,q)$ denote the number of distinct completions of an affine design with the parameters of $\mathrm{AG}(d,q)$ to a symmetric design with the parameters of $\mathrm{PG}(d,q)$. In [4], Jungnickel proved that

$$c_s(d,q) = (q^{d-1} + \cdots + q + 1)!^2 \sum_T \frac{1}{\mathrm{Aut}\, T}, \tag{6}$$

where T runs over a complete system of representatives of symmetric designs with the parameters of $\mathrm{PG}(d-1,q)$. One factor of $(q^{d-1} + \cdots + q+1)!$ comes from the fact that the points of D not in S can be freely permuted to arrive at distinct completions. A similar factor appears in Aut(S) because, as a substructure of D, these $q^{d-1} + \cdots + q + 1$ points can be freely permuted.

If S is the classical design $\mathrm{AG}(d,q)$ ($d \geqslant 3$), then $|\mathrm{Aut}(S)| = (q^{d-1} + \cdots + q+1)!\, |\mathrm{A\Gamma L}(d,q)|$. Since D contains $q^d + \cdots + q + 1$ derived affine designs, $N \leqslant |D(S)| \leqslant q^d + \cdots + q + 1$. Furthermore, by using $\mathrm{PG}(d-1,q)$ as T in the completion, and by observing that $\mathrm{Aut}(T) = \mathrm{P\Gamma L}(d,q)$, Jungnickel obtained the following bound for the number $S(d,q)$ of isomorphism classes of symmetric designs with the parameters of $PG(d,q)$:

THEOREM 3.1. (Jungnickel [4]).

$$S(d,q) \geqslant \frac{(q^{d-1} + \cdots + q + 1)!}{|\mathrm{P\Gamma L}(d,q)|\, |\mathrm{P\Gamma L}(d+1,q)|}. \tag{7}$$

We note that (7) is derived by assuming that the orbit sizes of all completions D under the action of $\mathrm{Aut}(S)$ are of full length. However, the subgroup of $\mathrm{Aut}(S)$ stabilizing all the parallel classes fixes every completion D of S. By considering the special case where $D = \mathrm{PG}(d, q)$, and using an argument similar to the one in the previous section, one can show that this subgroup has size $q^d(q-1)$. Moreover, from [6, Prop. 3.6], $\mathrm{Aut}(D)$ acts transitively on $D(\mathrm{AG}(d, q))$. Thus, $N=1$ instead of the value of $q^d + \cdots + q + 1$ that Jungnickel used in deriving (7). With $|G| = 1$ and $N=1$, (2) gives the following improved bound, which matches the one by Kantor:

THEOREM 3.2 (Kantor [6]).

$$S(d, q) \geqslant \frac{(q^{d-1} + \cdots + q + 1)!}{|\mathrm{P}\Gamma\mathrm{L}(d, q)|^2}. \tag{8}$$

When d is large, one can use another theorem of Kantor's to get a better bound. From [6, Thm. 1.1], if $q > 3$ and $d \geqslant 50$, there exist rigid designs with the parameters of $\mathrm{PG}(d, q)$ and $\mathrm{AG}(d, q)$. With $d > 50$, let D be a rigid symmetric design with the parameters of $\mathrm{PG}(d-1, q)$, and let A be a rigid design with the parameters of $\mathrm{AG}(d, q)$. Equation (6) gives $c_s(d, q) \geqslant (q^{d-1} + \cdots + q + 1)!^2$. With $|G| = 1$, $N = q^{d-1} + \cdots + q + 1$, and $|\mathrm{Aut}(S)| = (q^{d-1} + \cdots + q + 1)!$, (2) gives:

THEOREM 3.3. *If $q > 3$ and $d > 50$, then $S(d, q) \geqslant (q^{d-1} + \cdots + q)!$.*

Next, we apply the theory to some specific examples.

EXAMPLE 3.4. The parameters $d = 3$ and $q = 3$ correspond to completing 2-(27, 9, 4) affine resolvable designs to symmetric 2-(40, 13, 4) designs. The old bound (7) gives $S(3, 3) \geqslant 1$. The improved bound (8) gives $S(3, 3) \geqslant 197$. Both bounds are based on completing $S = \mathrm{AG}(3, 3)$. By using the computer program BDX [9], we actually found 252 non-isomorphic completions of $AG(3, 3)$.

Note that the best known bound, obtained by a direct construction of 2-(40, 13, 4) designs, is 389 [12, 15]. However, we can choose design number 68 from the list in [10]. Its automorphism group as a 2-(27, 9, 4) design is trivial. As a substructure in a 2-(40, 13, 4) design, its automorphism group has size 13!. Since the unique $T = \mathrm{PG}(2, 3)$ has an automorphism group of size 5616, (6) gives $c_s(3, 3) = (13!)^2/5616$. Next, we claim that $N = 1$. Since S is a residual design, it is obtained by choosing a block of D. If the choice of a different block of D gives another residual design isomorphic to S, then firstly, this block contains a non-zero block B of S as a subset, and secondly, by using this block, the derived design again contains 3 copies of $\mathrm{PG}(2, 3)$. Since the derived design contains every block three times, the intersection patterns of B with other blocks of S must also repeat with a multiplicity of three. Note that for this multiplicity calculation, we have to include the empty intersection of B with the original defining block of D, which is technically not in S. With our chosen S, there is no block B whose intersections with the other blocks have the correct repetition pattern. Therefore, none of the blocks of S can be chosen to define a design which can be a completion of S.

By using $N=1$ and $|G|=1$, (2) gives the bound $S(3,3) \geqslant (13!)^2/(13! \cdot 5616)$, or

$$S(3,3) \geqslant 1,108,800$$

4. HADAMARD DESIGNS OF ORDER $2n$ FROM ONES OF ORDER n

A symmetric 2-$(4n-1, 2n-1, n-1)$ design is also called a Hadamard design of order n. Given two Hadamard designs H_1 and H_2 of order n, one can construct a Hadamard design of order $2n$, which is shown in Fig. 1 in the form of an incidence matrix. Here H_2^c denotes the complement of H_2.

The substructure S is the bottom $4n$ rows. It contains the residual design with an extra column of all zeros. The residual design is resolvable and quasi-symmetric with intersection numbers $\{0, 2n\}$. Let c_h be the number of distinct ways that S can be completed. With an argument similar to the case of completing affine designs, c_h is given by

$$c_h(8n-1) = ((4n-1)!)^2 \sum_T \frac{1}{|\mathrm{Aut}\ T|}, \tag{9}$$

where T runs over a complete system of representatives of symmetric 2-$(4n-1, 2n-1, n-1)$ designs. The size of $\mathrm{Aut}(S)$ is the product of $(4n-1)!$ with the size of the automorphism group of the residual design in S as a 2-$(4n, 2n, 2n-1)$ design. The maximum number of isomorphic S's

$$\begin{array}{|c|c|c|} \hline 1 & & \\ \cdot & H_1 & H_1 \\ 1 & & \\ \hline 0 & & \\ \cdot & H_2 & H_2^c \\ 0 & & \\ \hline 0 & 1\ldots1 & 0\ldots0 \\ \hline \end{array}$$

FIG 1. Hadamard designs from two smaller ones.

that can occur in D is $8n-1$. Thus, (2) can be used to give a lower bound for the number of non-isomorphic completions of a particular S.

Consider the special case where both H_1 and H_2 are the Paley designs. From [5], the automorphism group of a Paley design of order $q > 11$, where q is a prime power $p^f = 4n-1$, has order $fq(q-1)/2$. Thus, (9) gives the number of distinct completions as

$$c_h(8n-1) = \frac{(4n-1)!^2}{f(4n-1)(2n-1)}.$$

Now, by using $|G| = 1$, $N = 8n-1$, and $\mathrm{Aut}(S) = (4n-1)!\, fq(q-1)/2$ in (2), we get:

THEOREM 4.1. *If* $4n-1 = p^f$ *is the power of a prime* p *and* $4n-1 > 11$, *then the number of non-isomorphic Hadamard* 2-$(8n-1, 4n-1, 2n-1)$ *designs is at least*

$$\frac{(4n-1)!}{(8n-1)\, f^2(4n-1)^2\, (2n-1)^2}.$$

Using the standard relationship between Hadamard 2-$(8n-1, 4n-1, 2n-1)$ designs, Hadamard 3-$(8n, 4n, 2n-1)$ designs and Hadamard matrices, we get:

COROLLARY 4.2. *If* $4n-1 = p^f$ *is the power of a prime* p *and* $4n-1 > 11$, *then the number of non-isomorphic Hadamard* 3-$(8n, 4n, 2n-1)$ *designs is at least*

$$\frac{(4n-1)!}{8n(8n-1)\, f^2\, (4n-1)^2\, (2n-1)^2},$$

and the number of non equivalent Hadamard matrices of order $8n$ *is at least*

$$\frac{(4n-1)!}{(8n)^2\, (8n-1)\, f^2\, (4n-1)^2\, (2n-1)^2}.$$

As for some specific examples, consider the extensions of designs from 2-(15, 7, 3) to 2-(31, 15, 7) and from 2-(19, 9, 4) to 2-(39, 19, 9).

There are five non-isomorphic 2-(15, 7, 3) designs given in [3, p. 11]. The order of their automorphism groups are 20160, 576, 96, 168 and 168. Thus, the number of distinct completions from (9) is $c_h(31) = 15! \cdot 31524292800$. Any of these five designs can also be used as H_2 to define the substructure S. With a similar argument to the one used in the previous section, N is at most equal to the number of blocks of S whose intersections with the other blocks repeats with a multiplicity of two. The fifth design in [3, p. 11] has no blocks with the correct repetition pattern and its automorphism group as a residual design is $168*16=2688$. Using it to define S, we have $|\mathrm{Aut}(S)|=15!\cdot 2688$. From (2), the number of non-isomorphic 2-(31, 15, 7) designs is at least $15!\cdot 31524292800/(15!\cdot 2688)$ or 11,727,788 which is better than the best known bound of 1,266,891 [3, 14].

The six 2-(19, 9, 4) designs are given in [3, p. 11]. Their automorphism group orders are 8, 6, 72, 24, 9 and 171. The number of distinct completions from (9) is $c_h(39)=19!\cdot 56465379210240000$. We use the second design in [3, p. 11] as the H_2. Its automorphism group as a residual design is $6*16=96$, which implies $|\mathrm{Aut}(S)|=19!\cdot 96$. Also, $N=1$, because there exists no block whose intersections with other blocks repeats with a multiplicity of two. From (2), the number of non-isomorphic 2-(39, 19, 9) designs is at least 5.87×10^{14}, which is better than the best known bound of 38 [3, 2]. Correspondingly, the number of inequivalent Hadamard matrices of order 40 is at least $5.87\times 10^{14}/40^2=3.66\times 10^{11}$.

APPENDIX

Below the base blocks of a non-classical symmetric (4, 4) net S are listed. The full automorphism group of S is of order 64, while the subgroup that fixes every of the 16 parallel classes and every of the 16 point classes, is the cyclic group of order 4.

Generating permutation: $\pi=(1, 2, 3, 4)(5, 6, 7, 8)\ldots(61, 62, 63, 64)$

Base blocks:

1	5	9	13	17	21	25	29	33	37	41	45	49	53	57	61
1	5	9	13	18	22	26	30	35	39	43	47	52	56	60	64
1	5	9	13	19	23	28	32	34	38	44	48	50	54	59	63
1	5	9	13	20	24	27	31	36	40	42	46	51	55	58	62
1	8	11	14	17	22	25	31	36	39	41	48	52	54	59	62
1	8	11	14	20	23	27	29	33	38	42	47	50	56	60	61
1	8	10	15	27	23	25	32	35	40	43	46	50	53	58	64
1	8	10	15	19	21	26	31	34	37	44	45	51	56	60	62
1	7	12	14	19	24	26	29	36	38	43	45	52	53	58	63
1	7	12	14	18	21	27	32	34	40	41	47	51	54	57	64
1	7	10	16	17	24	27	30	35	37	42	48	49	54	60	63
1	7	10	16	18	23	28	29	33	39	44	46	52	55	57	62
1	6	12	15	20	22	25	30	36	37	44	47	50	55	57	63
1	6	12	15	18	24	28	31	35	38	41	46	49	56	59	61
1	6	11	16	20	21	26	32	34	39	43	48	49	55	58	61
1	6	11	16	19	22	28	30	33	40	42	45	51	53	59	64

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