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A Generalization of Sperner's Lemma

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ABSTRACT

A brief proof is given of a generalization of Sperner's lemma to certain finite partially ordered sets.

Sperner's lemma [4] states the following: In the Boolean lattice of subsets of a set T with n elements, any set S of pairwise non-comparable elements has at most ${}_nC_m$ elements, where $m = [n/2]$. Thus a largest such S is the set of all m-element subsets of T. The theorem below states a generalization of Sperner's lemma to a useful type of partially ordered set. For the case in which the partially ordered set is in addition a geometric lattice, this theorem has been proved by Dilworth (unpublished), via the Unimodal Property. For the same case, Harper [2] has shown how the theorem can be obtained as a consequence of the Normalized Matching Property for regular bipartite graphs. The much briefer direct proof supplied by this note stems from Lubell's proof of Sperner's lemma [3].

THEOREM. *Let Q be a finite partially ordered set with universal bounds O, I and with a rank function ρ, such that for each $k \geqslant 0$ all elements of rank k*

(i) *are covered by the same number a_k of elements of rank $k + 1$, and*

(ii) *cover the same number b_k of elements of rank $k - 1$. If $e(k)$ is the number of elements of rank k in Q, then any set S of pairwise non-comparable elements in Q has at most M elements, where $M = \max_k e(k)$.*

PROOF: For each k, let $E(k)$ be the set of elements of Q of rank k. For $x \in Q$, let $p(x)$ be the probability that a randomly chosen maximal chain of Q will pass through x. In other words, let $p(x) = s(x)/t$, where t is the number of all maximal chains of Q and $s(x)$ is the number of those which pass through x.

(*1*) A first observation: For any given k, $p(x)$ is the same for all $x \in E(k)$. Indeed, if $\rho(x) = k$, $s(x)$ is the product of the number $b_k \cdots b_1$ of maximal chains of the interval $[O, x]$ and the number $a_k \cdots a_{n-1}$ of maximal chains of the interval $[x, I]$. (Here $n = \rho(I)$.) Thus $p(x) = b_k \cdots b_1 a_k \cdots a_{n-1}/t$, which is constant on $E(k)$.

(*2*) A second observation: $p(x) \geqslant 1/M$. For a given k, each maximal chain of Q passes through exactly one element of $E(k)$. Addition of probabilities for mutually exclusive events then gives $1 = \sum_{x \in E(k)} p(x)$. Since there are $e(k)$ summands, the common value of the $p(x)$, $x \in E(k)$, must be $1/e(k)$. Thus $p(x) = 1/e(k) \geqslant 1/M$.

(*3*) Finally, we note that each maximal chain of Q passes through at most one element of S. Therefore, the probability that a given maximal chain passes through *some* element of S is again obtained by addition and is $\sum_{x \in S} p(x)$. Since this probability cannot exceed 1, we have

$$1 \geqslant \sum_{x \in S} p(x) \geqslant \sum_{x \in S} 1/M = |S|/M,$$

i.e., $|S| \leqslant M$, as required.

The above result applies to finite Boolean algebras, finite projective geometries, finite affine geometries, and many other geometric lattices [1, Ch. IV]. It is still a matter of conjecture whether the conclusion is true for *all* finite geometric lattices [2].

References

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4. E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27** (1928), 544–548.