THE ARF INVARIANT OF A TOTALLY PROPER LINK

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Suppose L is an oriented link in S^3 such that each pair of components of L link each other an even number of times. Then the Arf invariant of L is equal to the sum (mod 2) of the Arf invariants of all sublinks of L plus a certain coefficient of the Conway polynomial of L. This result extends the formula recently given by Murasugi in the case when L has two components.

Introduction

In a recent paper [4], Murasugi proves the following theorem.

Theorem. Let L = \{K_1, K_2\} be an oriented proper link in S^3. Then

$$\text{Arf}(L) = \text{Arf}(K_1) + \text{Arf}(K_2) + \frac{1}{2} \sum_{i=1} \left[ \frac{d^2}{dt^2} \Delta_L(t, t) \right] \mod 2,$$

where \(\Delta_L(x, y)\) is the (suitably normalized) Alexander polynomial of L.

In this paper we shall give an alternate and somewhat simpler proof of this theorem as well as generalize it to a proper link of any number of components, provided the link has the additional property that all of its sublinks are proper. This is equivalent to saying that the linking number between any pair of components of L is even. We call such a link totally proper. The theorem we prove however, will involve the Conway polynomial \(V_L(z)\) rather than the Alexander polynomial. If L is a link of n components then its Conway polynomial is of the form

$$V_L(z) = z^{n-1} [a_0 + a_1 z^2 + \cdots + a_m z^{2m}].$$

Let

$$\phi_1(f) = a_1 = \frac{1}{2} \left[ \frac{d^2}{dz^2} (V_L(z) / z^{n-1}) \right]_{z=0}.$$
Then we prove

**Theorem 4.1.** Let $L$ be a totally proper oriented link in $S^3$. Then

$$\text{Arf}(L) = \sum_{L' \subseteq L} \text{Arf}(L') + \phi_1(L) \pmod{2}$$

$$= \sum_{L' \subseteq L} \phi_1(L') \pmod{2}.$$  

Here the first sum is taken over all sublinks of $L$, excluding $L$ itself, while the second sum is over all sublinks including $L$.

When $L$ has two components this is equivalent to Murasugi's theorem because the Conway polynomial and the reduced Alexander polynomial are essentially related via a change of variables. Both polynomials correspond to the infinite cyclic cover of the link exterior $X$ obtained by cutting $X$ open along a Seifert surface for $L$ and gluing infinitely many copies of this space together end to end.

There are, perhaps, many link invariants that can serve as the "error" term in this formula. For example, when $L$ has two components and their linking number is zero, Rachel Sturm has shown that Sato's invariant can replace $\phi_1(L)$. The advantage in working with the Conway polynomial rather than the Alexander polynomial is apparently due to the fact that $\phi_1(z)$ is inherently normalized.

In Section 1 we state some basic definitions and facts regarding the Arf invariant of a proper link. In Section 2 we recall the definition of the Conway polynomial and prove a technical lemma enumerating some of its properties. The main result is then proven in two stages: first when $L$ has two components and then in general. This is because the general proof proceeds by induction on $n$, the number of components of $L$. The case when $n = 2$ is proven first in order to start the induction. The proof of the inductive step however, is extremely similar to the proof when $n = 2$. Therefore, in Section 3 we give the proof when $n = 2$ and in Section 4 only outline the proof of the inductive step.

For convenience, we work in the smooth category. All knots and links are in $S^3$ and are assumed to be oriented. All congruence are mod 2.

1. Basic facts and definitions

Let $L = \{K_1, \ldots, K_n\}$ be an oriented link of $n$ components in $S^3$. We say that $L$ is related to a knot $J$ if there exists a smooth embedding of a planar surface $F$ in $S^3 \times I$ such that $F$ meets $S^3 \times \{0, 1\}$ transversely in $J$ and $L$ respectively. Given a link $L$, we may define its Arf invariant to be the Arf invariant of any knot $K$ related to it, provided that $L$ is proper, that is to say, that the sum of the linking numbers of any component of $I$ with all the other components is even. This was shown to be well defined by Robertello [5]. It is easy to construct examples of nonproper links which are related to knots with different Arf invariants. Notice that given a link $L$ we may produce a knot $K$ related to $L$ by simply band connecting together
all the components of $L$. (Of course the bands must respect the orientations of $L$.) Since the Arf invariant adds under connected sum we see that the Arf invariant of a totally split link is just the sum of the Arf invariants of its components. (A link of $n$ components is called *totally split* if there are $n$ disjoint balls in $S^3$ such that each ball contains exactly one component of the link. A link is *split* if there are two disjoint balls in $S^3$ such that each component of $L$ lies in one of the balls and each ball contains at least one component.)

If the orientations of all the components of a proper link are reversed the Arf invariant remains unchanged. However, if the orientations of only some of the components are reversed the Arf invariant may change.

A band move between two different components is called a *fusion* while its reverse (or one between two arcs of the same component) is called a *fission*. Fusions will preserve properness but fissions may not. However, if one proper link can be obtained from another proper link by a finite sequence of band moves, such that after each move a proper link results, then it is an easy matter to show that the Arf invariant is preserved. If two proper links are related in this manner then we shall say that they are related by a sequence of *proper band moves*.

We may use this principle to show that the Arf invariant of a boundary link is the sum of the Arf invariants of its components. For suppose that several knots bound disjoint Seifert surfaces. Each surface may be pictured as a disk with $2g$ bands attached where $g$ is the genus of the knot. The sequence of proper band moves illustrated in Fig. 1.1 shows that any crossing between bands of different surfaces may be changed without altering the Arf invariant of the link. Thus the original boundary link $L$ has the same Arf invariant as a totally split link $L'$ whose individual components are the same as those of $L$.

![Fig. 1.1.](image)

Actually, the above argument proves something more general. Suppose $F_1$ and $F_2$ are two disjoint oriented surfaces in $S^3$ such that $L = \partial F_1 \cup \partial F_2$ is a proper link. Then each $\partial F_i$ is proper and $\operatorname{Arf}(L) = \operatorname{Arf}(\partial F_1) + \operatorname{Arf}(\partial F_2)$.

2. The Conway polynomial

Before proving the main theorem we enumerate some properties of the Conway polynomial $\nabla_L(z)$ of an oriented knot or link $L$. The reader is referred to [1], [2] or [3] for a more detailed description of $\nabla_L(z)$. However, we briefly recall here the
definition of $\nabla_L(z)$ and the recursive scheme by which it may be calculated starting from a projection of $L$.

Given an oriented knot or link $L$ in $S^3$ let $V$ be the Seifert matrix associated to some Seifert surface $F$ for $L$. Then it is shown in [3] that $\det(x^{-1}V-xV^T)$ is a polynomial in $z=x-x^{-1}$ which is independent of the choice of $F$. We may define $\nabla_L(z)$ to be this polynomial. If $L$ is changed by an ambient isotopy then $\nabla_L(z)$ remains the same. If $L$ is the unknot then $\nabla_L(z) = 1$ and if $L$ is the unlink (with two or more components) then $\nabla_L(z) = 0$. Finally, suppose that $L'$, $L^-$, and $L'$ are three knots or links whose projections are identical except near a single crossing where they appear as shown in Fig. 2.1. Then their Conway polynomials are related by the equation

$$\nabla_{L'}(z) = \nabla_{L^-}(z) + z\nabla_{L^+}(z).$$

These properties allow one to compute $\nabla_L(z)$, starting from any projection of $L$, independent of its definition as a determinant. The diagram in Fig. 2.2 illustrates such a computation. The dots indicate the crossings that are to be changed or smoothed. We shall loosely refer to this process as 'Conway calculus'.

**Lemma 2.1.** Let $L = \{ K_1, K_2, \ldots, K_n \}$ be an oriented link of $n$ components. Then $\nabla_L(z)$ has the following properties.

i) $\nabla_L(z)$ has the form

$$\nabla_L(z) = z^{n-1}[a_0 + a_1 z^2 + \cdots + a_m z^{2m}]$$

ii) If $F_1$ and $F_2$ are two disjoint oriented surfaces, each having nonempty boundary, then $\nabla_L(z) = 0$ where $L = \partial F_1 \cup \partial F_2$.

iii) If $n = 1$ then $a_0 = 1$ and $a_1 = \text{Arf}(K_1)$.

iv) If $n = 2$ then $a_0 = 1k(K_1, K_2)$.

v) If $n = 3$ then

$$a_0 = l_{12}l_{13} + l_{12}l_{23} + l_{13}l_{23} \quad \text{where} \quad l_{ij} = 1k(K_i, K_j).$$

**Proof.** Both ii) and iii) as well as i) when $n = 1$ are proven in [3]. Notice that ii) includes the case when $L$ is split.
To prove i) for \( n > 1 \) we proceed by induction. Consider a projection of \( L \) and all undercrossings of \( K \) beneath the other components of \( L \). Suppose there are \( r \) right handed crossings labeled \( c_1, \ldots, c_r \) and \( s \) left handed crossings labeled \( c_{r+1}, \ldots, c_{r+s} \). Let \( L_1' \) result from changing \( c_1 \) and \( L_1'' \) from smoothing \( c_1 \). Now change \( c_2 \) in \( L_1' \) to get \( L_2' \) and smooth \( c_2 \) to get \( L_2'' \). Continue in this way until all the crossings \( c_i \) have been changed and smoothed.

Now \( L_{r+s}' \), is split since its first component lies above all the others. Hence, by ii), \( \nabla_{L_{r+s}''}(z) = 0 \). Therefore, we have the following formula:

\[
\nabla_L(z) = z \sum_{i=1}^{r} \nabla_{L_i'}(z) - z \sum_{i=r+1}^{r+s} \nabla_{L_i''}(z).
\]

Since each \( L_i'' \) is a link of one fewer component than \( L \) we have by assumption that each \( \nabla_{L_i''}(z) \) satisfies i). Hence, so does \( \nabla_L(z) \).
If $n = 2$ then the above equation becomes
\[
\nabla_L(z) = z[r + b_1z^2 + \cdots + b_2z^2] - z[s + c_1z^2 + \cdots + c_kz^{2k}]
\]
\[
= z[(r - s) + a_1z^2 + \cdots + a_mz^{2m}].
\]
Therefore, $a_0 = r - s = 1k(K_1, K_2)$. This proves iv).

If $n = 3$ then relabel $c_1, \ldots, c_r$ so that $c_1, \ldots, c_{r-1}$ are the right handed crossings of $K_1$ under $K_2$ and that $c_r, \ldots, c_{r+s}$ are the right handed crossings of $K_1$ under $K_3$. Let $r_2 = r - r_2$. Relabel the left handed crossings too so that $c_{r+1}, \ldots, c_{r+s}$ are the undercrossings of $K_1$ with $K_2$ and that $c_{r+s+1}, \ldots, c_{r+s}$ are the undercrossings of $K_1$ with $K_3$. Let $s_3 = s - s_2$. So $l_{12} = r_2 - s_2$ and $l_{13} = r_3 - s_3$. We now have,
\[
\nabla_L(z) = z \sum_{i=1}^{r} \nabla_{L_i}(z) + z \sum_{i=r_2+1}^{r} \nabla_{L_i}(z) - z \sum_{i=r+s_2+1}^{r+s} \nabla_{L_i}(z) - z \sum_{i=r+s+1}^{r+s} \nabla_{L_i}(z).
\]
Using iv) we have that for each $\nabla_{L_i}(z)$ with $1 \leq i \leq r$, $a_0 = l_{13} + l_{23}$. Similarly for $r_2 + 1 \leq i \leq r$ we have $a_0 = l_{12} - r_2 + l_{23}$, for $r + 1 \leq i \leq r + s_2$ we have $a_0 = l_{13} - r_3 + l_{23}$, and for $r + s_2 + 1 \leq i \leq r + s$ we have $a_0 = l_{23}$. Thus for $L$ we have
\[
a_0 = r_2(l_{13} + l_{23}) + r_3(l_{12} - r_2 + l_{23}) - s_2(l_{13} - r_3 + l_{23}) - s_3l_{23} = l_{12}l_{13} + l_{12}l_{23} + l_{13}l_{23}. \]

3. The main result

If $L$ is a link of $n$ components in $S^3$ then $\nabla_L(z) = z^{n-1}[a_0 + a_1z^2 + \cdots + a_mz^{2m}]$. Let $\phi_i(L) = a_r$.

**Theorem 3.1.** Let $L = \{K_1, K_2\}$ be an oriented proper link. Then
\[
\text{Arf}(L) = \text{Arf}(K_1) + \text{Arf}(K_2) + \phi_1(L) \pmod{2}
\]
\[
= \phi_1(K_1) + \phi_1(K_2) + \phi_1(L) \pmod{2}.
\]

**Proof.** We may picture a Seifert surface $F$ for $L$ as shown in Fig. 3.1.

![Fig. 3.1.](image)

Assume that each band is lying flat so that only one side of $F$ is visible. Every crossing between bands of $F$ looks like one of the crossings shown in Fig. 3.2. A
circle has been placed on those strands which belong to \( K_2 \), while those that belong to \( K_1 \) have been left unmarked. Only crossings of type \( d \) or \( e \) introduce linking between \( K_1 \) and \( K_2 \). Since \( L \) is proper there must be an even number of crossings of type \( d \) and \( e \) together. In other words, there is an even number of twists in the band carrying \( K_2 \).

We shall induct on the number of crossings of type \( b \). Suppose there are none. Then the band carrying \( K_2 \) lies above all the other bands. Hence, \( K_1 \) is actually a connected sum and \( L \) may be pictured as shown in Fig. 3.3.

Suppose \( 1k(K_1, K_2) = 2k \). We may add \( 2k \) twists to the band carrying \( K_2 \) by the sequence of proper band moves illustrated in Fig. 3.4. This creates a link \( L' = \{K'_1, K'_2\} \) having the same Arf invariant. Now \( 1k(K'_1, K'_2) = 0 \), so it is not hard to see that \( L' \) is a boundary link. But the components of \( L' \) are individually the same as those of \( L \). Hence

\[
\text{Arf}(L) = \text{Arf}(L') = \text{Arf}(K'_1) + \text{Arf}(K'_2) = \text{Arf}(K_1) + \text{Arf}(K_2).
\]
Thus it only remains to show that $\phi_1(L) \equiv 0$. There are two ways to see this. First, consider a Seifert matrix for $L$. Since there are no crossings of type $b$ and since the band carrying $K_2$ has $-2k$ twists the matrix must look like

$$V = \begin{pmatrix} A & 0 \\ 0 & -2k \end{pmatrix}.$$ 

where $A$ is a Seifert matrix for $J$. Now

$$\det(x^{-1}V - xV^T) = \det\begin{pmatrix} x^{-1}A - xA^T & 0 \\ 0 & (x-x^{-1})2k \end{pmatrix}.$$ 

This gives,

$$\nabla_L(z) = 2k z \nabla_J(z). \tag{3.1}$$

Thus, $\phi_1(L) = 2k \phi(J) \equiv 0$.

Alternatively, we may arrive at equation 3.1 by using Conway calculus and the fact that $\nabla_L(z) = 0$ since $L'$ is a boundary link.

This starts the induction.

Now suppose that there are $n$ crossings of type $b$ but that the theorem is true for $n-1$ or fewer such crossings. Using proper band moves, change one of the crossings as shown in Fig. 3.5. Now $\text{Arf}(L') = \text{Arf}(L)$. But $L'$ has $n-1$ crossings of type $b$, so by assumption

$$\text{Arf}(L') = \text{Arf}(K'_1) + \text{Arf}(K_2) + \phi_1(L').$$

Thus, it remains to show that

$$\text{Arf}(K'_1) + \phi_1(L') = \text{Arf}(K_1) + \phi_1(L). \tag{3.2}$$

Now consider the Conway polynomial calculations associated to the diagram in Fig. 3.6. We start with the link $L$ and only make changes near the band crossings that we focussed on in Fig. 3.5. Again, strands belonging to the same knot are similarly marked. Note that in each of $J_3$, $J_4$, and $J_5$ there is a strand which may be marked in one of two different ways, depending on the global nature of $L$. This gives

$$\nabla_L(z) = \nabla_L(z) + z \nabla_{J_4}(z) - z \nabla_{J_2}(z) - z^2 \nabla_{J_5}(z) - z \nabla_{J_4}(z) + z \nabla_{J_5}(z).$$
Hence we have,
\[ \phi_1(L) = \phi_1(L') + \phi_1(J_1) - \phi_1(J_2) - \phi_0(J_3) - \phi_0(J_4) + \phi_0(J_5). \] (3.3)

But, \( \phi_1(J_1) = \text{Arf}(J_1) \) and \( \phi_1(J_2) = \text{Arf}(J_2) \) which are the same, as seen by the proper band moves illustrated in Fig. 3.7. Hence (3.3) reduces to
\[ \phi_1(L) = \phi_1(L') + \phi_0(J_3) + \phi_0(J_4) + \phi_0(J_5). \] (3.4)
Let $K'_i$ be obtained from $K'_1$ by the sequence of proper band moves shown in Fig. 3.8. This shows that

$$\text{Arf}(K'_1) = \text{Arf}(K''_i).$$  \hspace{1cm} (3.5)

Now the Conway polynomial calculations associated to the diagram in Fig. 3.9 gives equation 3.6.

$$\text{Arf}(K''_i) = \text{Arf}(K_1) + \phi_0(J_6).$$  \hspace{1cm} (3.6)

Combining equations 3.2–3.6 we see that it remains to prove

$$\phi_0(J_3) + \phi_0(J_4) + \phi_0(J_5) + \phi_0(J_6) = 0.$$  \hspace{1cm} (3.7)

The numbers involved in equation 3.7 depend only on the linking numbers of the various links. So far we have been concerned only with the projections of the
various knots and links near the band crossing of \( L \) that has been changed. However, taking into consideration the entire link \( L \), we see that the 4 strands near this band crossing can be connected in two different ways as illustrated in Fig. 3.10.

![Fig. 3.10.](image)

We begin by considering case I. Consider the link of four components obtained from \( L \) by altering \( L \) near the band crossing as shown in Fig. 3.11. Let \( X_{ij} = 1k(\alpha_i, \alpha_j) \). Considering \( L \) we see that

\[
1k(K_1, K_2) = X_{14} + X_{24} + X_{34} = 0. \tag{3.8}
\]

Considering \( J_3 \) gives,

\[
\phi_0(J_3) = 1k(N_1, N_2) = X_{13} + X_{14} + X_{23} + X_{24}. \tag{3.9}
\]

Considering \( J_4 \) we have,

\[
1k(N'_1, N'_2) = X_{12} + X_{13},
1k(N'_1, K_2) = X_{14},
1k(N'_2, K_2) = X_{24} + X_{34}.
\]

Hence, using both the lemma and 3.8, we have

\[
\phi_0(J_4) = X_{14}(X_{12} + X_{13}) + X_{14}(X_{24} + X_{34}) + (X_{12} + X_{13})(X_{24} + X_{34}) = X_{14}^2. \tag{3.10}
\]
Considering $J_5$ we have,

$$\lk(N_1^r, N_2^r) = X_{12} + X_{13}$$
$$\lk(N_1^r, K_2) = X_{24}$$
$$\lk(N_2^r, K_2) = X_{14} + X_{34}. $$

Hence,

$$\phi_0(J_5) = X_{24}(X_{12} + X_{23}) + X_{24}(X_{14} + X_{34}) + (X_{12} + X_{23})(X_{14} + X_{34}) = X_{24}^2. \tag{3.11}$$

Finally, considering $J_6$ gives,

$$\phi_0(J_6) = \lk(N_1^r, \hat{K}_1) = X_{13} + X_{23}. \tag{3.12}$$

Combining equations 3.9–3.12 we have

$$\phi_0(J_3) + \phi_0(J_4) + \phi_0(J_5) + \phi_0(J_6) = X_{13} + X_{14} + X_{23} + X_{24} + X_{14}^2 + X_{24}^2$$
$$= X_{13} + X_{23} + X_{14}(X_{14} + 1) + X_{24}(X_{24} + 1)$$
$$= X_{13} + X_{23}$$
$$= \phi_0(J_6).$$

This completes the proof of the theorem in case I. Case II is similar and is left to the reader. □

4. Links with more components

It should be possible to apply the techniques used in the previous section to an arbitrary link $L$. However, unlike the case when $L$ has two components, the various sublinks of an arbitrary proper link may or may not themselves be proper links. This phenomenon apparently blocks the direct generalization of the proof given in Section 3 to an arbitrary proper link. But, for a proper link $L$, every sublink of which is also proper, we can prove the following generalization of Theorem 3.1. We shall only outline the proof, since it proceeds in a manner similar to the one given previously, and furthermore employs no significantly new or different ideas.

**Theorem 4.1.** Suppose $L = \{K_1, K_2, \ldots, K_n\}$ is an oriented totally proper link. Then

$$\text{Arf}(L) \equiv \sum_{L' \subseteq L} \text{Arf}(L') + \phi_1(L) \pmod{2}$$

$$\equiv \sum_{L' \subseteq L} \phi_1(L') \pmod{2}. $$

**Outline of proof.** We proceed by induction on $n$, the case $n = 2$ having been already proven.
Again, picture a Seifert surface for $L$ similar to the one shown in Fig. 3.1. This time though, $F$ has $n - 2$ more 'special' bands each carrying an additional component of $L$. To prove the inductive step we will in turn induct on the total number of crossings of special bands under 'normal' bands.

Suppose there are no such crossings. Then $L$ appears as in Fig. 3.3, but again with more special bands. Now perform a fusion of $K_{n-1}$ with $K_n$ to create a link $L' = \{K_1, K_2, \ldots, K'_{n-1}\}$ with the same Arf invariant as $L$. Repeatedly applying the theorem to $L'$ and its sublinks we may show that

$$\text{Arf}(L) = \text{Arf}(L') = \phi_1(L_{n-1}) + \phi_1(L_n) + \phi_1(L) + \sum_{L'' \leq L} \phi_1(L'').$$

(4.1)

Here $L_i$ denotes the sublink of $L$ gotten by deleting $K_i$. But the same argument given before, using the Seifert matrix of $F$, shows that $\phi_1(L) = 0$. Thus it only remains to prove that

$$\phi_1(L_{n-1}) + \phi_1(L_n) = 0.

But if we repeat this argument this time fusing $K_i$ and $K_n$, we get a formula similar to 4.1 but with $i$ and $j$ replacing $n - 1$ and $n$. Thus, adding the two formulas gotten by fusing $K_i$ and $K_j$ and $K_j$ and $K_k$ reveals that

$$\phi_1(L_i) + \phi_1(L_k) = 0 \quad \text{for all } i \neq k.

This starts the induction.

Now suppose that some special band, say the one carrying $K_2$, crosses beneath a normal band. Change this crossing, again as illustrated in Fig. 3.5, to obtain a link $L' = \{K'_1, K_2, \ldots, K_n\}$. Now $L'$ has one fewer such crossing and so by our inductive hypothesis the theorem is true for $L'$. Hence,

$$\text{Arf}(L) = \text{Arf}(L') = \sum_{L'' \leq L} \phi_1(L'').

So it remains to show,

$$\sum_{L'' \leq L} \phi_1(K'_1, L'') = \sum_{L'' \leq L} \phi_1(K_1, L '').$$

(4.2)

Now again, consider the Conway polynomial calculation associated to the diagram in Fig. 3.6. This gives,

$$\phi_1(K_1, K_2, L'') = \phi_1(K'_1, K_2, L'') + \phi_1(J_1, L'') + \phi_1(J_2, L'') + \phi_0(J_3, L'') + \phi_0(J_4, L'') + \phi_0(J, L''),$$

(4.3)

where $L''$ is any sublink (possibly empty) of $L_{12}$. The same band moves as before, shown in Fig. 3.7, coupled with the inductive hypothesis can be used to show that

$$\phi_1(J_1, L'') = \phi_1(J_2, L'') \quad \text{for all } L'' \leq L_{12}.$$
Thus, 4.3 reduces to

$$\phi_1(K_1, K_2, L'') = \phi_1(K'_1, K_2, L'') + \phi_0(J_3, L'') + \phi_0(J_4, L'') + \phi_0(J_5, L'')$$

(4.4)

for any $L'' \subset L_{12}$.

Now let $K''$ be obtained from $K'_1$ as before. This shows

$$Arf(K'_1, L'') = Arf(K''_1, L'') \text{ for all } L'' \subset L_{12}. \quad (4.5)$$

The Conway polynomial calculation associated to Fig. 3.9 gives

$$\phi_1(K''_1, L'') = \phi_1(K_1, L'') + \phi_0(J_6, L'') \text{ for all } L'' \subset L_{13}. \quad (4.6)$$

Now by inducting on the number of components of $L''$ and making use of the inductive hypothesis on $n$, we may derive from equation 4.5 the following formula:

$$\phi_1(K''_1, L'') = \phi_1(K''_1, L'') \text{ for all } L'' \subset L_{13}. \quad (4.7)$$

Now using 4.4, 4.6 and 4.7 we can show that 4.2 reduces to

$$\sum_{L'' \subset L_{13}} \phi_0(J_3, L'') + \phi_0(J_4, L'') + \phi_0(J_5, L'') + \phi_0(J_6, L'') = 0. \quad (4.8)$$

To prove that this is true we again consider the two cases illustrated in Fig. 3.10. The following generalization of Lemma 2.1 is now needed. Its proof is similar to the proof of Lemma 2.1.

**Lemma 4.2.** Let $L = \{K_1, \ldots, K_n\}$ be an oriented link of at least four components such that $lk(K_i, K_j) = 0$ for all $3 \leq i, j \leq n$ and $lk(K_1, K_i) + lk(K_2, K_i) = 0$ for $i \geq 3$. Then $\phi_0(L) = 0$.

This lemma greatly simplifies the left hand side of equation 4.8. In fact, it reduces it to equation 3.7 which was already shown to be true. \(\square\)

We close with two questions:

1) Is there an analogous formula for $Arf(L)$ when $L$ is proper but not totally proper?

2) For an arbitrary link $L$, $Arf(L)$ exists only if $L$ is proper whereas

$$\sum_{L \in L} \phi_1(L')$$

exists regardless. What is the significance of this sum when $L$ is not proper (or totally proper)?

**References**

Added in proof

I have recently learned that Sturm has also given a proof of Murasugi’s theorem (see [6]).

Hitoshi Murakami has also given a proof of Theorem 4.1 as well as answered question 1 for links with 4 or less components. See: “The Arf invariant and the Conway polynomial of links”: (Preprint).