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Trees of integral triangles with given rectangular defect

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ABSTRACT

The rectangular defect of a triangle with side lengths a, b and c is $a^2+b^2-c^2$ where $a, b \leq c$. For a given integer d we examine the set PIT(d) consisting of all primitive integral triangles with rectangular defect equal to d. There are simple transformations τ_1, τ_2 and τ_3 which produce new elements of PIT(d) from any triangle with defect d. They determine a partial ordering on PIT(d) in which applying any τ_i moves upward. We will show that the poset PIT(d) has finitely many components and that each of these components is isomorphic to one of two rooted trees $\mathbb{T}$ or $\widetilde{\mathbb{T}}$ (where $\mathbb{T}$ is the regular rooted tree of valence three and $\widetilde{\mathbb{T}}$ is a subtree of it). It follows that the minimal elements of PIT(d) form a finite set from which any triangle in PIT(d) can be uniquely obtained by applying a finite sequence of the τ_i's.

In order to prove these statements we will analyze a larger poset $\Sigma(d)$ which contains copies of both PIT(d) and its inverse $-$PIT(d) as subposets. The elements of $\Sigma(d)$ are equivalence classes of solutions to the equation

$$x_1^2 + x_2^2 + x_3^2 - 2x_2x_1 - 2x_2x_3 = d.$$

The key result will assert that the complement of $\pm$PIT(d) in $\Sigma(d)$ is a finite poset, denoted by Core(d). The proof of this key result is very different according to whether d is nonpositive (the obtuse case) or d is positive (the acute case), and the two cases must be analyzed separately. In the obtuse case we will see that the components of Core(d) are singletons while in the acute case they are poset segments or poset circuits (these are the finite connected posets in which each element has at most two neighbors). For all values of d the analysis of $\Sigma(d)$ will produce algorithms for constructing both Core(d) and the minimal elements of PIT(d).

1. Introduction

Let $\Delta(a, b, c)$ denote a triangle in the Euclidean plane with side lengths a, b and c where $c \geq a$ and $c \geq b$.[1] The *rectangular defect* of $\Delta(a, b, c)$ is the real number

$$d = a^2 + b^2 - c^2. \tag{1.1}$$

By the law of cosines, $d = 2ab\cos(\gamma)$ where γ is the interior angle of $\Delta(a, b, c)$ opposite the side of length c. As γ is the largest interior angle in $\Delta(a, b, c)$, the defect is respectively positive, zero, or negative according to whether $\Delta(a, b, c)$ is an acute triangle, a right triangle, or an obtuse triangle. Thus the defect provides a crude measure of how close a triangle is to being a right triangle. A precise geometric interpretation is illustrated in Fig. 1: Let $\Delta = \Delta(a, b, c)$ be a triangle with defect d, and let A, B and C be the vertices of Δ opposite the sides of length a, b and c respectively. Choose a point A' in the plane so that CA' is perpendicular to CA and $|CA'| = |CA| = b$. Then $|d| = 4\text{Area}(\Delta')$ where Δ' is the triangle with vertices A', B and C. (While the choice of the vertex A' is not unique, the area of Δ' is independent of that choice.)

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[1] Congruent triangles are always considered to be the same in this work. In writing $\Delta(a, b, c)$ it is assumed that the third coordinate c is the maximal side length of the triangle however the other two side lengths are unordered. So $\Delta(b, a, c) = \Delta(a, b, c)$ with this notation.

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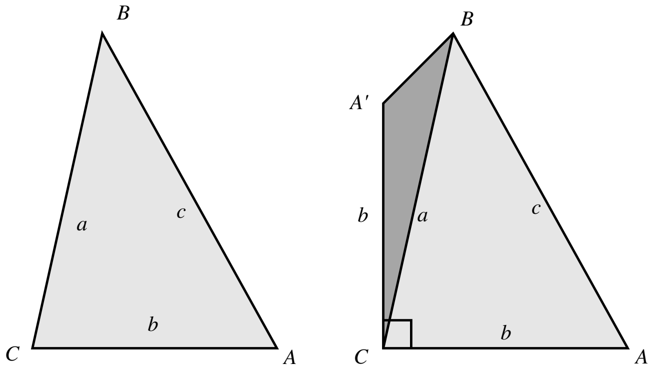

Fig. 1. Geometric interpretation of rectangular defect.

Table 1
The root triangles with defect between −10 and 10.

d	Root(d)
10	$\Delta(5, 7, 8)$, $\Delta(11, 17, 20)$, $\Delta(15, 19, 24)$
9	$\Delta(3, 4, 4)$, $\Delta(3, 5, 5)$, $\Delta(3, 7, 7)$, $\Delta(3, 8, 8)$, $\Delta(7, 9, 11)$, $\Delta(11, 12, 16)$, $\Delta(13, 18, 22)$
8	$\Delta(5, 8, 9)$, $\Delta(8, 13, 15)$, $\Delta(12, 15, 19)$
7	$\Delta(4, 4, 5)$, $\Delta(8, 8, 11)$, $\Delta(10, 14, 17)$
6	$\Delta(5, 9, 10)$, $\Delta(9, 11, 14)$
5	$\Delta(4, 5, 6)$, $\Delta(7, 10, 12)$
4	$\Delta(2, 3, 3)$, $\Delta(2, 5, 5)$, $\Delta(6, 7, 9)$
3	$\Delta(4, 6, 7)$
2	$\Delta(3, 3, 4)$
1	$\Delta(1, 1, 1)$, $\Delta(1, 2, 2)$
0	$\Delta(3, 4, 5)$
−1	$\Delta(2, 2, 3)$
−2	$\Delta(3, 5, 6)$, $\Delta(7, 7, 10)$
−3	$\Delta(2, 3, 4)$, $\Delta(5, 6, 8)$
−4	$\Delta(3, 6, 7)$, $\Delta(6, 9, 11)$, $\Delta(10, 11, 15)$
−5	$\Delta(2, 4, 5)$, $\Delta(8, 10, 13)$
−6	$\Delta(3, 7, 8)$, $\Delta(9, 13, 16)$, $\Delta(13, 15, 20)$
−7	$\Delta(2, 5, 6)$, $\Delta(11, 14, 18)$, $\Delta(3, 3, 5)$, $\Delta(5, 7, 9)$, $\Delta(9, 9, 13)$
−8	$\Delta(3, 8, 9)$, $\Delta(12, 17, 21)$, $\Delta(16, 19, 25)$, $\Delta(4, 5, 7)$, $\Delta(7, 8, 11)$
−9	$\Delta(2, 6, 7)$, $\Delta(14, 18, 23)$
−10	$\Delta(3, 9, 10)$, $\Delta(15, 21, 26)$, $\Delta(19, 23, 30)$

A triangle $\Delta(a, b, c)$ is *integral* if each of its side lengths a, b and c are integers, and it is *primitive* if in addition we have $\gcd(a, b, c) = 1$. Note that the defect of an integral triangle is itself an integer. Our intent in this paper is to examine the set PIT(d) which consists of all primitive integral triangles whose defect equals d for a given integer d. To study this set one is led to consider the transformations

$$\begin{aligned} \tau_1(a, b, c) &= (a - 2b + 2c, 2a - b + 2c, 2a - 2b + 3c), \\ \tau_2(a, b, c) &= (a + 2b + 2c, 2a + b + 2c, 2a + 2b + 3c), \quad \text{and} \\ \tau_3(a, b, c) &= (-a + 2b + 2c, -2a + b + 2c, -2a + 2b + 3c) \end{aligned} \tag{1.2}$$

which are easily seen to preserve both the maximality of third coordinates and the defect of $\Delta(a, b, c)$. As a consequence these transformations can be used to generate new elements of PIT(d) from old ones. They also determine a partial ordering on PIT(d) by declaring $\Delta(u)$ to be less than $\Delta(\tau_j u)$ for each $j = 1, 2$ and 3. Our main results will show that each primitive integral triangle with defect d can be uniquely obtained from a minimal element of the poset PIT(d) by applying a sequence of transformations from $\{\tau_1, \tau_2, \tau_3\}$, and we will establish that the set Root(d) of minimal elements in PIT(d) is finite and nonempty for all values of d. For example, the 'root triangles' whose defects have absolute value at most ten are displayed in Table 1.

Of course the special case of integral right triangles, which have defect $d = 0$, has been extensively studied. These triangles correspond to positive 'Pythagorean triples' which are triples of positive integers (a, b, c) satisfying $a^2 + b^2 = c^2$. Here it is well known that PIT(0) forms a regular rooted tree $\mathbb{T}$ of valence three (as shown in Fig. 3), and Root(0) consists of the single right triangle $\Delta(3, 4, 5)$. This means that every primitive integral right triangle can be obtained from $\Delta(3, 4, 5)$ by applying a unique sequence of transformations from $\{\tau_1, \tau_2, \tau_3\}$. For example, applying $\tau_3\tau_1\tau_3\tau_2\tau_1^3$ to $\Delta(3, 4, 5)$ produces the primitive integral right triangle $\Delta(12\,360, 7009, 14\,209)$. The identification of PIT(0) with $\mathbb{T}$ has been discovered and/or examined by many different authors including Berggren [3], Barning [2], Hall [5], Alperin [1], McCullough [9] and

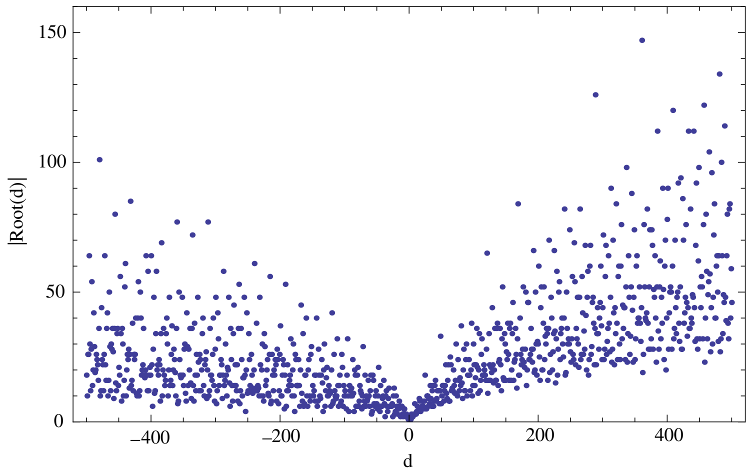

Fig. 2. The number of root triangles with defect d between -500 and 500.

Lonnemo [8]. In [2] Barning also described the primitive integral triangles with defects $d = \pm 1$, which he calls 'almost Pythagorean'. He showed that PIT(-1) is a rooted tree $\widetilde{\mathbb{T}}$ (as seen in Fig. 3) and that PIT(1) is the disjoint union of $\mathbb{T}$ and $\widetilde{\mathbb{T}}$. So Root(-1) and Root(1) have one and two elements respectively, as indicated in Table 1. In fact all values of d for which Root(d) has cardinality one or two are included in this table. More generally, the data in Fig. 2 suggests that the number of root triangles in PIT(d) grows without bound as a function of $|d|$.

Our approach to studying PIT(d) is to identify it with a subposet $\Sigma_\Delta(d)$ of a larger partially ordered set $\Sigma(d)$ whose elements are equivalence classes of primitive integer solutions (x_1, x_2, x_3) to the quadratic Diophantine equation

$$x_1^2 + x_2^2 + x_3^2 - 2x_2x_1 - 2x_2x_3 = d. \tag{1.3}$$

Eq. (1.3) is equivalent to (1.1) under the linear substitution $x_1 = c - a$, $x_2 = c$ and $x_3 = c - b$. At first glance (1.3) may not look like an improvement over (1.1) but we will see that it is algebraically more versatile and carries significant geometric content independent of (1.1). The left hand side of (1.3) is invariant under the three transformations σ_1, σ_2 and σ_3 given by

$$\begin{aligned} \sigma_1(x_1, x_2, x_3) &= (2x_2 - x_1, x_2, x_3), \\ \sigma_2(x_1, x_2, x_3) &= (x_1, 2x_1 + 2x_3 - x_2, x_3) \quad \text{and} \\ \sigma_3(x_1, x_2, x_3) &= (x_1, x_2, 2x_2 - x_3). \end{aligned} \tag{1.4}$$

These transformations can be used to construct the aforementioned transformations τ_1, τ_2 and τ_3 (see Eqs. (4.3)), but are considerably easier to work with. They may be viewed as reflections on a geometric space,[2] and the reflection group that they generate is a right-angled Coxeter group. The associated Coxeter diagram is the tree with three vertices (and both edges labeled ∞), and the Coxeter group is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2) * \mathbb{Z}_2$.

The right-angled Coxeter group perspective creates an informative parallel with another well-studied quadratic Diophantine equation, the 'Descartes circle formula', which may be expressed as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4 = 0. \tag{1.5}$$

In fact, the left hand sides of both (1.3) and (1.5) can be recognized as the quadratic forms determined by the standard bilinear form associated with a Coxeter group. In the case of (1.5) the associated Coxeter diagram is the complete graph on four vertices (with edges labeled by ∞) and the Coxeter group is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. Primitive integer solutions to the Descartes circle formula have been extensively examined in a series of papers [7,4] by Lagarias, Mallows, Wilks and co-authors (see also [10]). Although, in distinction from our setting, there are infinitely many 'root' solutions to (1.5). Another distinction is that partial orderings necessarily play a central role in our investigations but were only hinted at in [4] through the use of the term 'root quadruple'. Nevertheless the results in sections 3 and 4 of [4] have particularly inspired some of the calculations in the present paper. As in [4] these calculations are elementary and do not explicitly involve the right-angled Coxeter group interpretation.

To prove our main results we will show that the poset $\Sigma_\Delta(d)$ is a disjoint collection of rooted trees isomorphic to $\mathbb{T}$ or $\widetilde{\mathbb{T}}$, and that $\Sigma(d)$ can be constructed by attaching these rooted trees and their inverses to a finite subposet Core(d) $\subset \Sigma(d)$. Since no more than four rooted trees will be attached at a given element of Core(d), the collection of rooted trees in $\Sigma_\Delta(d)$

[2] As written they are affine reflections on $\mathbb{R}^3$ but they can also be realized as isometric reflections across the three sides of a right triangle with two ideal vertices in the hyperbolic plane.

must be finite. Therefore $\mathrm{PIT}(d) \cong \Sigma_\Delta(d)$ is a disjoint union of finitely many rooted trees, and its set of minimal elements $\textsc{Root}(d)$ is finite. The most difficult step in this process will be to verify that $\mathrm{Core}(d)$ is finite, and in addressing this it will become clear that there is a significant dichotomy between the cases of obtuse triangles (where $d \le 0$) and acute triangles (where $d > 0$). As evidence of the dichotomy, we shall see that when $d \le 0$ each component of $\Sigma(d)$ is one of a finite number of rooted trees or their inverses, and each component of $\mathrm{Core}(d)$ is a singleton. But when $d > 0$, no component of $\Sigma(d)$ is a rooted tree and there are infinitely many different isomorphism types for the components of $\mathrm{Core}(d)$. On the other hand the components of $\mathrm{Core}(d)$ always have a particularly nice form. For example, when $d > 4$ they are always 'interval' or 'circuit' posets (see Theorem 6.3). In both of the cases $d \le 0$ and $d > 0$ we will give algorithms which construct the elements of $\textsc{Root}(d)$, and these can be used to generate data such as given in Fig. 2 and Tables 1–3.

I would like to thank Ted Swang for discussions instrumental to the development of this work.

Glossary of terms:

Term	Defined in	Description
$\mathrm{PIT}(d)$	Introduction	Poset of primitive integral triangles with defect d
$\textsc{Trip}(d)$	(3.1)	Poset of d-triples
$\Sigma(d)$	(3.5)	Poset of equivalence classes of d-triples
$\mathcal{R}(d)$	(3.6)	Set of components of $\Sigma(d)$
$\Sigma_\Delta(d)$	(4.1)	Subposet of $\Sigma(d)$ corresponding to $\mathrm{PIT}(d)$
$\mathrm{Core}(d)$	(4.6)	The core of $\Sigma(d)$
Preferred d-triple	After (3.5)	Special representative for an element of $\Sigma(d)$
$\sigma_1, \sigma_2, \sigma_3$	(1.4)	Fundamental reflections on $\textsc{Trip}(d)$
μ	(3.2), (4.5)	Negation involution on $\textsc{Trip}(d)$ and $\Sigma(d)$
η	(3.2)	Coordinate interchange involution on $\textsc{Trip}(d)$
$\ell(x)$	(3.3)	Length of a d-triple
$S_i(x), S(x)$	(3.4)	Determines direction that σ_i moves at $x \in \textsc{Trip}(d)$
$N(x)$	(3.7)	Norm of a d-triple
$\mathcal{N}[x]$	(4.2)	Set of neighbors of $[x] \in \Sigma(d)$
$\widehat{\chi}$	(4.4)	Transformation from $\mathrm{PIT}(d)$ to $\Sigma(d)$

2. Rooted trees and indicable relations

Partial orderings play a central role in this work and in this section we sketch some basic terminology. Stanley's book [11] is a good source for background details.

A *poset* consists of a set P and a partial ordering $\le$ on P. The *inverse of* P is the poset $-P = (P, \ge)$. We say P is *locally finite* if for each x, y the interval $[x, y] = \{z \mid x \le z \le y\}$ is finite, and it satisfies the *descending chain condition* if every decreasing sequence $x_1 > x_2 > x_3 > \cdots$ is finite. If $x < y$ and there is no z with $x < z < y$ then x is an *immediate predecessor of* y, and y is an *immediate successor of* x. Two elements of P are *neighbors* if one is an immediate successor of the other. The number of immediate successors of an element of P is its *upward valence*, while the number of immediate predecessors is its *downward valence*. An element is *minimal* if no element is smaller than it, and *maximal* if no element is larger. If $(P, \le)$ is a poset and $Q \subseteq P$ then $(Q, \le)$ is a *subposet* of $(P, \le)$. The subposet Q is an *upward ideal* if $x \in Q$ and $y > x$ implies that $y \in Q$, a *downward ideal* if $x \in Q$ and $z < x$ implies that $z \in Q$, and a *full subposet* if $[x, y] \subseteq Q$ for each $x, y \in Q$.

Via the association of a poset with its Hasse diagram one may apply graph theory terminology to posets. A *path* from x to y in a poset P is a sequence $x = x_0, x_1, \ldots, x_n = y$ in which x_i is a neighbor of x_{i-1} for $i = 1, \ldots, n$, and P is *connected* if any two of its elements can be joined by a path. The relation on P where $x \equiv y$ if and only if there is a path from x to y is an equivalence relation whose equivalence classes are the *components of* P.

A *rooted tree* is a connected poset P satisfying the descending chain condition with the property that each element has no more than one immediate predecessor. By the descending chain condition, if x is an element of a rooted tree then the sequence $x = x_0, x_1, x_2, \ldots$ where x_{i+1} is the immediate predecessor of x_i (if it has one) must terminate at a minimal element after finitely many steps. Thus every rooted tree contains a minimal element. On the other hand it cannot have more than one minimal element since a path between distinct minimal elements would contain an element with more than one immediate predecessor. The unique minimal element in a rooted tree is called its *root*. A rooted tree is *regular of degree n* if the upward valence of each element equals n. An *uprooted tree* is a poset whose inverse is a rooted tree—it has a unique maximal element, which is also called its *root*. It is not hard to verify that rooted and uprooted trees are locally finite.

The regular rooted tree $\mathbb{T}$ of degree three plays a key role in our investigations. (See Fig. 3.) It can be described formally as the set of finite strings $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ with entries $\alpha_i \in \{-1, 0, 1\}$ where $\beta \le \alpha$ if β is an initial substring of α. The immediate successors of $\alpha_1 \ldots \alpha_n$ are $\alpha_1 \ldots \alpha_n\alpha_{n+1}$ where $\alpha_{n+1} \in \{-1, 0, 1\}$, and its immediate predecessor is $\alpha_1 \ldots \alpha_{n-1}$ provided that $n > 0$. The empty string is the root of $\mathbb{T}$. The function $\omega : \{-1, 0, 1\} \to \{-1, 0, 1\}$ given by $\omega(j) = -j$ extends to a poset isomorphism $\omega : \mathbb{T} \to \mathbb{T}$ with $\omega(\alpha_1 \ldots \alpha_n) = \omega(\alpha_1) \ldots \omega(\alpha_n)$. The quotient set $\widetilde{\mathbb{T}} = \mathbb{T}/\omega$, with elements

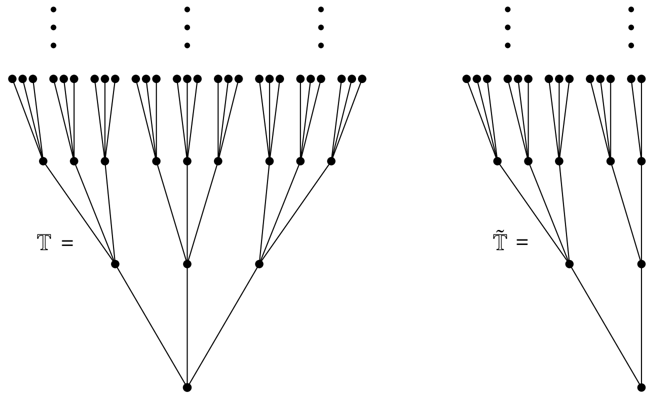

Fig. 3. The rooted trees $\mathbb{T}$ and $\widetilde{\mathbb{T}}$.

$[\alpha] = \{\alpha, \omega(\alpha)\}$, also forms a rooted tree. Here $[\beta] \le [\alpha]$ when $\beta \le \alpha$ or $\beta \le \omega(\alpha)$. The tree $\widetilde{\mathbb{T}}$ can also be realized as the subposet of $\mathbb{T}$ consisting of strings whose first nonzero entry is -1. A string in $\widetilde{\mathbb{T}}$ has upward valence two if all of its entries are 0 and three otherwise. The set of strings of 0's forms a maximal chain C which we call the *axis ray* of $\widetilde{\mathbb{T}}$. In this fashion $\widetilde{\mathbb{T}}$ can be characterized as the unique rooted tree containing a maximal chain C such that each element in C has upward valence two and each element outside C has upward valence three.

The partial orderings that occur in this work arise via a construction we call 'indicable closure', described as follows. If $\prec$ is a relation on S then its *RT-closure* is the relation $\le$ where $x \le y$ if there is a sequence $x = s_0, s_1, \ldots, s_n = y$ with $n \ge 0$ and $s_{i-1} \prec s_i$ for $i = 1, \ldots, n$. The RT-closure of $\prec$ is the smallest reflexive and transitive relation containing $\prec$. An *index function* for $\prec$ is a function $\lambda : S \to \mathbb{Z}$ with $\lambda(x) < \lambda(y)$ if $x \prec y$ and $x \ne y$. A relation is *indicable* if it admits an index function $\lambda : S \to \mathbb{Z}$. Indicable relations are anti-symmetric because if $x \le y \le x$ and $x \ne y$ then $\lambda(x) < \lambda(y) < \lambda(x)$ violating the anti-symmetry of the Archimedean ordering on $\mathbb{Z}$. If $\lambda : S \to \mathbb{Z}$ is an index function for $\prec$ then it is also an index function for the RT-closure $\le$. Therefore the RT-closure of an indicable relation is a partial ordering. An indicable poset is locally finite if it admits an index function whose point inverses are finite, and it satisfies the descending chain condition if it admits an index function whose image is bounded below in $\mathbb{Z}$.

To formally verify that the relation on PIT(d) described in the introduction is a partial ordering we use the indicable closure construction. For $\Delta(u), \Delta(v) \in \mathrm{PIT}(d)$ write $\Delta(u) \prec \Delta(v)$ if $v = \tau_j(u)$ for some $j \in \{1, 2, 3\}$. The 'perimeter function' $\rho(\Delta(a, b, c)) = a + b + c$ is an index function for $\prec$ since if $u = (a, b, c)$ then

$$\begin{aligned} \rho(\Delta(\tau_1 u)) - \rho(\Delta(u)) &= 4a + 6(c - b) > 0, \\ \rho(\Delta(\tau_2 u)) - \rho(\Delta(u)) &= 4a + 4b + 6c > 0, \quad \text{and} \\ \rho(\Delta(\tau_3 u)) - \rho(\Delta(u)) &= 4b + 6(c - a) > 0. \end{aligned}$$

Therefore the RT-closure of $\prec$ forms a partial ordering $\le$ on PIT(d). As there are only finitely many integral triangles with given perimeter, ρ has finite point inverses and PIT(d) is locally finite. Also the image of the perimeter function is bounded below by 0 and PIT(d) satisfies the descending chain condition.

3. *d*-triples and the poset $\Sigma(d)$

An ordered triple of integers $x = (x_1, x_2, x_3)$ with $\gcd(x_1, x_2, x_3) = 1$ which satisfies Eq. (1.3) will be called a *d-triple*. We define

$$\mathrm{Trip}(d) = \left\{ x \in \mathbb{Z}^3 \mid x \text{ is a } d\text{-triple} \right\}. \tag{3.1}$$

Each of the five transformations η and μ, given by

$$\begin{aligned} \eta(x_1, x_2, x_3) &= (x_3, x_2, x_1) \quad \text{and} \\ \mu(x_1, x_2, x_3) &= (-x_1, -x_2, -x_3), \end{aligned} \tag{3.2}$$

and the fundamental reflections σ_1, σ_2, and σ_3 defined in (1.4) are involutions on $\mathbb{Z}^3$. Straightforward calculations show that each of them leave Trip(d) invariant. The *length function* on Trip(d) is the transformation $\ell : \mathrm{Trip}(d) \to \mathbb{Z}$ given by

$$\ell(x_1, x_2, x_3) = x_1 + x_2 + x_3. \tag{3.3}$$

We define a relation $\prec$ on Trip(d) by setting

$$x \prec y \text{ if } y = \sigma_j(x) \text{ for some } j \in \{1, 2, 3\} \text{ and } \ell(x) < \ell(y).$$

Clearly ℓ is an index function with respect to $\prec$, so the RT-closure of $\prec$ forms a partial ordering $\leq$ on $\textsc{Trip}(d)$. With this partial ordering $\textsc{Trip}(d)$ is locally finite.[3]

To keep track of whether applying σ_j to a d-triple moves upward or downward in $\textsc{Trip}(d)$ we introduce functions S_1, S_2 and S_3 defined for $x = (x_1, x_2, x_3)$ by

$$\begin{aligned} S_1(x) &= \operatorname{sign}(x_2 - x_1), \\ S_2(x) &= \operatorname{sign}(x_1 + x_3 - x_2), \\ S_3(x) &= \operatorname{sign}(x_2 - x_3), \end{aligned} \tag{3.4}$$

where $\operatorname{sign}(t)$ equals $+$, 0, or $-$ according to whether $t > 0$, $t = 0$ or $t < 0$. Thus $S_i(x) = +$ if and only if σ_i increases the value of $\ell(x)$, indeed $S_i(x) = \operatorname{sign}(\ell(\sigma_i x) - \ell(x))$ for each $i \in \{1, 2, 3\}$. We also let $S(x)$ denote the ordered triple consisting of $S_1(x), S_2(x)$ and $S_3(x)$ written as a string of length three. For example, $x = (4, 3, -2)$ is an element of $\textsc{Trip}(17)$ with $S(x) = --+$. (See Fig. 4.)

From definitions it is readily seen that $\sigma_1\sigma_3 = \sigma_3\sigma_1, \sigma_1\eta = \eta\sigma_3$ and $\sigma_2\eta = \eta\sigma_2$. Also $\sigma_1\eta$ has order four, and $\langle \sigma_1, \sigma_3, \eta \rangle = \langle \sigma_1\eta, \eta \rangle$ is a dihedral group of order eight acting on the set $\textsc{Trip}(d)$. Let $\Sigma(d)$ denote the set of orbits of this action

$$\Sigma(d) = \{\, [x] \mid x \in \textsc{Trip}(d)\} \tag{3.5}$$

where

$$[x] = \{x, \sigma_1(x), \sigma_3(x), \sigma_3\sigma_1(x), \eta(x), \eta\sigma_1(x), \eta\sigma_3(x), \eta\sigma_3\sigma_1(x)\},$$

and let

$$\mathcal{R}(d) = \{\text{components of } \Sigma(d)\}. \tag{3.6}$$

A d-triple x is *preferred* provided that $S_1(x) \geq 0$ and $S_3(x) \geq 0$. Note that each element of $\Sigma(d)$ has a preferred representative: for example, if $S_1(x) = -$ then $\sigma_1 x$ is a representative of $[x]$ with $S_1(\sigma_1 x) = +$, and if $S_3(x) = -$ then $S_3(\sigma_3 x) = +$. If x is preferred then so is ηx, so preferred representatives are only unique up to η.

Next we describe a partial ordering on $\Sigma(d)$. This is somewhat delicate because partial orderings do not descend to quotients except in special circumstances. Indicability plays a key role in surmounting this difficulty but a new index function is needed for the quotient. Here that role is filled by $\lambda : \Sigma(d) \to \mathbb{Z}$ where

$$\lambda[x_1, x_2, x_3] = x_2.$$

For elements $[x]$ and $[y]$ in $\Sigma(d)$ we write

$$[x] \prec [y] \text{ if } [x] \neq [y] \text{ and } x' \prec y' \text{ for some } d\text{-triples } x' \in [x] \text{ and } y' \in [y].$$

Lemma 3.1. *$\prec$ is an indicable relation on $\Sigma(d)$ with index function λ.*

Proof. Let $[x]$ and $[y]$ be elements of $\Sigma(d)$ with $[x] \prec [y]$ and $x \prec y$. Then $y = \sigma_j(x)$ for some $j \in \{1, 2, 3\}$, but $j = 2$ because $[x] \neq [y]$. Therefore $y = (x_1, y_2, x_3)$ and $x_2 = \ell(x) - x_1 - x_3 < \ell(y) - x_1 - x_3 = y_2$. □

Thus the RT-closure $\leq$ of $\prec$ is a partial ordering on $\Sigma(d)$. It is not hard to verify that $\Sigma(d)$ is a locally finite poset.[4] The natural projection $\textsc{Trip}(d) \to \Sigma(d)$ is a poset map, which means that $[x] \leq [y]$ in $\Sigma(d)$ whenever $x < y$ in $\textsc{Trip}(d)$. To illustrate these definitions portions of the Hasse diagrams of $\textsc{Trip}(17)$ and $\Sigma(17)$ are shown in Fig. 4, and the projection $\textsc{Trip}(17) \to \Sigma(17)$ can be seen as collapsing the highlighted 'diamonds' in $\textsc{Trip}(17)$ to points in $\Sigma(17)$. A more global picture of $\Sigma(17)$ is given in Fig. 8.

We end this section with two lemmas that will be useful as we proceed. The *norm* of a triple $x = (x_1, x_2, x_3)$ is

$$N(x) = x_1^2 + x_2^2 + x_3^2. \tag{3.7}$$

Lemma 3.2. *Let x be a d-triple with $\ell(x) > 0$ and $N(x) > d$. For each $j \in \{1, 2, 3\}$ either $\ell(\sigma_j(x)) > 0$ or $N(\sigma_j(x)) \leq d$.*

Proof. Suppose that x is a d-triple with $\ell(x) > 0$ and $N(x) > d$. Let $y = \sigma_j x$ for $j \in \{1, 2, 3\}$ and assume that $N(y) > d$. To prove the lemma we must show that $\ell(y) > 0$. As $2x_2(x_1 + x_3) = N(x) - d > 0$ and $x_2 + (x_1 + x_3) = \ell(x) > 0$, both x_2 and $x_1 + x_3$ are positive. If $j = 2$ then $y_1 + y_3 = x_1 + x_3 > 0$ and if $j \neq 2$ then $y_2 = x_2 > 0$. Thus one of $y_1 + y_3$ or y_2 is positive, and so they are both positive because $2y_2(y_1 + y_3) = N(y) - d > 0$. It follows that $\ell(y) = y_2 + (y_1 + y_3) > 0$. □

[3] Eq. (1.3) can be written as $4x_1^2 + 6x_1x_3 + 4x_3^2 - 4\ell(x)x_1 - 4\ell(x)x_3 + \ell(x)^2 = d$. For a fixed value of $\ell(x)$, the graph in the x_1x_3-plane of this equation is an ellipse which can only pass through finitely many integer lattice points. Thus the point-inverses of $\ell : \textsc{Trip}(d) \to \mathbb{Z}$ are finite.

[4] Eq. (1.3) may be expressed as $(x_1 - x_2)^2 + (x_3 - x_2)^2 = x_2^2 + d$ whose graph is a (possibly degenerate) circle in the x_1x_3-plane with center (x_2, x_2) for a fixed value of $x_2 = \lambda(x)$. As the circle contains at most finitely many integer lattice points, $\lambda^{-1}(x_2)$ is a finite subset of $\textsc{Trip}(d)$.

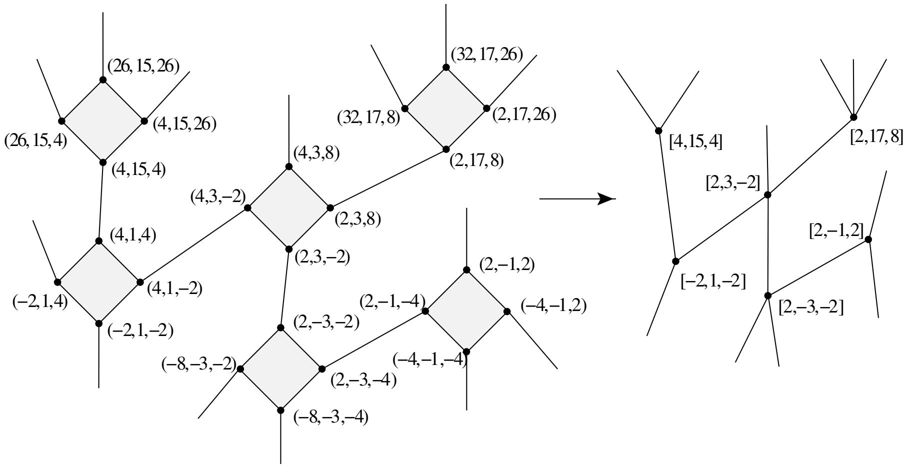

Fig. 4. Corresponding portions of the posets Trip(17) and $\Sigma(17)$.

Lemma 3.3. *Let $x = (x_1, x_2, x_3)$ be a d-triple with $x_1 \geq 0$ and $x_3 \geq 0$.*

(a) *If $S_2(x) = 0$ then x is a preferred d-triple.*
(b) *If $S_2(x) \neq 0$ and $S(x) \neq +-+$ then $S(\sigma_2 x) = +-+$.*

Proof. Let $x_1 \geq 0$ and $x_3 \geq 0$. If $S_2(x) = 0$ then $x_2 = x_1 + x_3$, $S_1(x) = \text{sign}(x_3) \geq 0$ and $S_3(x) = \text{sign}(x_1) \geq 0$, and (a) holds. Assume that $S_2(x) \neq 0$ and $S(x) \neq +-+$. If $S_2(x) = -$ then either $S_1(x) \leq 0$ or $S_3(x) \leq 0$. But if $S_1(x) \leq 0$ then $x_1 + x_3 < x_2 \leq x_1$, contradicting the assumption that $x_3 \geq 0$. If $S_3(x) \leq 0$ then $x_1 < 0$ which is also impossible. Therefore $S_2(x) = +$. Now

$$S_1(\sigma_2 x) = \text{sign}((2x_1 + 2x_3 - x_2) - x_1) \geq \text{sign}(x_1 + x_3 - x_2) = S_2(x) = +$$

and $S_3(\sigma_2 x) = +$ similarly. Since $S_2(\sigma_2 x) = -S_2(x)$, we have $S(\sigma_2 x) = +-+$. $\square$

4. The posets $\Sigma_\Delta(d)$ and Core(d)

Let $\Sigma_\Delta(d)$ be the subposet of $\Sigma(d)$ defined by

$$\Sigma_\Delta(d) = \{[x] = [x_1, x_2, x_3] \in \Sigma(d) \mid S(x) = +-+, \quad x_1 \geq 0 \text{ and } x_3 \geq 0\}. \tag{4.1}$$

If $[y]$ is a neighbor of $[x]$ in $\Sigma(d)$ then $[y] = [\sigma_2 \gamma x]$ for some $\gamma \in \langle \sigma_1, \sigma_3, \eta \rangle$, but γ may be chosen from $\langle \sigma_1, \sigma_3 \rangle$ because $[\sigma_2 \eta \gamma x] = [\eta \sigma_2 \gamma x] = [\sigma_2 \gamma x]$. Therefore the set of neighbors of $[x]$ is a subset of

$$\mathcal{N}[x] = \{[\sigma_2 x], [\sigma_2 \sigma_1 x], [\sigma_2 \sigma_3 \sigma_1 x], [\sigma_2 \sigma_3 x]\}. \tag{4.2}$$

Lemma 4.1. *Let x be a preferred d-triple with $[x] \in \Sigma_\Delta(d)$.*

(a) *$[\sigma_2 x]$ is an immediate predecessor of $[x]$ in $\Sigma(d)$.*
(b) *$[\sigma_2 \sigma_1 x]$, $[\sigma_2 \sigma_3 \sigma_1 x]$ and $[\sigma_2 \sigma_3 x]$ are immediate successors of $[x]$ in $\Sigma(d)$ and elements of $\Sigma_\Delta(d)$.*

Proof. Let x be a d-triple representing an element of $\Sigma_\Delta(d)$ with $S(x) = +-+$, $x_1 \geq 0$ and $x_3 \geq 0$. Let y be one of $\sigma_1 x$, $\sigma_3 \sigma_1 x$ or $\sigma_3 x$. As y is not preferred and $y_i \geq x_i \geq 0$, then $S_2(y) \neq 0$ by Lemma 3.3(a), $[\sigma_2 y]$ is a successor of $[y] = [x]$ by Lemma 3.3(b), and $[\sigma_2 y] \in \Sigma_\Delta(d)$. Thus $[\sigma_2 x]$ is a predecessor of $[x]$ while the other elements of $\mathcal{N}[x]$ are successors. As every neighbor of $[x]$ is in $\mathcal{N}[x]$, $[\sigma_2 x]$ is an immediate predecessor of $[x]$ in $\Sigma(d)$ and (a) holds. Since $[\sigma_2 y] \in \Sigma_\Delta(d)$, $[\sigma_2 \sigma_2 y] = [y] = [x]$ is an immediate predecessor of $[\sigma_2 y]$ by (a), and (b) holds. $\square$

Let $\chi : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the bijective transformation $\chi(a, b, c) = (c - a, c, c - b)$. Using definitions (1.2) and (1.4) it is easily verified that

$$\begin{aligned} \chi\, \tau_1 &= \sigma_2 \sigma_1 \chi, \\ \chi\, \tau_2 &= \sigma_2 \sigma_3 \sigma_1 \chi, \quad \text{and} \\ \chi\, \tau_3 &= \sigma_2 \sigma_3 \chi. \end{aligned} \tag{4.3}$$

These equations may be viewed as defining the transformations τ_1, τ_2 and τ_3 in terms of the more elementary involutions σ_1, σ_2 and σ_3. If (a, b, c) satisfies (1.1) then $\chi(a, b, c)$ satisfies (1.3), and we define $\widehat{\chi} : \text{PIT}(d) \to \Sigma(d)$ by

$$\widehat{\chi}(\Delta(u)) = [\chi(u)]. \tag{4.4}$$

Theorem 4.2. *$\widehat{\chi}$ defines a poset isomorphism from* PIT(d) *to* $\Sigma_\Delta(d)$.

Proof. Let $\Delta(a,b,c) \in \text{PIT}(d)$ and let $x = \chi(a,b,c)$. Then $x_1 = c - a \geq 0$ and $x_3 = c - b \geq 0$. In addition $S_1(x) = \text{sign}(a) = +$, $S_3(x) = \text{sign}(c) = +$, and $S_2(x) = \text{sign}(c-a-b) = -$ by the triangle inequality. This shows that $[x] \in \Sigma_\Delta(d)$ and the image of $\widehat{\chi}$ is in $\Sigma_\Delta(d)$. The correspondence $[x] \mapsto \Delta(\chi^{-1}(x))$ where x is a preferred representative for $[x]$ is the inverse of $\widehat{\chi}$, and $\widehat{\chi} : \text{PIT}(d) \to \Sigma_\Delta(d)$ is a bijection. By definition $\Delta(u) \prec \Delta(v)$ in PIT(d) if and only if $v = \tau_j(u)$ for some $j \in \{1,2,3\}$, and $\chi(v) = \sigma_2\gamma\chi(u)$ for some $\gamma \in \{\sigma_1, \sigma_3\sigma_1, \sigma_3\}$ by (4.3). This in turn is equivalent to writing $\widehat{\chi}(u) \prec \widehat{\chi}(v)$ since $\widehat{\chi}(u) \neq [\sigma_2\gamma\chi(u)]$ by Lemma 4.1(a). Therefore $\Delta(u) \prec \Delta(v)$ if and only if $\widehat{\chi}(u) \prec \widehat{\chi}(v)$. It follows that $\widehat{\chi}$ is a poset isomorphism with respect to the RT-closures of the relations $(\text{PIT}(d), \prec)$ and $(\Sigma(d), \prec)$. □

Theorem 4.3. *$\Sigma_\Delta(d)$ is an upward ideal in $\Sigma(d)$ and each component C of $\Sigma_\Delta(d)$ is a rooted tree isomorphic to $\mathbb{T}$ or $\widetilde{\mathbb{T}}$. If $x = (x_1, x_2, x_3)$ is a preferred representative for the root element of C then C is isomorphic to $\widetilde{\mathbb{T}}$ if and only if $x_1 = x_3$.*[5]

Proof. The restriction of λ to $\Sigma_\Delta(d)$ is an index function whose image is bounded below by 0, showing that $\Sigma_\Delta(d)$ satisfies the descending chain condition. By Lemma 4.1 each component of $\Sigma_\Delta(d)$ is a rooted tree in which each element has upward valence 3 or less. In addition, each immediate successor of an element of $\Sigma_\Delta(d)$ is in $\Sigma_\Delta(d)$, and $\Sigma_\Delta(d)$ is an upward ideal in $\Sigma(d)$.

To complete the proof we must show that each component C of $\Sigma_\Delta(d)$ is isomorphic to $\mathbb{T}$ or to $\widetilde{\mathbb{T}}$. Let $x = (x_1, x_2, x_3)$ be a preferred representative for an element $[x]$ in C. By Lemma 4.1 the immediate successors of $[x]$ are

$$[\sigma_2\sigma_1 x] = [2x_2 - x_1, 3x_2 - 2x_1 + 2x_3, x_3],$$
$$[\sigma_2\sigma_3 x] = [x_1, 3x_2 + 2x_1 - 2x_3, 2x_2 - x_3], \quad \text{and}$$
$$[\sigma_2\sigma_3\sigma_1 x] = [2x_2 - x_1, 7x_2 - 2x_1 - 2x_3, 2x_2 - x_3].$$

If two of these coincide then their second coordinates are equal. But $7x_2 - 2x_1 - 2x_3$ cannot equal $3x_2 - 2x_1 + 2x_3$ or $3x_2 + 2x_1 - 2x_3$ as otherwise $x_2 - x_3 = 0$ or $x_2 - x_1 = 0$, contradicting the assumption that $S_3(x) = S_1(x) = +$. Thus the upward valence of $[x]$ is 2 or 3. Moreover, the upward valence of $[x]$ is 2 if and only if $x_1 = x_3$ since $3x_2 - 2x_1 + 2x_3$ and $3x_2 + 2x_1 - 2x_3$ are equal only when $x_1 = x_3$. When $x_1 = x_3$ the first and third coordinates of both $\sigma_2 x$ and $\sigma_2\sigma_3\sigma_1 x$ will also be equal, but the first and third coordinates of $\sigma_2\sigma_1 x$ will not be equal. So, if the upward valence of $[x]$ is 2 then one of its immediate successors has upward valence 2 and the other has upward valence 3, and, if $[x]$ has an immediate predecessor in $\Sigma_\Delta(d)$ then its immediate predecessor also has upward valence 2. It follows that if C contains an element with upward valence 2 then the elements in C with upward valence 2 form a maximal ascending chain starting at the root element, and C is isomorphic to $\widetilde{\mathbb{T}}$. Otherwise every element of C has upward valence 3 and C is isomorphic to $\mathbb{T}$. □

The involution $\mu : \text{Trip}(d) \to \text{Trip}(d)$ (defined in (3.2)) and its projection $\mu : \Sigma(d) \to \Sigma(d)$ given by

$$\mu[x] = [-x] \tag{4.5}$$

are order-reversing poset isomorphisms. We denote $\mu(\Sigma_\Delta(d))$ by $-\Sigma_\Delta(d)$. Observe that $\Sigma_\Delta(d)$ is disjoint from $-\Sigma_\Delta(d)$ since λ takes positive values on $\Sigma_\Delta(d)$ and negative values on $-\Sigma_\Delta(d)$. By Theorem 4.3, $-\Sigma_\Delta(d)$ is a downward ideal in $\Sigma(d)$ and each component of $-\Sigma_\Delta(d)$ is an uprooted tree isomorphic to $-\mathbb{T}$ or $-\widetilde{\mathbb{T}}$. The *core of* $\Sigma(d)$ is the subposet

$$\text{Core}(d) = \Sigma(d) - \big(\Sigma_\Delta(d) \cup -\Sigma_\Delta(d)\big). \tag{4.6}$$

Any immediate successor of an element of $\Sigma_\Delta(d)$ is in $\Sigma_\Delta(d)$ by Theorem 4.3. Therefore the only possible elements of $\Sigma_\Delta(d)$ which are adjacent to Core(d) are the roots of $\Sigma_\Delta(d)$. Similarly the only elements of $-\Sigma_\Delta(d)$ neighboring Core(d) must be roots. This shows that $\Sigma(d)$ can be constructed from Core(d) by attaching disjoint trees isomorphic to $\pm\mathbb{T}$ or $\pm\widetilde{\mathbb{T}}$, and no more than four trees are attached at each element of Core(d) since an element of $\Sigma(d)$ has at most four neighbors.

In the illustrations of $\Sigma(d)$ displayed throughout the paper, Core(d) is highlighted in bold, and the tree components of $\Sigma_\Delta(d)$ are indicated by dashed circles labeled with $\pm\mathbb{T}$ or $\pm\widetilde{\mathbb{T}}$ as appropriate. The set of root triangles Root(d) of defect d is in 1–1 correspondence with the set of dashed circles labeled $\mathbb{T}$ or $\widetilde{\mathbb{T}}$.

Corollary 4.4. *For each integer d,* Core(d) *is a full subposet of $\Sigma(d)$. When $d \neq 1$ each component of $\Sigma(d)$ intersects* Core(d) *in a component of* Core(d).

Proof. Let $[x], [y] \in \text{Core}(d)$. If $[y] \leq [z] \leq [x]$ then $[z] \notin \Sigma_\Delta(d)$ because $\Sigma_\Delta(d)$ is an upward ideal and $[z] \notin -\Sigma_\Delta(d)$ because $-\Sigma_\Delta(d)$ is an downward ideal. Therefore Core(d) is full in $\Sigma(d)$. Let C be a component of $\Sigma(d)$ where $d \neq 1$. Any path joining points of $C \cap \text{Core}(d)$ can be shortened to one that does not enter $\Sigma_\Delta(d) \cup -\Sigma_\Delta(d)$. Thus $C \cap \text{Core}(d)$ is connected, and if it is nonempty it will be a component of Core(d). Suppose that $C \cap \text{Core}(d) = \emptyset$. Then $C \subset \Sigma_\Delta(d) \cup -\Sigma_\Delta(d)$ and C contains a root element $[x]$ of a component of $\Sigma_\Delta(d)$ or $-\Sigma_\Delta(d)$. By replacing C with $-C$ we may assume that $[x] \in \Sigma_\Delta(d)$. The immediate predecessor of $[x]$ is $[\sigma_2 x]$ which is in $-\Sigma_\Delta(d)$ since it is not in $\Sigma_\Delta(d)$. Then $[-\sigma_2 x] = [-x_1, -2x_1 - 2x_3 + x_2, -x_3] \in \Sigma_\Delta(d)$ and both $-x_1$ and $-x_3$ are nonnegative. It follows that $x_1 = x_3 = 0$, and $x = (0,1,0)$ and $d = 1$, which is a contradiction. □

[5] In Section 7 we will derive a formula which counts the number of $\widetilde{\mathbb{T}}$ components in $\Sigma_\Delta(d)$.

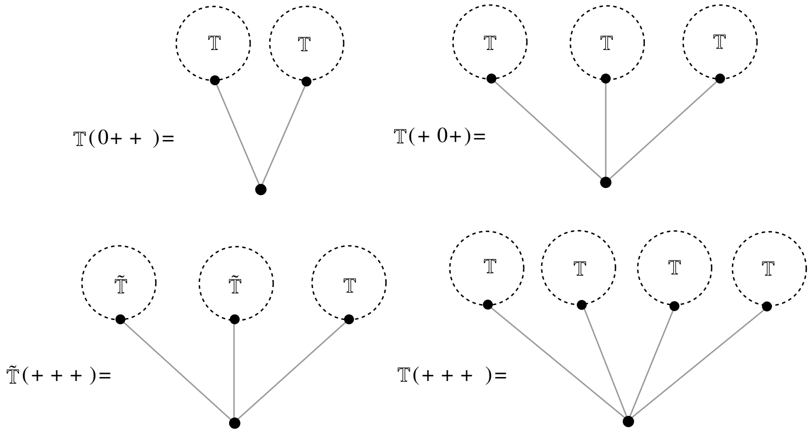

Fig. 5. The four forms for rooted tree components of $\Sigma(d)$ when $d < -2$.

There is a simple algorithm that will produce the root triangle of the component of PIT(d) containing a given triangle $\Delta(a, b, c)$. By applying $\widehat{\chi}$ we describe this algorithm as taking place in the component C of $\Sigma_\Delta(d)$ which contains $[x]$ where $x = \chi(a, b, c)$. Construct a sequence

$$x = x^{(0)} > x^{(1)} > x^{(2)} > \cdots$$

of d-triples with $S(x^{(i)}) = + - +$ as follows. Given $x^{(i-1)}$, let $y^{(i)} = \sigma_2 x^{(i-1)} < x^{(i-1)}$ and let $x^{(i)}$ be the preferred d-triple $\gamma y^{(i)}$ with $\gamma \in \langle \sigma_1, \sigma_3 \rangle$. Thus $\gamma = \sigma_3^{\epsilon_3} \sigma_1^{\epsilon_1}$ for $\epsilon_1, \epsilon_3 \in \{0, 1\}$ where $\epsilon_k = 1$ if $S_k(y^{(i)}) = -$. Now check that $S_2(x^{(i)}) = -$, $x_1^{(i)} \geq 0$ and $x_3^{(i)} \geq 0$. If one of these conditions fails then $[x^{(i)}] \notin \Sigma_\Delta(d)$, and the algorithm stops with $[x^{(i-1)}]$ as the root of C. Otherwise $[x^{(i)}] \in \Sigma_\Delta(d)$ and we continue on with the sequence. Each step decreases the value of $\ell(x^{(i)}) > 0$ so the procedure will halt after a finite number of iterations.

To illustrate the algorithm, consider $\Delta(193, 318, 372) \in \mathrm{PIT}(-11)$, where $x = \chi(193, 318, 372) = (179, 372, 54)$. Then $y^{(1)} = \sigma_2(x) = (179, 94, 54)$ and $S(y^{(1)}) = - + +$ so we obtain $x^{(1)} = \sigma_1(179, 94, 54) = (9, 94, 54)$. Continuing in this fashion, we get $y^{(2)} = (9, 32, 54)$, $x^{(2)} = \sigma_3(y^{(2)}) = (9, 32, 10)$, $y^{(3)} = (9, 6, 10)$, $x^{(3)} = \sigma_3\sigma_1(y^{(3)}) = (3, 6, 2)$. Finally $y^{(4)} = (3, 4, 2) = x^{(4)}$ and $S_2(3, 4, 2) = +$, and the process stops at the root $[3, 6, 2] \in \Sigma_\Delta(-11)$. So $\Delta(193, 318, 372)$ is obtained from $\Delta(\chi^{-1}(3, 6, 2)) = \Delta(3, 4, 6) \in \mathrm{Root}(-11)$ by applying τ_2, τ_3 and τ_1 in that order.

5. The obtuse case

In this section we examine the structure of $\Sigma(d)$ where $d \leq 0$. If $x = (x_1, x_2, x_3)$ is a d-triple then Eq. (1.3) gives $2x_2(x_1 + x_3) = x_1^2 + x_2^2 + x_3^2 - d > -d \geq 0$. Thus x_2, $x_1 + x_3$ and $\ell(x)$ are non-zero and have the same sign, when $d \leq 0$. Two integers $i, j \in \{1, 2, 3\}$ are *adjacent* if $|i - j| = 1$.

Lemma 5.1. *Let $d \leq 0$ and let x be a d-triple with $\ell(x) > 0$.*

(a) *If y is in the same component of* Trip(d) *as x then $\ell(y) > 0$ and $y_2 > 0$.*
(b) *If i and j are adjacent and $S_i(x) = -$ then $S_j(x) = +$.*

Proof. For (a), there is a sequence of d-triples from x to y in which each term is a neighbor of the following term. So it is enough to assume that x and y are neighbors. In this case $y = \sigma_j(x)$ for some $j \in \{1, 2, 3\}$. Since $d \leq 0$ and the norm of any d-triple is positive, Lemma 3.2 shows that $\ell(y) > 0$, and (a) holds.

Assume that $S_1(x)$ and $S_2(x)$ are non-positive with at least one of the two being negative. Then $x_2 \leq x_1$ and $x_1 + x_3 \leq x_2$ and one of the two inequalities is strict. It follows that $x_1 + x_3 < x_1$, and $x_3 < 0$. On the other hand rewriting (1.3) as

$$(x_2 - x_1)^2 + x_3^2 - 2x_2x_3 = d$$

gives $x_2x_3 \geq 0$. Thus $x_2 \leq 0$, but $2x_2 \geq x_2 + x_1 + x_3 = \ell(x) > 0$ which is a contradiction. So if one of $S_1(x)$ or $S_2(x)$ is negative then the other is positive. Similarly if one of $S_2(x)$ or $S_3(x)$ is negative then one is positive. Part (b) follows since the only pairs of adjacent elements of $\{1, 2, 3\}$ are $\{1, 2\}$ or $\{2, 3\}$. □

Theorem 5.2. *For $d \leq 0$ the components of $\Sigma(d)$ are rooted or uprooted trees whose root elements form* Core(d). *When $d < -2$ each rooted tree component of $\Sigma(d)$ is isomorphic to $\mathbb{T}(0 + +)$, $\pm\mathbb{T}(+0+)$, $\widetilde{\mathbb{T}}(+ + +)$ or $\mathbb{T}(+ + +)$ as shown in Fig. 5, and each uprooted tree component is isomorphic to the inverse of one of these.*

Proof. Let x be a preferred d-triple with $\ell(x) > 0$ and let C be the component of $\Sigma(d)$ containing $[x]$. By Lemma 5.1(a), $\lambda(C)$ is bounded below by 0 and C satisfies the descending chain condition. Let y be $\sigma_1 x$, $\sigma_3\sigma_1 x$ or $\sigma_3 x$ and assume that $y \neq x$. Then one of $S_1(y)$ or $S_3(y)$ is negative and $S_2(y) = +$ by Lemma 5.1(b). Thus $[\sigma_2(y)]$ is a successor to $[y] = [x]$, and $[\sigma_2 x]$ is the only possible immediate predecessor of $[x]$ in $\Sigma(d)$. This shows that C is a rooted tree. Since $S_2(\sigma_2 y) = -$, Lemma 5.1(b) implies that $S(\sigma_2 y) = + - +$ and $[\sigma_2 y] \in \Sigma_\Delta(d)$. If $[\sigma_2 x]$ is a successor of $[x]$ in $\Sigma(d)$ then $S_2(x) = +$, which means that $S(\sigma_2 x) = + - +$ and $[\sigma_2 x] \in \Sigma_\Delta(d)$. These observations show that each element of $\mathcal{N}[x]$ which is a successor of $[x]$ is in $\Sigma_\Delta(d)$ and the only element of C in Core(d) is its root.

Now assume that $d < -2$ and that $[x]$ is the root of C. Then $S_1(x) \geq 0$, $S_2(x) \geq 0$ and $S_3(x) \geq 0$. Replacing x with ηx if necessary, we assume that $S_1(x) \leq S_3(x)$. If $S_1(x) = S_3(x) = 0$ then $x = (1, 1, 1)$ and $d = -1$, which is a contradiction. If $S_1(x) = S_2(x) = 0$ then $x = (1, 1, 0)$ and $d = 0$, which is also impossible. Three cases remain: (i) $S(x) = 0++$, (ii) $S(x) = +0+$ and (iii) $S(x) = +++$.

Case i: $x_1 = x_2 > x_3 > 0$. Here $\mathcal{N}[x] = \{[\sigma_2 x], [\sigma_2\sigma_3 x]\}$ and $[\sigma_2 x] \neq [\sigma_2\sigma_3 x]$ (because $\lambda[\sigma_2\sigma_3 x] = 5x_1 - 2x_3 > x_1 + 2x_3 = \lambda[\sigma_2 x]$). Each of $[\sigma_2 x]$ and $[\sigma_2\sigma_3 x]$ is the root of a component of $\Sigma_\Delta(d)$ isomorphic to $\mathbb{T}$ so neither can succeed the other. Thus both are immediate successors of $[x]$, and C is isomorphic to $\mathbb{T}(0++)$.

Case ii: $S(x) = +0+$ and $x_2 = x_1 + x_3$. If $x_1 = x_3$ then $x_2 = 2x_1$, $x = (1, 2, 1)$ and $d = -2$, which contradicts the assumption that $d < -2$. Thus $x_1 \neq x_3$ and the elements of $\mathcal{N}[x] = \{[x], [\sigma_2\sigma_1 x], [\sigma_2\sigma_3\sigma_1 x], [\sigma_2\sigma_3 x]\}$ can be seen to be distinct. Moreover $[\sigma_2\sigma_1 x]$, $[\sigma_2\sigma_3\sigma_1 x]$ and $[\sigma_2\sigma_3 x]$ are immediate successors of $[x]$ as each is the root of a $\mathbb{T}$ component of $\Sigma_\Delta(d)$. It follows that C is isomorphic to $\mathbb{T}(+0+)$.

Case iii: $S(x) = +++$. Here $[\sigma_2 x]$, $[\sigma_2\sigma_1 x]$, $[\sigma_2\sigma_3 x]$ and $[\sigma_2\sigma_3\sigma_1 x]$ are successors of $[x]$. As each is the root of a component of $\Sigma_\Delta(d)$ they are immediate successors of $[x]$. The only possible redundancy among the four is that $[\sigma_2\sigma_1 x] = [\sigma_2\sigma_3 x]$ when $x_1 = x_3$. In this case $[x]$ has three distinct immediate successors $[\sigma_2 x]$, $[\sigma_2\sigma_1 x]$ and $[\sigma_2\sigma_3\sigma_1 x]$. The component of $\Sigma_\Delta(d)$ whose root is the first and third of these is $\widetilde{\mathbb{T}}$ while the component whose root is the second of these is $\mathbb{T}$. So $C \cong \widetilde{\mathbb{T}}(+++)$ when $x_1 = x_3$. On the other hand if $x_1 \neq x_3$ then $[x]$ has four immediate successors all of which are roots of $\Sigma_\Delta(d)$ isomorphic to $\mathbb{T}$, and $C \cong \mathbb{T}(+++)$.

Finally, if C' is a component of $\Sigma(d)$ containing $[x]$ where $\ell(x) < 0$ then $\mu(C')$ is a rooted tree as examined above. Therefore C' is an uprooted tree intersecting the core of $\Sigma(d)$ in its root element and isomorphic to the inverse of $\mu(C')$. □

For $d \leq 0$ let $\mathcal{M}(d)$ be the set consisting of all triples of integers (A, x_1, x_3) which satisfy $A^2 = 2x_1x_3 + d$, $x_1 \geq x_3 \geq A \geq 0$ and $\gcd(A, x_1, x_3) = 1$. If $(A, x_1, x_3) \in \mathcal{M}(d)$ then $-d = 2x_1x_3 - A^2 \geq 2A^2 - A^2 = A^2$. Hence A is bounded above by $\sqrt{-d}$. Also, given A and d there are only finitely many factorizations x_1x_3 of $(A^2 - d)/2$. This shows that $\mathcal{M}(d)$ is finite for each $d \leq 0$.

Theorem 5.3. *Let $d \leq 0$. The function $(A, x_1, x_3) \mapsto [x_1, x_1 + x_3 - A, x_3]$ is a one-to-one correspondence between $\mathcal{M}(d)$ and the set of minimal elements in $\Sigma(d)$.*

Proof. If $(A, x_1, x_3) \in \mathcal{M}(d)$ then it is easy to show that $x = (x_1, x_1 + x_3 - A, x_3)$ is a d-triple with $\ell(x) > 0$. Also $S_1(x)$, $S_2(x)$ and $S_3(x)$ are non-negative, and $[x]$ is minimal in $\Sigma(d)$. Conversely suppose that x is a preferred representative for a minimal element of $\Sigma(d)$. Then $x_2 \geq x_1$ and $x_2 \geq x_3$. By replacing x with ηx we may assume that $x_1 \geq x_3$. Notice that $S_2(x) \geq 0$ since otherwise $[\sigma_2(x)]$ would be less than $[x]$ and $[x]$ would not be minimal. Eq. (1.3) can be rewritten as

$$x_2^2 - 2(x_1 + x_3)x_2 + (x_1^2 + x_3^2 - d) = 0$$

and the quadratic formula gives $x_2 = x_1 + x_3 \pm \sqrt{2x_1x_3 + d}$. In fact

$$x_2 = x_1 + x_3 - \sqrt{2x_1x_3 + d}$$

because $\operatorname{sign}(x_1 + x_3 - x_2) = S_2(x) \geq 0$. Thus $A = \sqrt{2x_1x_3 + d}$ is an integer and $x = (x_1, x_1 + x_3 - A, x_3)$. Also $x_1 \geq x_3 \geq A$ as $\operatorname{sign}(x_3 - A) = \operatorname{sign}(x_2 - x_1) \geq 0$, and $\gcd(x_1, x_3, A) = \gcd(x_1, x_3, x_2) = 1$. This shows that $(A, x_1, x_3) \in \mathcal{M}(d)$. □

Let C be the rooted tree component containing the minimal element of $\Sigma(d)$ associated with $(A, x_1, x_3) \in \mathcal{M}(d)$. If $x = (x_1, x_1 + x_3, -A, x_3)$ then $S(x) = +0+$ if $A = 0$, $S(x) = 0++$ if $x_1 > x_3 = A$, and $S(x) = +++$ if $x_3 > A$. Thus

$$C \cong \begin{cases} T(0++) & \text{if } x_1 > x_3 = A > 0 \\ T(+0+) & \text{if } A = 0 \\ \widetilde{T}(+++) & \text{if } x_1 = x_3 > A > 0 \\ T(+++) & \text{if } x_1 > x_3 > A > 0. \end{cases} \tag{5.1}$$

It is not hard to design a computer routine which produces the elements of $\mathcal{M}(d)$ for $d < -2$ and collates them into the four forms for C. As the number of components of $\Sigma_\Delta(d)$ in C is 2 if $C \cong T(0++)$, 3 if $C \cong T(+0+)$ or $\widetilde{T}(+++)$, and 4 if $C \cong T(+++)$, the routine can easily be extended to enumerate the elements of Root(d). (See Table 2.)

Corollary 5.4. *For each integer $d \leq 0$, Root(d) is a nonempty finite set and there are infinitely many primitive integral triangles with defect d.*

Table 2
Cardinalities of $\mathcal{R}(d) = \{\text{components of } \Sigma(d)\}$ and Root(d) for $-25 \le d \le -3$.

d	Number of components of $\Sigma(d)$ isomorphic to:				$\lvert\mathcal{R}(d)\rvert$	$\lvert$Root$(d)\rvert$
	$\mathbb{T}(0++)$	$\mathbb{T}(+0+)$	$\widetilde{\mathbb{T}}(+++)$	$\mathbb{T}(+++)$		
−3	1	0	0	0	2	2
−4	0	1	0	0	2	3
−5	1	0	0	0	2	2
−6	0	1	0	0	2	3
−7	1	0	1	0	4	5
−8	1	1	0	0	4	5
−9	1	0	0	0	2	2
−10	0	1	0	0	2	3
−11	1	0	0	1	4	6
−12	0	2	0	0	4	6
−13	1	0	0	0	2	2
−14	0	1	1	0	4	6
−15	2	0	0	1	6	8
−16	1	1	0	0	4	5
−17	1	0	1	0	4	5
−18	0	1	0	0	2	3
−19	1	0	0	1	4	6
−20	0	2	0	1	6	10
−21	2	0	0	0	4	4
−22	0	1	0	0	2	3
−23	1	0	1	2	8	13
−24	2	2	0	0	8	10
−25	1	0	0	0	2	2

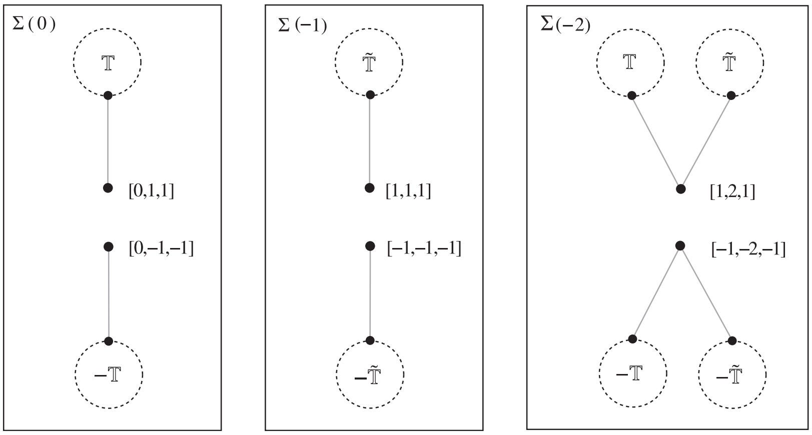

Fig. 6. $\Sigma(d)$ for $d = 0, -1, -2$.

Proof. Let $d \le 0$. Since Root(d) has no more than $4|\mathcal{M}(d)|$ elements it is a finite set. If d is even then $(A, x_1, x_3) = (0, -d/2, 1) \in \mathcal{M}(d)$ and if d is odd then $(A, x_1, x_3) = (1, (1-d)/2, 1) \in \mathcal{M}(d)$. This shows that $\mathcal{M}(d)$ and Root(d) are nonempty. As PIT(d) has at least one component isomorphic to $\mathbb{T}$ or $\widetilde{\mathbb{T}}$ it contains an ascending chain with infinitely many elements. □

Theorem 5.3 can be used to describe $\Sigma(d)$ for $d = 0, -1$ and -2. Here $\mathcal{M}(0) = \{(0, 1, 0)\}$, $\mathcal{M}(-1) = \{(1, 1, 1)\}$ and $\mathcal{M}(-2) = \{(0, 1, 1)\}$, and this easily leads to the pictures displayed in Fig. 6. As another example consider $\Sigma(-11)$ where $\mathcal{M}(-11) = \{(1, 3, 2), (1, 6, 1)\}$. There are two rooted tree components in $\Sigma(-11)$, one isomorphic to $T(+++)$ with root [3, 4, 2] and the other isomorphic to $T(0++)$ with root [6, 6, 1] by (5.1). Thus $\Sigma_\Delta(-11)$ has six minimal elements [3, 6, 2], [5, 10, 2], [6, 14, 3], [6, 18, 5], [6, 8, 1] and [11, 28, 6] (the first four are immediate successors of [3, 4, 2] and the last two are immediate successors of [6, 6, 1]). Applying $\widehat{\chi}^{-1}$ shows that Root(-11) consists of the six triangles $\Delta(3, 4, 6)$, $\Delta(5, 8, 10)$, $\Delta(8, 11, 14)$, $\Delta(12, 13, 18)$, $\Delta(2, 7, 8)$, and $\Delta(17, 22, 28)$.

6. The acute case

In this section we examine the structure of $\Sigma(d)$ when $d > 0$.

Lemma 6.1. *If x is a d-triple with $d > 0$ then there are integers $j, k \in \{1, 2, 3\}$ such that $S_j(x) = -$ and $S_k(x) = +$.*

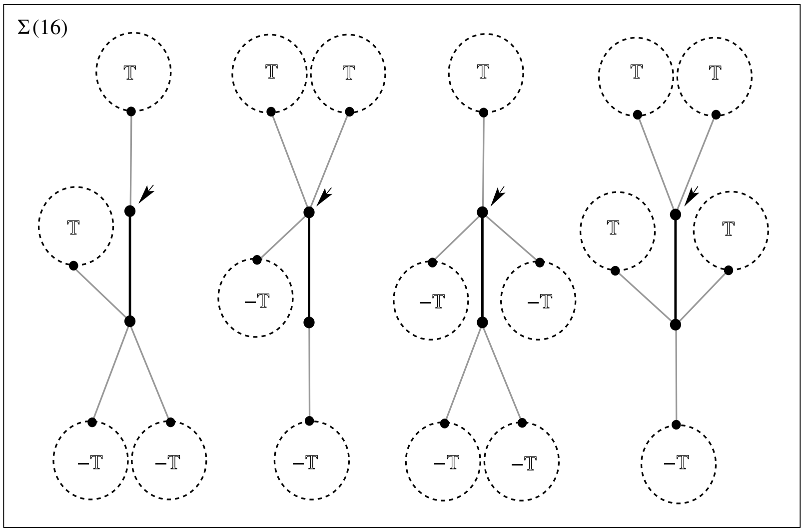

Fig. 7. $\Sigma(16)$.

Proof. Let x be a d-triple where $d > 0$. Suppose that $S_1(x) \geq 0$, $S_2(x) \geq 0$ and $S_3(x) \geq 0$. Then $x_2 \geq x_1$, $x_1 + x_3 - x_2 \geq 0$ and $x_2 \geq x_3$. From (1.3) we have

$$2x_1x_3 + d = (x_1 + x_3 - x_2)^2 \geq 0, \tag{6.1}$$

and so $x_1 + x_3 - x_2 = \sqrt{2x_1x_3 + d}$. Also $x_3 \geq x_1 + x_3 - x_2 = \sqrt{2x_1x_3 + d} \geq 0$ and $x_1 \geq x_1 + x_3 - x_2 = \sqrt{2x_1x_3 + d} \geq 0$. Hence $x_1x_3 \geq (\sqrt{2x_1x_3 + d})^2 = 2x_1x_3 + d$. Rewriting this inequality gives $x_1x_3 \leq -d < 0$ which contradicts the fact that $x_1 \geq 0$ and $x_3 \geq 0$. We conclude that $S_j(x) = -$ for some $j \in \{1, 2, 3\}$. Now suppose that $S_1(x) \leq 0$, $S_2(x) \leq 0$ and $S_3(x) \leq 0$. Then $-x$ is a d-triple with $S_1(-x) \geq 0$, $S_2(-x) \geq 0$ and $S_3(-x) \geq 0$, which is impossible by the argument above. Thus $S_k(x) = +$ for some $k \in \{1, 2, 3\}$. □

By the lemma $\Sigma(d)$ has no minimal or maximal element when $d > 0$. Moreover each component of Trip(d) has d-triples with positive length and d-triples with negative length. So Lemma 5.1(a) fails for $d > 0$. Lemma 5.1(b) fails as well. For example $(4, 3, -2)$ is a 17-triple with $\ell(4, 3, -2) > 0$ and $S(4, 3, -2) = --+$.

Lemma 6.2. *Let $d > 0$.*

(a) *Let x be a preferred d-triple with $x_2 \geq 0$. Then $[x] \in \mathrm{Core}(d)$ if and only if at least one of x_1 or x_3 is negative.*
(b) *If x is a d-triple with $N(x) \leq d$ then either $[x] \in \mathrm{Core}(d)$ or $x = \pm(0, 1, 0)$.*
(c) *Every element of $\mathrm{Core}(d)$ is comparable to an element $[x]$ with $N(x) \leq d$.*

Proof. Let x be a preferred d-triple with $x_2 \geq 0$. After replacing x by ηx, we may assume that $x_1 \geq x_3$, and $S_3(x) \geq S_1(x) \geq 0$. By Lemma 6.1, $S_3(x) = +$ and $S_2(x) = -$, so that $x_2 > x_3$ and $x_1 + x_3 - x_2 < 0$. If $x_2 = 0$ or $S_1(x) = 0$ then $[x] \in \mathrm{Core}(d)$ and $x_3 < 0$, and (a) holds in these cases. So assume that $x_2 > 0$ and $S_1(x) = +$. Then $S(x) = +-+$, and $[x] \in \Sigma_\Delta(d)$ if and only if $x_3 \geq 0$. Therefore $[x]$ is an element of $\mathrm{Core}(d)$ if and only if $x_3 < 0$, verifying (a).

Let x be a d-triple with $N(x) \leq d$. Thus $2x_2(x_1 + x_3) = N(x) - d \leq 0$. Suppose that $x_2 > 0$. Then $x_1 + x_3 \leq 0$, and one of x_1 or x_3 or $x_1 = x_3 = 0$. If $x_1 = x_3 = 0$ then $x = (0, 1, 0)$. If $x_1 < 0$ or $x_3 < 0$ then $[x] \in \mathrm{Core}(d)$ by (a). In either case (b) holds. If $x_2 = 0$ then the preferred representative of $[x]$ has a negative first or third coordinate, and again $[x] \in \mathrm{Core}(d)$. Finally, if $x_2 < 0$ then $-x$ is a d-triple with $N(-x) \leq d$ and $-x_2 > 0$, and $[-x] \in \mathrm{Core}(d)$ or $x = (0, -1, 0)$, proving (b).

Let y be a d-triple with $[y] \in \mathrm{Core}(d)$. Suppose that $\ell(y) > 0$. By Lemma 6.1 we can construct a descending sequence $y = y^{(1)} > y^{(2)} > y^{(3)} > \cdots$ in Trip(d). Let i_0 be the smallest integer such that $\ell(y^{(i_0+1)}) \leq 0$. If $N(y^{(i_0)}) \leq d$ then put $x = y^{(i_0)}$. If $N(y^{(i_0)}) > d$ then $N(y^{(i_0+1)}) \leq d$ by Lemma 3.2 and put $x = y^{(i_0+1)}$. In either case $[y]$ is comparable to $[x]$ and $N(x) \leq d$. Then $[x] \in \mathrm{Core}(d)$ by (c), and $[y^{(i)}] \in \mathrm{Core}(d)$ for each i because $\mathrm{Core}(d)$ is a full subposet of $\Sigma(d)$. This verifies (d) when $\ell(y) > 0$. Now suppose that $\ell(y) < 0$. Then $\ell(-y) > 0$ and $[-y] = \mu([y]) \in \mathrm{Core}(d)$. By the above, $[-y]$ is comparable to some $[x] \in \mathrm{Core}(d)$ with $N(x) \leq d$ and $[y] = \mu[-y]$ is comparable to $[-x] \in \mathrm{Core}(d)$. Finally, if $\ell(y) = 0$ then $N(y) = d + 2y_2(y_1 + y_3) = d - 2y_2^2 \leq d$. □

A *circuit* is a finite connected poset in which every element has exactly two distinct neighbors. A *segment* is a finite connected poset in which each element has at most two neighbors and two elements have just one neighbor.

Theorem 6.3. *If $d > 4$ then $\mathrm{Core}(d)$ has finitely many components each of which is a segment or a circuit.*

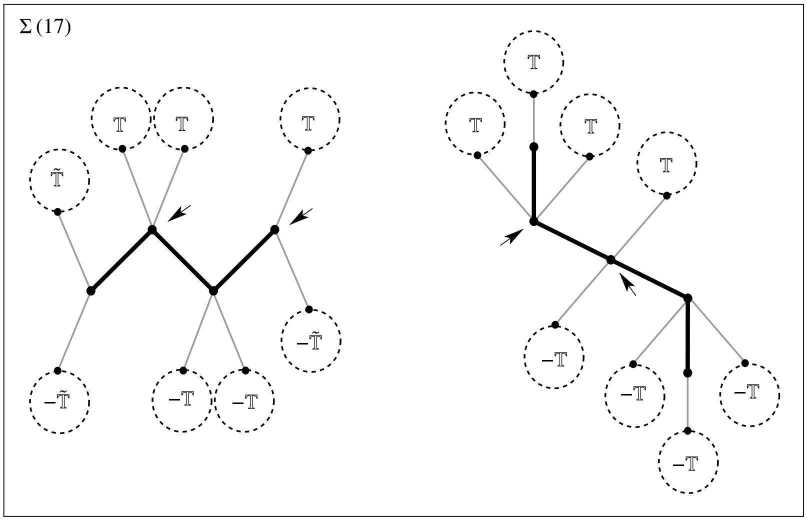

Fig. 8. $\Sigma(17)$.

Proof. We first claim that each element $[x] \in \mathrm{Core}(d)$ has at most two neighbors. We may arrange that $x_2 \geq 0$ (after replacing $[x]$ with $\mu[x]$) and that x is a preferred d-triple with $x_1 \geq x_3$. Hence $x_2 \geq x_1$ and $x_3 < 0$ by Lemma 6.2(a). Recall that each neighbor of $[x]$ in $\Sigma(d)$ is an element of $\mathcal{N}[x] = \{[\sigma_2 x], [\sigma_2\sigma_1 x], [\sigma_2\sigma_3\sigma_1 x], [\sigma_2\sigma_3 x]\}$.

Consider the d-triple $y = \sigma_2\sigma_3\sigma_1 x = (2x_2 - x_1, 7x_2 - 2x_1 - 2x_3, 2x_2 - x_3)$ with

$$y_2 = 2x_2 - x_3 + 2x_2 + 2(x_2 - x_1) + (x_2 - x_3) > 2x_2 - x_3 = y_3 \geq y_1,$$

and $S(y) = + - +$. Then $[y] \in \Sigma_\Delta(d)$ because $y_1 = x_2 + (x_2 - x_1) \geq 0$ and $y_3 \geq -x_3 > 0$. This shows that $[\sigma_2\sigma_3\sigma_1 x] \in \mathcal{N}[x] - \mathrm{Core}(d)$.

Suppose that $x_1 > 0$ and let y and z be the d-triples

$$y = \sigma_2 x = (x_1, 2x_1 + 2x_3 - x_2, x_3) \quad \text{and}$$
$$z = \sigma_2\sigma_3 x = (x_1, 3x_2 + 2x_1 - 2x_3, 2x_2 - x_3).$$

Then $[y] \in \mathrm{Core}(d)$ by Lemma 6.2, and $[y] \neq [x]$. On the other hand $[z] \in \Sigma_\Delta(d)$ because $z_1 > 0$, $z_2 = x_2 + 2(x_2 + x_1 - x_3) > x_2$, $z_3 > 0$ and $S(z) = + - +$.

Suppose that $x_1 \leq 0$. Let y and z be the d-triples

$$y = \sigma_2 x = (x_1, 2x_1 + 2x_3 - x_2, x_3) \quad \text{and}$$
$$z = \sigma_2\sigma_1 x = (2x_2 - x_1, 3x_2 - 2x_1 + 2x_3, x_3).$$

Then $S(y) = - + -$ and $y_2 = 2x_1 + 2x_3 - x_2 < 0$, which implies that $[y] \in -\Sigma_\Delta(d)$ and $[y] \notin \mathrm{Core}(d)$. On the other hand, $z_1 = x_2 + (x_2 - x_1) > 0$ while $z_3 < 0$. This shows that $[z] \in \mathrm{Core}(d)$, however $[z]$ will equal $[x]$ when $x_2 = z_2$. In the latter case, we have $x_2 = x_1 - x_3$ and

$$\sigma_2\sigma_3 x = (x_1, 3x_2 + 2x_1 - 2x_3, 2x_2 - x_3) = (x_1, 5x_2, 2x_2 - x_3).$$

Since $2x_2 - x_3 > 0$, $[\sigma_2\sigma_3 x]$ is an element of $\mathrm{Core}(d)$ which is distinct from $[x]$ provided that $x_1 \neq 0$ and $x_2 \neq 0$. If $x_1 = 0$ then $x = (0, 1, -1)$ and if $x_2 = 0$ then $x = (-1, 0, -1)$. In either case $d \leq 4$ contradicting assumption.

The previous paragraphs show that at least one, and no more than two, of the elements of $\mathcal{N}[x]$ are in $\mathrm{Core}(d)$. This proves that each element has valence one or two in $\mathrm{Core}(d)$ as claimed. By Lemma 6.2(c), each element of $\mathrm{Core}(d)$ is in a maximal chain which also contains an element $[x]$ with $N(x) \leq d$. As each element of $\mathrm{Core}(d)$ has valence one or two, $[x]$ is contained in at most two distinct maximal chains of $\mathrm{Core}(d)$. There are only finitely many d-triples x with $N(x) \leq d$ and so $\mathrm{Core}(d)$ is the union of a finite number of maximal chains. To complete the proof we will show that every maximal chain in $\mathrm{Core}(d)$ is finite.

For contradiction suppose that $\mathrm{Core}(d)$ has a maximal chain with infinitely many elements. Then $\mathrm{Core}(d)$ contains an infinite ascending sequence

$$[x^{(1)}] < [x^{(2)}] < [x^{(3)}] < \cdots \tag{6.2}$$

in which $x^{(i)}$ is a preferred d-triple for each i and $[x^{(i+1)}]$ is an immediate successor of $[x^{(i)}]$. (If $\mathrm{Core}(d)$ contains a descending sequence then applying μ will produce an ascending sequence.) Each step in the sequence increases $\lambda[x^{(i)}]$ so by replacing (6.2) with a tail we may assume that $x_2^{(i)} = \lambda[x^{(i)}] > 0$ for each i. As $[x^{(i+1)}]$ is an immediate successor of $[x^{(i)}]$, $x^{(i+1)} =$

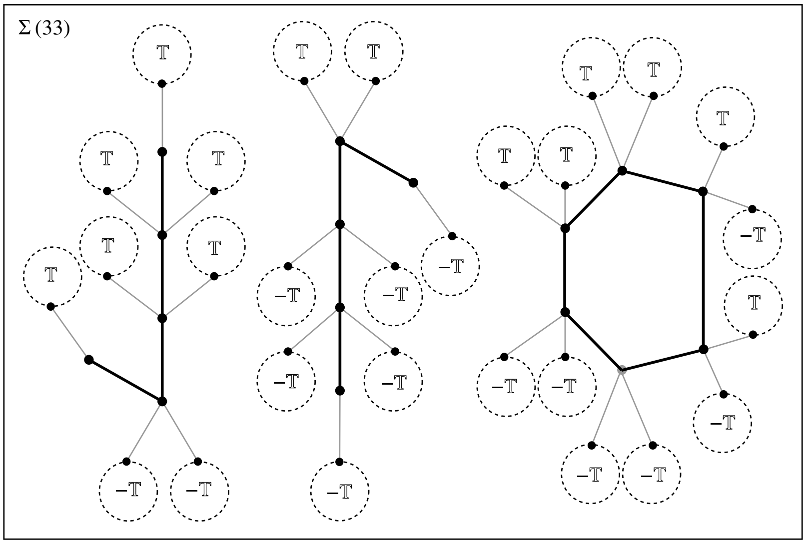

Fig. 9. $\Sigma(33)$.

$\gamma_2\sigma_2\gamma_1 x^{(i)}$ for some $\gamma_1, \gamma_2 \in \langle \sigma_1, \sigma_3 \rangle$. In fact each γ_1 equals σ_1 or σ_3 because if $\gamma_1 = 1$ then $[x^{(i+1)}] = [\sigma_2 x^{(i)}] \leq [x^{(i)}]$ and if $\gamma_1 = \sigma_3\sigma_1$ then $[x^{(i+1)}] = [\sigma_2\sigma_3\sigma_1 x^{(i)}] \notin \mathrm{Core}(d)$. Moreover if $\gamma_2 = \sigma_3$ then $\eta x^{(i+1)} = \eta\gamma_2\sigma_2\sigma_3 x^{(i)} = \gamma_2'\sigma_2\sigma_1\eta x^{(i)}$ where $\gamma_2' = \eta\gamma_2\eta \in \langle \sigma_1, \sigma_3 \rangle$, and by replacing $x^{(i+1)}$ and $x^{(i)}$ with $\eta x^{(i+1)}$ and $\eta x^{(i)}$ we may arrange that $\gamma_1 = \sigma_1$ for each i.

For convenience of notation we write $x = x^{(i)}$ and $y = x^{(i+1)} = \gamma_2\sigma_2\sigma_1 x$. Since x is a preferred representative of $[x]$ which is an immediate predecessor of $[y]$, $S_1(x) = +$ and $S_2(\sigma_1 x) = +$, implying that $x_2 - x_1 > 0$ and $x_2 - x_1 + x_3 > 0$. Also

$$\sigma_2\sigma_1 x = (2x_2 - x_1, 3x_2 - 2x_1 + 2x_3, x_3)$$

and $S_3(\sigma_2\sigma_1 x) = \mathrm{sign}(x_2 + (x_2 - x_1) + (x_2 - x_1 + x_3)) = +$. As $y = \gamma_2\sigma_2\sigma_1 x$ is a preferred d-triple, it follows that $\gamma_2 \in \langle \sigma_1 \rangle$. Therefore y equals $\sigma_2\sigma_1 x$ or $\sigma_1\sigma_2\sigma_1 x$. If $y = \sigma_2\sigma_1 x$ then $y_2 - y_1 = (x_2 - x_1) + 2x_3 < x_2 - x_1$. If $y = \sigma_1\sigma_2\sigma_1 x$ then

$$\sigma_1\sigma_2\sigma_1 x = (4x_2 - 3x_1 + 4x_3, 3x_2 - 2x_1 + 2x_3, x_3)$$

and $y_2 - y_1 = (x_2 - x_1) - 2(x_2 - x_1 + x_3) < x_2 - x_1$. This shows that each step in the sequence (6.2) strictly decreases the value of $x_2^{(i)} - x_1^{(i)}$. Thus there is an integer i_1 for which $x_2^{(i_1)} - x_1^{(i_1)}$ is negative. But then $S_1(x^{(i_1)}) = -$ which contradicts the assumption that $x^{(i_1)}$ is a preferred d-triple. This completes the proof. □

Circuit components in Core(d) are rare, at least for small values of d. The first one occurs in Core(33), and the circuit is a hexagon (see Fig. 9).[6] The next smallest values of d where Core(d) has a circuit are 37 (octagon), 57 (decagon), 65 (decagon), 79 (two squares), 101 (dodecagon) and 105 (two octagons).

A different proof of Theorem 6.3 can be given by analyzing possible configurations for the neighbors in $\Sigma(d)$ of each $[x] \in \mathrm{Core}(d)$. For $d > 4$ and $x_2 \geq 0$ one finds ten different 'local isomorphism types' describing these neighbors. Each of the ten types can be parametrized by a finite subset of $\mathbb{Z}^3$ and Core(d) has finitely many elements of each type. The illustrations of $\Sigma(d)$ in Figs. 7, 8 and 10 have been chosen so that all ten local types appear. The small arrows pointing to certain vertices indicate these types. This approach to Theorem 6.3 also provides the basis for a computer procedure which constructs the poset $\Sigma(d)$ when $d > 4$. The next theorem points the way to another procedure which achieves the same end.

For $d > 0$, let $\mathcal{M}(d)$ be the set of triples $(A, x_1, x_3) \in \mathbb{Z}^3$ with $A^2 - d = 2x_1x_3, -A \leq x_1 + x_3 \leq 0, x_1 \geq x_3$ and $\gcd(A, x_1, x_3) = 1$. If $(A, x_1, x_3) \in \mathcal{M}(d)$ then either $x_1x_3 \leq 0$ and $A^2 = 2x_1x_3 + d \leq d$, or $x_1x_3 > 0$ and

$$A^2 \leq A^2 + (x_1 - x_3)^2 = A^2 + (x_1 + x_3)^2 - 4x_1x_3 \leq 2A^2 - 2(A^2 - d) = 2d.$$

In either case $A \leq \sqrt{2d}$. For fixed A and d, $(A^2 - d)/2$ has only finitely many factorizations x_1x_3. This shows that $\mathcal{M}(d)$ is finite for each $d > 0$.

Theorem 6.4. *Let $x = (x_1, x_2, x_3)$ be a d-triple with $d > 0, x_2 \geq 0$ and $x_1 \geq x_3$. Then $N(x) \leq d$ if and only if $x = (x_1, x_1 + x_3 + A, x_3)$ for some $(A, x_1, x_3) \in \mathcal{M}(d)$.*

[6] The hexagonal circuit in Core(33) consists of the vertices [2, −7, −6], [4, −1, −4], [4, 1, −4], [6, 7, −2], [2, 5, −2], and [2, −5, −2].

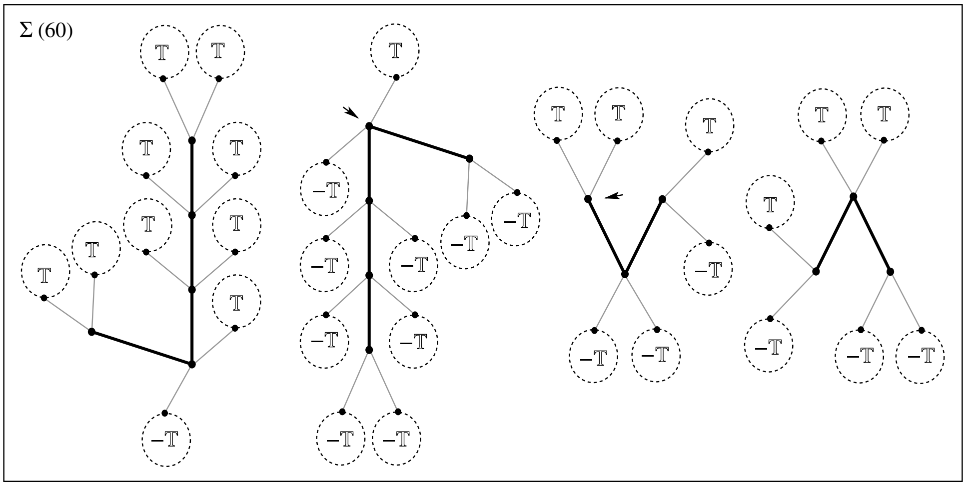

Fig. 10. $\Sigma(60)$.

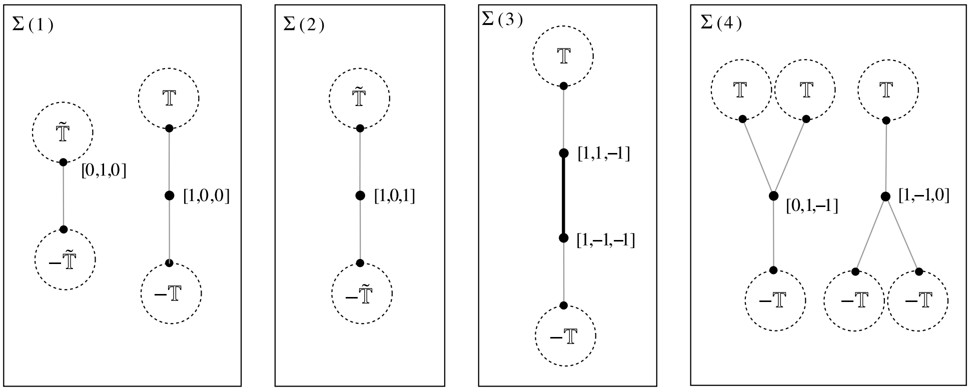

Fig. 11. $\Sigma(d)$ for $d = 1, 2, 3, 4$.

Proof. If $(A, x_1, x_3) \in \mathcal{M}(d)$ then it is easy to see that $x = (x_1, x_1 + x_3 + A, x_3)$ is a d-triple with $N(x) \leq d$. Conversely, suppose x is a d-triple with $N(x) \leq d, x_2 \geq 0$ and $x_1 \geq x_3$. Let $A = x_2 - x_1 - x_3$. By (1.3), $A^2 = 2x_1x_3 + d$ and $\gcd(x_1, x_3, A) = \gcd(x_1, x_2, x_3) = 1$. Also, $x_1 + x_3 = x_2 - A \geq -A$ and $x_1 + x_3 \leq 0$ because $2x_2(x_1 + x_3) = N(x) - d \leq 0$. This shows that $(A, x_1, x_3) \in \mathcal{M}(d)$. □

In distinction from the obtuse case, the surjection from the finite set $\mathcal{M}(d)$ to the set of components of $\Sigma(d)$ need not be injective. So additional work is needed to pare the image of $\mathcal{M}(d)$ down to a unique set of component representatives of $\Sigma(d)$. A computer routine which constructs the Hasse diagram for $\Sigma(d)$ can be created by first enumerating $\mathcal{M}(d)$ and the associated elements of Core(d), and enlarging this set step-by-step to include neighboring elements in Core(d). Such a routine produced the data shown in Table 3.

Corollary 6.5. *For each integer $d > 0$,* Root(d) *is a nonempty finite set, and there are infinitely many primitive integral triangles with defect d.*

Proof. Choose integers A, x_1 and x_3 with $A = d \pmod 2$, $x_1x_3 = (A^2 - d)/2$ and $\gcd(x_1, x_3, A) = 1$. (For example one might take A to be 0 or 1 according to the parity of d, $x_1 = 1$ and $x_3 = (A^2 - d)/2$.) Even though A, x_1 and x_2 may not satisfy the condition $-A \leq x_1 + x_3 \leq 0$ required to be an element of $\mathcal{M}(d)$, the proof of Theorem 6.4 shows that $x = (x_1, x_1 + x_3 + A, x_3)$ is a d-triple. The component of $\Sigma(d)$ containing $[x]$ has at least one component of $\Sigma_\Delta(d)$ in it. □

Fig. 11 displays the Hasse diagrams for the posets $\Sigma(d)$ where $1 \leq d \leq 4$ which are obtained by first observing that $\mathcal{M}(1) = \{(1, 0, -1), (1, 0, 0)\}$, $\mathcal{M}(2) = \{(0, 1, -1), (2, -1, -1)\}$, $\mathcal{M}(3) = \{(1, 1, -1)\}$ and $\mathcal{M}(4) = \{(2, 0, -1)\}$.

The last portion of the proof of Theorem 6.3 suggests that there are positive integers d for which Core(d) contains arbitrarily long chains. Here is an example: Let $d = b^2 + 2$ where b is an odd positive integer. Then $(1, b, -1)$ is a d-triple.

Table 3
Cardinalities of $\mathcal{R}(d) = \{\text{components of } \Sigma(d)\}$ and Root(d) for $3 \le d \le 25$.

d	$\lvert\mathcal{R}(d)\rvert$	$\lvert\text{Root}(d)\rvert$
1	2	2
2	1	1
3	1	1
4	2	3
5	1	2
6	1	2
7	2	3
8	2	3
9	3	7
10	1	3
11	1	3
12	2	4
13	1	6
14	2	4
15	2	4
16	4	9
17	2	8
18	1	4
19	1	5
20	2	6
21	2	8
22	1	6
23	2	5
24	4	8
25	5	18
26	1	5
27	1	5
28	2	10
29	1	10
30	2	6
31	2	9
32	2	7
33	3	14
34	3	9
35	2	6
36	4	12
37	2	14
38	1	6
39	2	8
40	4	14
41	2	16
42	2	8
43	1	9
44	2	12
45	2	12
46	2	12
47	2	7
48	4	12
49	8	33
50	1	7
51	2	10

By successively applying $\sigma_2\sigma_1$ to $(1, b, -1)$ we obtain a maximal chain C in $\Sigma(d)$

$$[1, b, -1] < [2b-1, 3b-4, -1] < \cdots < [(b^2+1)/2, (b^2+1)/2, -1]$$

($x_2 - x_1$ decreases by 2 for each step up in C). By Lemma 6.2, $C \subset \text{Core}(d)$, and $C \cup \mu C$ is a chain of length $b+1$ in Core(d). Similarly, if $d = b^2+2$ where b is an even positive integer then there is a chain of length b in Core(d). This example confirms that there are infinitely many different isomorphism types of components of Core(d) for $d > 0$.

7. Isosceles integral triangles

In this section we consider the isosceles triangles in PIT(d). If $\Delta(a, b, c)$ is isosceles then it has one of the forms $\Delta(a, c, c)$ or $\Delta(a, a, c)$. So $\widehat{\chi}$ carries the set of isosceles triangles in PIT(d) onto the intersection of $\Sigma_\Delta(d)$ with $I_1(d) \cup I_2(d)$ where

$$I_1(d) = \{[x_1, x_2, x_3] \in \Sigma(d) \mid x_3 = 0\} \quad \text{and}$$
$$I_2(d) = \{[x_1, x_2, x_3] \in \Sigma(d) \mid x_1 = x_3\}.$$

We will give brief descriptions of $I_1(d)$ and $I_2(d)$ for $|d| > 2$.

Let Q be a connected infinite subposet of a poset P such that each pair of neighbors in Q are also neighbors in P. Then Q is a *line in P* if each element of Q has valence two in Q. A line is *vertical* if each element has upward valence 1, and *bent* if it has exactly one element with upward valence not equal to 1.

If $[x_1, x_2, 0] \in I_1(d)$ then $d = (x_2 - x_1)^2$. So $I_1(d)$ is nonempty only when d is a perfect square, and we write $d = D^2$ for $D > 1$. If $x = (x_1, x_2, 0)$ where $x_2 - x_1 = \pm D$, $-D < x_2 < D$ and $x_1x_2 < 0$ then $N(x) = (x_1 - x_2)^2 + 2x_1x_2 < d$, and $[x] \in I_1(d) \cap \mathrm{Core}(d)$. It is not hard to show that

$$L_x = \left\{[(\sigma_2\sigma_1)^k x] \mid x \in Z\right\} \subset I_1(d)$$

is a vertical line in $\Sigma(d)$ that intersects $\mathrm{Core}(d)$ in the single point $[x]$ and $[x]$ is an endpoint of a segment component of $\mathrm{Core}(d)$. Furthermore $I_1(d)$ is the disjoint union of $2\phi(D)$ of these vertical lines L_x.

If $[x_1, x_2, x_1] \in I_2(d)$ then $d = 2(x_2 - x_1)^2 - x_2^2$. So $I_2(d)$ is nonempty only when $d = 2B^2 - A^2$ for relatively prime integers A and B. The 'generalized Pell equation'

$$A^2 - 2B^2 = -d \tag{7.1}$$

is known to have a primitive integer solution (A, B) only if d is not divisible by 4 or any prime congruent to ± 3 modulo 8 (see Section 16.3 of [6] for example). Assume that d has this form and that $|d| > 2$. If $[x] = [x_1, x_2, x_1] \in I_2(d)$ then

$$L'_x = \left\{[(\sigma_2\sigma_3\sigma_1)^k (x)] \mid k \in \mathbb{Z}\right\} \subset I_2(d)$$

is a line in $\Sigma(d)$. If $(x_1, x_2, x_1) \in \textsc{Trip}(d)$ then $(A, B) = (x_2, x_2 - x_1)$ is a primitive solution to the Pell equation (7.1). The family of solutions to (7.1) corresponding to the line L'_x forms precisely what is known as a 'class' of primitive solutions to (7.1).[7] Such a class contains a unique 'fundamental solution' and the set of primitive fundamental solutions has cardinality 2^s where s is the number of odd prime divisors of d. (Details may be found in Section 16.3 of [6].) So there are 2^s lines L'_x. It is not hard to show that these lines are pairwise disjoint and constitute all of $I_2(d)$. Each L'_x intersects $\mathrm{Core}(d)$ in a single point whose complement in L'_x is the union of the axis rays of two components of $\Sigma_\Delta(d) \cup -\Sigma_\Delta(d)$ isomorphic to $\pm\widetilde{\mathbb{T}}$. If $d > 2$ then each L'_x is a vertical line intersecting $\mathrm{Core}(d)$ in an endpoint of a segment component of $\mathrm{Core}(d)$. If $d < -2$ then each L'_x is a bent line which is the intersection of $I_2(d)$ with a component of $\Sigma(d)$ isomorphic to $\pm\widetilde{\mathbb{T}}(+++)$, and the root element of L coincides with the root element of that component of $\Sigma(d)$.

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[7] This is usually formulated by pairing a solution (A, B) to (7.1) with $A + B\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. The class of solutions containing (A, B) corresponds to the set $\left\{(A + B\sqrt{2})(3 + 2\sqrt{2})^k \mid k \in \mathbb{Z}\right\}$. Here $3 + 2\sqrt{2}$ is the fundamental solution to the Pell equation $X^2 - 2Y^2 = 1$.