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Solitary and edge-minimal bases for representations of the simple lie algebra G_2

Robert G. Donnelly, Scott J. Lewis, Robert Pervine

Department of Mathematics and Statistics, Murray State University, Murray, KY 42071, USA

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Abstract

We consider two families of weight bases for "one-rowed" irreducible representations of the simple Lie algebra G_2 constructed in Donnelly et al [Constructions of representations of $\mathfrak{o}(2n+1, \mathbb{C})$ that imply Molev and Reiner–Stanton lattices are strongly Sperner, Discrete Math. 263 (2003) 61–79] using two corresponding families of distributive lattices (called "supporting graphs"), here denoted $L_{G_2}^{\mathrm{LM}}(k)$ and $L_{G_2}^{\mathrm{RS}}(k)$. We exploit the relationship between these bases and their supporting graphs to give combinatorial proofs that the bases enjoy certain uniqueness and extremal properties (the "solitary" and "edge-minimal" properties, respectively). Our application of the combinatorial technique we develop in this paper to obtain these results relies on special total orderings of the elements and edges of the lattices. We also apply this technique to another family of lattice supporting graphs to re-derive results obtained in Donnely et al. using different, more algebraic methods.

Keywords: Distributive lattice; Supporting graph; Solitary basis; Edge-minimal basis

1. Introduction

This paper is a sequel to [2]. In [2] we derived combinatorial results (namely, "rank symmetry," "rank unimodality," and the "strong Sperner property") for two families of distributive lattices, there denoted $L_B^{\mathrm{Mol}}(k, 2n)$ and $L_B^{\mathrm{RS}}(k, 2n)$, by explicitly constructing actions of the odd orthogonal Lie algebras $\mathfrak{o}(2n+1, \mathbb{C})$ on these lattices. As a consequence we were also able to obtain new constructions of certain families of finite-dimensional irreducible representations of the simple Lie algebra G_2 on two families of "G_2 lattices." We call these the "one-rowed" representations of G_2. The combinatorial setting of the lattices $L_B^{\mathrm{Mol}}(k, 2n)$ and $L_B^{\mathrm{RS}}(k, 2n)$ was nice enough to allow us to draw certain conclusions about our odd orthogonal representation constructions: in particular, we showed that the bases we found for the representing spaces enjoy certain uniqueness and extremal properties [2, Corollary 3.1]. However, at the time we could not extend those methods to establish the same uniqueness and extremal properties of the bases obtained in our G_2 constructions. In this paper we provide a new combinatorial technique (Theorem 4.1), applied in the setting of our two families of G_2 lattices, to prove that the bases for the one-rowed representations of G_2 found in [2] are "solitary" and "edge-minimal" (Theorem 5.1). A basis with the solitary property is uniquely identified in a precise way by its "supporting graph"; a basis with the edge-minimal property has a supporting graph which does not contain as a

E-mail address: scott.lewis@murraystate.edu (S.J. Lewis).

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subgraph the supporting graph for any other basis. (See Section 3 for precise definitions; see [1] for further discussion of these properties.) We see Theorem 5.1 as additional evidence that the solitary property (uniqueness) is closely connected to the edge-minimal property (extremal). We also see this as further evidence that while the solitary property is necessarily rare, many, and possibly all, irreducible representations of the simple Lie algebras have at least one basis which enjoys this combinatorial property.

Our new combinatorial technique for demonstrating that certain families of bases enjoy these uniqueness and extremal properties extends methods of [1]. The methods of [1] apply to a number of bases, but only when these bases behave in a particularly nice way when one restricts to the action of Lie subalgebras ("restricts irreducibly for a chain of Levi subalgebras," in the parlance of that paper). Here we use Theorem 4.1 to demonstrate, for the first time, uniqueness and extremal properties for two families of bases—those bases associated to our two families of G_2 lattices—which do not meet the specialized algebraic conditions required by the methods of [1]. We also apply Theorem 4.1 to re-derive in Theorem 6.1 the analogous results for our odd orthogonal constructions obtained in [2]. In the future we hope to apply this new technique to other bases, such as those mentioned in Table 1 of [1] or those found in [6].

2. Combinatorial setting for our results; G_2 lattices

In this paper, we identify a poset with its Hasse diagram [8, p. 98], and all posets will be finite. For elements $\mathbf{s}$ and $\mathbf{t}$ of a poset P, there is a directed edge $\mathbf{s} \to \mathbf{t}$ in the Hasse diagram if and only if $\mathbf{t}$ *covers* $\mathbf{s}$, i.e. $\mathbf{s} < \mathbf{t}$ and there is no $\mathbf{x}$ in P such that $\mathbf{s} < \mathbf{x} < \mathbf{t}$. Let I be any set. A poset P is an *edge-colored poset with edges colored by the set I* if there is a function assigning to each edge of the Hasse diagram of P an element from the set I. If an edge $\mathbf{s} \to \mathbf{t}$ in P is assigned color $i \in I$, we write $\mathbf{s} \xrightarrow{i} \mathbf{t}$. When we depict the Hasse diagram for a poset, its edges are directed "up." We say the vertex $\mathbf{s}$ and the edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ are *below* $\mathbf{t}$, and the vertex $\mathbf{t}$ and the edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ are *above* $\mathbf{s}$. The vertex $\mathbf{s}$ is a *descendant* of $\mathbf{t}$, and $\mathbf{t}$ is an *ancestor* of $\mathbf{s}$. We let $\mathscr{V}(P)$ denote the vertex set of P, let $\mathscr{E}_i(P)$ denote the set of edges in P of color i, and set $\mathscr{E}(P) := \bigcup_{i \in I} \mathscr{E}_i(P)$. If J is a subset of I, remove all edges from P whose colors are not in J; connected components of the resulting edge-colored poset are called *J-components* of P. Let Q be another edge-colored poset with edge colors from I. If the vertices of Q are a subset of the vertices of P and the edges of Q of color i are a subset of the edges of P of color i for each $i \in I$, then Q is an *edge-colored subgraph* of P. Two edge-colored posets are isomorphic if there is a bijection between their vertex sets that preserves edges and edge colors. For a poset P, a *rank function* is a surjective function $\rho : P \longrightarrow \{0, \ldots, l\}$ $(l \geqslant 0)$ with the property that if $\mathbf{s} \to \mathbf{t}$ in the Hasse diagram for P, then $\rho(\mathbf{s}) + 1 = \rho(\mathbf{t})$. We call l the *length* of P with respect to ρ. For any $\mathbf{s}$ in P, $\rho(\mathbf{s})$ is the *rank* of $\mathbf{s}$. The set $\rho^{-1}(i)$ is the *ith rank* of P. A poset which possesses a rank function is called a *ranked poset*. A ranked poset that is connected has a unique rank function. See [8] for definitions of other combinatorial terms.

We say a poset P *has no open vees* if: (1) Whenever $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ are elements in P for which $\mathbf{s}$ and $\mathbf{t}$ both cover $\mathbf{r}$, then there exists a unique $\mathbf{u}$ in P which covers both $\mathbf{s}$ and $\mathbf{t}$, and (2) whenever $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{u}$ are elements in P for which $\mathbf{u}$ covers both $\mathbf{s}$ and $\mathbf{t}$, then there exists a unique $\mathbf{r}$ in P which is covered by both $\mathbf{s}$ and $\mathbf{t}$. Note that modular lattices, and therefore distributive lattices, are ranked lattices with no open vees. A *path from* $\mathbf{s}$ *to* $\mathbf{t}$ in a poset P is a sequence $(\mathbf{s}_0 = \mathbf{s}, \mathbf{s}_1, \ldots, \mathbf{s}_r = \mathbf{t})$ such that for $1 \leqslant p \leqslant r$ it is the case that either $\mathbf{s}_{p-1} \to \mathbf{s}_p$ or $\mathbf{s}_p \to \mathbf{s}_{p-1}$. We say this path has *length r*. The distance $\mathbf{dist}(\mathbf{s}, \mathbf{t})$ between $\mathbf{s}$ and $\mathbf{t}$ in a connected poset P is the minimum length achieved when all paths from $\mathbf{s}$ to $\mathbf{t}$ in P are considered. If P is a ranked poset and if $\mathbf{s} \leqslant \mathbf{t}$ in P, then $\mathbf{dist}(\mathbf{s}, \mathbf{t}) = \rho(\mathbf{t}) - \rho(\mathbf{s})$. If P is a ranked lattice with no open vees, then one can see that for any $\mathbf{s}$ and $\mathbf{t}$, $\mathbf{dist}(\mathbf{s}, \mathbf{t}) = 2\rho(\mathbf{u}) - \rho(\mathbf{s}) - \rho(\mathbf{t}) = \rho(\mathbf{s}) + \rho(\mathbf{t}) - 2\rho(\mathbf{r})$, where $\mathbf{u}$ (respectively, $\mathbf{r}$) is the unique least upper bound (resp. greatest lower bound) of $\mathbf{s}$ and $\mathbf{t}$ in P; hence P is a modular lattice.

Throughout this paper k denotes a positive integer. Following [2], we let $L_{G_2}^{\mathrm{RS}}(k)$ (respectively, $L_{G_2}^{\mathrm{LM}}(k)$) denote the set of all 7-tuples $\mathbf{s} = (s_1, \ldots, s_7)$ of nonnegative integers such that $\sum s_i = k$ and satisfying the rule that $s_1 \neq 0$ only if $s_7 = 0$ (respectively, the rule that $s_4 \in \{0, 1\}$). For any such $\mathbf{s} = (s_1, \ldots, s_7)$ and $\mathbf{t} = (t_1, \ldots, t_7)$, let $S_j := \sum_{i=1}^{j} s_i$ and $T_j := \sum_{i=1}^{j} t_i$ $(1 \leqslant j \leqslant 7)$. Observe that $\mathbf{s}$ and $\mathbf{t}$ are uniquely identified by the sequences of partial sums $(S_1, \ldots, S_6)$ and $(T_1, \ldots, T_6)$, respectively. (We ignore $S_7 = \sum_{i=1}^{7} s_i$ and $T_7 = \sum_{i=1}^{7} t_i$ since both sums are equal to k.) Declare that $\mathbf{s} \leqslant \mathbf{t}$ if and only if $S_j \leqslant T_j$ for $1 \leqslant j \leqslant 6$. One can see that the poset $L_{G_2}^{\mathrm{RS}}(k)$ (respectively, $L_{G_2}^{\mathrm{LM}}(k)$) coincides with the distributive lattice denoted $L_G^{\mathrm{RS}}(2, k\omega_1)$ (respectively, $L_G^{\mathrm{Lit}}(2, k\omega_1)$) in [2]. The elements of $L_{G_2}^{\mathrm{RS}}(k)$ correspond naturally with certain combinatorial objects called "tableaux" appearing in [4]; there, these tableaux are associated

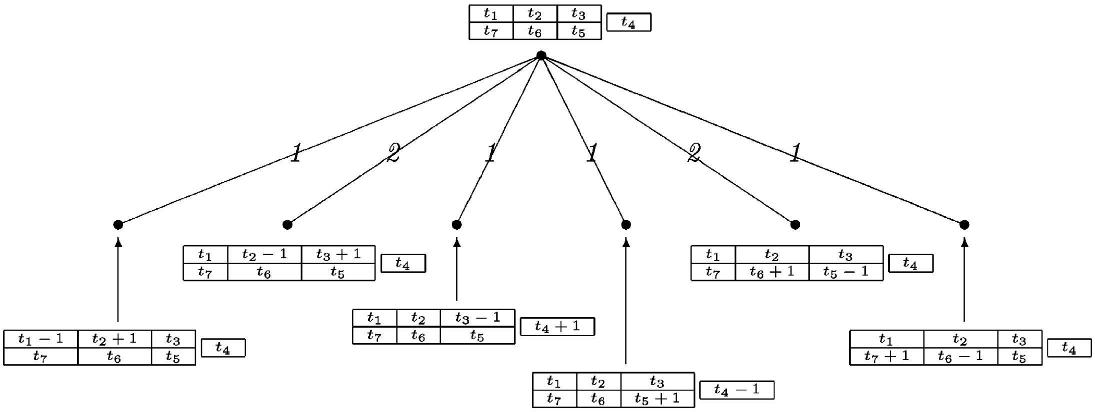


Fig. 1.

with the simple Lie algebra of type B_3. The distributive lattice $L^{\mathrm{RS}}_{G_2}(k)$ first appeared in [7]. The elements of $L^{\mathrm{LM}}_{G_2}(k)$ correspond naturally with tableaux appearing in [5]. See [2] for a discussion of the connection of the lattices $L^{\mathrm{LM}}_{G_2}(k)$ with Molev's work. For these reasons we call $L^{\mathrm{RS}}_{G_2}(k)$ and $L^{\mathrm{LM}}_{G_2}(k)$ "Reiner–Stanton" and "Littelmann–Molev" G_2 lattices, respectively. We define the *boundary for* $L^{\mathrm{RS}}_{G_2}(k)$ to be the set

$$\{(b_1, \ldots, b_7) \in L^{\mathrm{RS}}_{G_2}(k) | b_j + b_{j+1} = k \text{ where } 1 \leqslant j \leqslant 6\}.$$

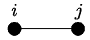
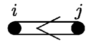
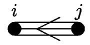
For $L^{\mathrm{LM}}_{G_2}(k)$, the *boundary* is the set

$$\left\{ (b_1, \ldots, b_7) \in L^{\mathrm{LM}}_{G_2}(k) \;\middle|\; \begin{array}{l} b_j + b_{j+1} = k \text{ where } j = 1, 2, 5, \text{ or } 6; \\ \text{or } b_3 + b_4 + b_5 = k \end{array} \right\}.$$

Let L be one of $L^{\mathrm{LM}}_{G_2}(k)$ or $L^{\mathrm{RS}}_{G_2}(k)$. Observe that in L, $\mathbf{s} \vee \mathbf{t}$ is the element identified with the sequence of partial sums $(\max(S_1, T_1), \ldots, \max(S_6, T_6))$, while $\mathbf{s} \wedge \mathbf{t}$ is the element identified with $(\min(S_1, T_1), \ldots, \min(S_6, T_6))$. It is easy to see that $\rho(\mathbf{s}) = \sum_{i=1}^{6} S_i$ is the unique rank function. It follows that $\mathbf{dist}(\mathbf{s}, \mathbf{t}) = \sum_{i=1}^{6} |S_i - T_i|$. The unique maximal element in L is $(k, 0, 0, 0, 0, 0, 0)$; the unique minimal element is $(0, 0, 0, 0, 0, 0, k)$. So L has length $6k$. At times it is convenient to view an element $\mathbf{t} = (t_1, \ldots, t_7)$ in L on the following "grid": $\begin{array}{|c|c|c|}\hline t_1 & t_2 & t_3 \\ \hline t_7 & t_6 & t_5 \\ \hline\end{array}\ \boxed{t_4}$. There is a covering relation $\mathbf{s} \rightarrow \mathbf{t}$ if $\mathbf{s}$ is one of $\begin{array}{|c|c|c|}\hline t_1-1 & t_2+1 & t_3 \\ \hline t_7 & t_6 & t_5 \\ \hline\end{array}\ \boxed{t_4}$, $\begin{array}{|c|c|c|}\hline t_1 & t_2 & t_3-1 \\ \hline t_7 & t_6 & t_5 \\ \hline\end{array}\ \boxed{t_4+1}$, $\begin{array}{|c|c|c|}\hline t_1 & t_2 & t_3 \\ \hline t_7 & t_6 & t_5+1 \\ \hline\end{array}\ \boxed{t_4-1}$, or $\begin{array}{|c|c|c|}\hline t_1 & t_2 & t_3 \\ \hline t_7+1 & t_6-1 & t_5 \\ \hline\end{array}\ \boxed{t_4}$ (in which case we give this edge color 1), or if $\mathbf{s}$ is one of the two vertices $\begin{array}{|c|c|c|}\hline t_1 & t_2-1 & t_3+1 \\ \hline t_7 & t_6 & t_5 \\ \hline\end{array}\ \boxed{t_4}$ or $\begin{array}{|c|c|c|}\hline t_1 & t_2 & t_3 \\ \hline t_7 & t_6+1 & t_5-1 \\ \hline\end{array}\ \boxed{t_4}$ (in which case we give this edge color 2). Thus, the six possible descendants of $\mathbf{t}$ in L can be depicted as in Fig. 1. One can see that there is a one-to-one correspondence between the elements of the boundary set and the ranks of L. Define a *total order* on the elements of L as follows: for distinct $\mathbf{s}$ and $\mathbf{t}$ in L, we say $\mathbf{s}$ *precedes* $\mathbf{t}$ in the total order if (1) $\rho(\mathbf{s}) > \rho(\mathbf{t})$; or (2) $\rho(\mathbf{s}) = \rho(\mathbf{t})$ and $\mathbf{dist}(\mathbf{s}, \mathbf{b}) < \mathbf{dist}(\mathbf{t}, \mathbf{b})$, where $\mathbf{b}$ is the unique boundary element of L for which $\rho(\mathbf{b}) = \rho(\mathbf{s}) = \rho(\mathbf{t})$; or (3) $\rho(\mathbf{s}) = \rho(\mathbf{t})$, $\mathbf{dist}(\mathbf{s}, \mathbf{b}) = \mathbf{dist}(\mathbf{t}, \mathbf{b})$, and there exists a j such that $s_i = t_i$ for $i > j$ and $s_j < t_j$. If $\mathbf{s}$ is a nonmaximal element of L, let i be the largest index $(1 \leqslant i \leqslant 6)$ for which $\mathbf{t} = (s_1, \ldots, s_{i-1}, s_i + 1, s_{i+1} - 1, s_{i+2}, \ldots, s_7)$ is an element of the lattice L; in this case we write $\mathbf{t} = \mathbf{rightmost\_decrease}(\mathbf{s})$. See Section 7 for examples.

3. Lie algebra actions on edge-colored posets

Some of the notation and language of [1] are needed in this paper to develop the algebraic aspects of Theorems 4.1 and 5.1. Throughout this paper n is a positive integer, and we use $\mathfrak{g}$ to denote the semisimple Lie algebra of rank n with Chevalley generators $\{x_i, y_i, h_i\}_{i=1}^{n}$ satisfying the Serre relations associated to a Dynkin diagram $\mathscr{D}$ with n nodes. For the simple Lie algebras, we number the nodes of the Dynkin diagram as in [3, p. 58]. In what follows the number $\mathscr{D}_{j,i}$ can be found in the chart below by looking at the subgraph of $\mathscr{D}$ determined by the choice of distinct nodes i and j.

Subgraph	$\mathcal{D}_{i,j}$	$\mathcal{D}_{j,i}$
	0	0
	−1	−1
	−1	−2
	−1	−3

(With $i = 1$ and $j = 2$, the diagram in the last row of this chart is the Dynkin diagram for the simple Lie algebra G_2.) We use $\{\omega_1, \ldots, \omega_n\}$ to denote the corresponding fundamental weights. The jth simple root α_j can be identified with $\sum_{i=1}^{n} \mathscr{D}_{j,i}\omega_i$. Finite-dimensional irreducible $\mathfrak{g}$-modules are in one-to-one correspondence with *highest weights*, i.e. the nonnegative linear combinations of the fundamental weights. The *one-rowed* representations of G_2 are the irreducible representations corresponding to the highest weight $k\omega_1$. (From here on, vector spaces in this paper will be assumed to be finite-dimensional.) If V is a $\mathfrak{g}$-module, then there is at least one basis $\mathscr{B} := \{v_{\mathbf{s}}\}_{\mathbf{s}\in P}$ (where P is an indexing set with $|P| = \dim V$) consisting of eigenvectors for the actions of the h_i's: for any $\mathbf{s}$ in P and $1 \leqslant i \leqslant n$, there exists an integer $m_i(\mathbf{s})$ such that $h_i.v_{\mathbf{s}} = m_i(\mathbf{s})v_{\mathbf{s}}$. The *weight* of the basis vector $v_{\mathbf{s}}$ is the sum $wt(v_{\mathbf{s}}) := \sum_{i=1}^{n} m_i(\mathbf{s})\omega_i$. We say $\mathscr{B}$ is a *weight basis* for V. Form an edge-colored directed graph on the vertex set P which indicates the supports of the actions of the generators on the basis $\mathscr{B}$ as follows: a directed edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ of color i is placed from index $\mathbf{s}$ to index $\mathbf{t}$ if $c_{\mathbf{t},\mathbf{s}}v_{\mathbf{t}}$ (with $c_{\mathbf{t},\mathbf{s}} \neq 0$) appears as a term in the expansion of $x_i.v_{\mathbf{s}}$ as a linear combination in the basis $\{v_{\mathbf{x}}\}$, or if $d_{\mathbf{s},\mathbf{t}}v_{\mathbf{s}}$ (with $d_{\mathbf{s},\mathbf{t}} \neq 0$) appears when we expand $y_i.v_{\mathbf{t}}$ in the basis $\{v_{\mathbf{x}}\}$. The resulting edge-colored directed graph, which is also denoted by P, is the *supporting graph for the basis $\mathscr{B}$ of V*, or simply a *supporting graph for V*. Disregarding edge colors, a supporting graph is always the Hasse diagram for a ranked poset [1, Lemma 3.1.E]. To keep track of the actions of the generators on vectors of the basis $\mathscr{B}$ we sometimes attach the two coefficients $c_{\mathbf{t},\mathbf{s}}$ (the "x"-coefficient) and $d_{\mathbf{s},\mathbf{t}}$ (the "y"-coefficient) to each edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ of P. In this case,

$$x_i.v_{\mathbf{s}} = \sum_{\mathbf{t}:\mathbf{s}\xrightarrow{i}\mathbf{t}} c_{\mathbf{t},\mathbf{s}}v_{\mathbf{t}} \quad \text{and} \quad y_i.v_{\mathbf{t}} = \sum_{\mathbf{s}:\mathbf{s}\xrightarrow{i}\mathbf{t}} d_{\mathbf{s},\mathbf{t}}v_{\mathbf{s}}. \tag{1}$$

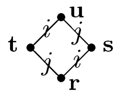
The supporting graph P together with the coefficients $\{(c_{\mathbf{t},\mathbf{s}}, d_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s}\xrightarrow{i}\mathbf{t}\in\mathscr{E}(P)}$ is the *representation diagram* (also denoted by P) for the basis $\mathscr{B}$ of V. If the coefficients $c_{\mathbf{t},\mathbf{s}}$ and $d_{\mathbf{s},\mathbf{t}}$ are positive rational numbers we say that the weight basis $\mathscr{B}$ is *positive rational*. A supporting graph P of V is *positive rational* if there is a positive rational basis for V which has P as its supporting graph. A supporting graph for a basis $\mathscr{B}$ of V is *edge-minimal* if no other weight basis for V has its supporting graph appearing as a proper edge-colored subgraph in the supporting graph for $\mathscr{B}$. Two weight bases $\{v_{\mathbf{s}}\}_{\mathbf{s}\in P}$ and $\{w_{\mathbf{t}}\}_{\mathbf{t}\in Q}$ for V are *diagonally equivalent* if there is an ordering on these bases with respect to which the corresponding change of basis matrix is diagonal; the bases are *scalar equivalent* if this diagonal matrix is a scalar multiple of the identity. The supporting graph for the basis $\mathscr{B}$ is *solitary* if no weight basis for V has the same supporting graph as $\mathscr{B}$ other than those bases that are diagonally equivalent to $\mathscr{B}$. Observe that, up to diagonal equivalence, the representation V can have at most a finite number of solitary bases. The adjectives *edge-minimal* and *solitary* apply to weight bases as well as supporting graphs. Up to diagonal equivalence, then, a solitary weight basis is uniquely identified by its supporting graph.

Let P be a ranked poset whose Hasse diagram edges are colored with colors taken from the set $\{1, \ldots, n\}$. For $\mathbf{s}$ in P, set $m_i(\mathbf{s}) := 2\rho_i(\mathbf{s}) - l_i(\mathbf{s})$, where $l_i(\mathbf{s})$ is the length of the i-component containing $\mathbf{s}$ and $\rho_i(\mathbf{s})$ is the rank of $\mathbf{s}$ within this i-component. Following [2], we say P satisfies the *structure condition* for $\mathfrak{g}$ if for $1 \leqslant i \leqslant n$, we have $m_i(\mathbf{s}) + \mathscr{D}_{j,i} = m_i(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{j} \mathbf{t}$ with $i \neq j$. Set $wt_P(\mathbf{s}) := \sum_{i=1}^{n} m_i(\mathbf{s})\omega_i$. Then P satisfies the structure condition if and only if $wt_P(\mathbf{s}) + \alpha_j = wt_P(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{j} \mathbf{t}$ in P. This condition depends not only on $\mathfrak{g}$ (information from the corresponding Dynkin diagram) but also on the combinatorics of P. An *edge-labelled poset P with colors from* $\{1, \ldots, n\}$ is an edge-colored ranked poset with edge colors from the set $\{1, \ldots, n\}$ together with an assignment of edge coefficients $\{(c_{\mathbf{t},\mathbf{s}}, d_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathscr{E}(P)}$. We call $\pi_{\mathbf{s},\mathbf{t}} := c_{\mathbf{t},\mathbf{s}} d_{\mathbf{s},\mathbf{t}}$ the *edge product* associated to a given edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in the edge-labelled poset P. The edge-labelled poset P satisfies the *crossing condition* if for any $\mathbf{s}$ in P and any color i ($1 \leqslant i \leqslant n$), we have

$$\sum_{\mathbf{r}:\mathbf{r} \xrightarrow{i} \mathbf{s}} \pi_{\mathbf{r},\mathbf{s}} - \sum_{\mathbf{t}:\mathbf{s} \xrightarrow{i} \mathbf{t}} \pi_{\mathbf{s},\mathbf{t}} = m_i(\mathbf{s}). \tag{2}$$

A relation of form (2) is a *crossing relation*. The edge-labelled poset P satisfies the *diamond condition* if for any pair of vertices $\mathbf{s}$ and $\mathbf{t}$ in P and any colors i and j ($1 \leqslant i, j \leqslant n$), we have

$$\sum_{\mathbf{u}:\mathbf{s} \xrightarrow{j} \mathbf{u} \text{ and } \mathbf{t} \xrightarrow{i} \mathbf{u}} c_{\mathbf{u},\mathbf{s}} d_{\mathbf{t},\mathbf{u}} = \sum_{\mathbf{r}:\mathbf{r} \xrightarrow{i} \mathbf{s} \text{ and } \mathbf{r} \xrightarrow{j} \mathbf{t}} d_{\mathbf{r},\mathbf{s}} c_{\mathbf{t},\mathbf{r}}, \tag{3}$$

where an empty sum is zero. For $\mathbf{s}$ and $\mathbf{t}$ in P, suppose there is a unique element $\mathbf{u}$ such that $\mathbf{s} \xrightarrow{j} \mathbf{u}$ and $\mathbf{t} \xrightarrow{i} \mathbf{u}$, and suppose there is a unique element $\mathbf{r}$ such that $\mathbf{r} \xrightarrow{i} \mathbf{s}$ and $\mathbf{r} \xrightarrow{j} \mathbf{t}$. Then we have this subgraph in P: 

The diamond condition in this case implies that

$$c_{\mathbf{u},\mathbf{s}} d_{\mathbf{t},\mathbf{u}} = d_{\mathbf{r},\mathbf{s}} c_{\mathbf{t},\mathbf{r}}, \quad c_{\mathbf{u},\mathbf{t}} d_{\mathbf{s},\mathbf{u}} = d_{\mathbf{r},\mathbf{t}} c_{\mathbf{s},\mathbf{r}} \quad \text{and} \quad \pi_{\mathbf{u},\mathbf{s}} \pi_{\mathbf{t},\mathbf{u}} = \pi_{\mathbf{r},\mathbf{s}} \pi_{\mathbf{t},\mathbf{r}}. \tag{4}$$

Any relation of form (3) or (4) is a *diamond relation*. Let $V[P]$ be the complex vector space with basis $\{v_{\mathbf{s}}\}_{\mathbf{s} \in P}$. The following lemma is a reformulation of Proposition 3.4 of [1].

Lemma 3.1. *Let P be an edge-labelled poset with colors from $\{1, \dots, n\}$ having the property that at least one of the two coefficients ($c_{\mathbf{t},\mathbf{s}}$ or $d_{\mathbf{s},\mathbf{t}}$) assigned to any given edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in P is nonzero. Then $V[P]$ is a $\mathfrak{g}$-module with the action of $\mathfrak{g}$ induced by the actions on $V[P]$ of the x_i's and y_i's described at* (1) *and the edge-labelled poset P is a representation diagram for the weight basis $\{v_{\mathbf{s}}\}_{\mathbf{s} \in P}$ of $V[P]$ if and only if P satisfies the diamond, crossing, and structure conditions. In this case, $h_i.v_{\mathbf{s}} = m_i(\mathbf{s}) v_{\mathbf{s}}$ for any $\mathbf{s}$ in P and $1 \leqslant i \leqslant n$, so $wt(v_{\mathbf{s}}) = \sum_{i=1}^{n} m_i(\mathbf{s}) \omega_i = wt_P(\mathbf{s})$.*

The main G_2 result of [2] was:

Theorem. *With edge coefficients assigned to the edges of $L_{G_2}^{\mathrm{LM}}(k)$ and $L_{G_2}^{\mathrm{RS}}(k)$ as in [2], the edge-labelled posets $L_{G_2}^{\mathrm{LM}}(k)$ and $L_{G_2}^{\mathrm{RS}}(k)$ are representation diagrams for the one-rowed representation of G_2 corresponding to highest weight $k\omega_1$.*

4. A combinatorial technique for demonstrating uniqueness and extremal properties of bases

The main result of this section allows us in certain circumstances to reduce the question of demonstrating uniqueness and extremal properties for a given basis for a simple Lie algebra representation to the problem of demonstrating that the supporting graph meets certain combinatorial conditions. We will normally only apply this result when actions of Lie algebra generators on the basis have been explicitly identified in such a way that we have explicit descriptions of the edge coefficients associated to the edges of the supporting graph. In this paper, the main application of Theorem 4.1 is in the proof of Theorem 5.1. (The total ordering on elements of $L_{G_2}^{\mathrm{LM}}(k)$ and $L_{G_2}^{\mathrm{RS}}(k)$ and the function **rightmost_decrease** defined in Section 2 will play the respective roles of the "reverse linear extension" and the "ancestor function" in Theorem 4.1.) We also apply Theorem 4.1 in Section 6 to re-derive some of the results of Corollary 3.1 from [2] for the odd orthogonal lattices $L_B^{\mathrm{Mol}}(k, 2n)$ and $L_B^{\mathrm{RS}}(k, 2n)$. We believe Theorem 4.1 can be applied to many of the other bases and supporting graphs mentioned in Table 1 of [1].

Let $(L, \leqslant)$ be a connected edge-labelled poset with rank function ρ and with edges colored by the set $\{1, \dots, n\}$. A *reverse linear extension $\mathscr{T}$ of L* is a total ordering on the elements of L (with the relation in $\mathscr{T}$ denoted by "$\leqslant_{\mathscr{T}}$") such that if $\mathbf{s} \leqslant \mathbf{t}$ in L then $\mathbf{t} \leqslant_{\mathscr{T}} \mathbf{s}$ in $\mathscr{T}$, i.e. $\mathbf{t}$ *precedes* $\mathbf{s}$ *in the total order* $\mathscr{T}$. An *ancestor function f* is a function $f : \mathscr{V}(L) \backslash \{\text{vertices of maximal rank}\} \rightarrow \mathscr{V}(L)$ with the property that if $\mathbf{s}$ is not of maximal rank in $\mathscr{V}(L)$, then $f(\mathbf{s})$ is an ancestor of $\mathbf{s}$. Let $\mathbf{x} \rightarrow \mathbf{y}$ and $\mathbf{s} \rightarrow \mathbf{t}$ be edges in L. The next definition is inductive: We say that $\pi_{\mathbf{x},\mathbf{y}}$ *can be* *expressed in terms of edge products prior to* $\mathbf{s} \to \mathbf{t}$ if (1) $\mathbf{y}$ precedes $\mathbf{t}$ in the total order, or (2) $\mathbf{y} = \mathbf{t}$ and $\mathbf{x}$ precedes $\mathbf{s}$, or (3) $\pi_{\mathbf{x},\mathbf{y}}$ is part of a diamond or crossing relation such that any other edge $\mathbf{p} \to \mathbf{q}$ involved in the relation has the property that $\pi_{\mathbf{p},\mathbf{q}}$ can be expressed in terms of edge products prior to $\mathbf{s} \to \mathbf{t}$. If $\mathbf{s}$ is a descendant of $\mathbf{t}$ in L with $\mathbf{s} \xrightarrow{i} \mathbf{t}$, then $\mathbf{s}$ is a *diamond descendant* (relative to the function *f*) if $f(\mathbf{s}) \neq \mathbf{t}$; we say $\mathbf{s}$ is a *crossing descendant* of $\mathbf{t}$ if $f(\mathbf{s}) = \mathbf{t}$ and all other descendants of $\mathbf{t}$ along edges of color i precede $\mathbf{s}$ in the total order $\mathscr{T}$; otherwise, $\mathbf{s}$ is an *exceptional descendant* of $\mathbf{t}$. The edge-labelled poset L together with $\mathscr{T}$ and *f* is *diamond-and-crossing friendly* if (1) whenever $\mathbf{s}$ is a diamond descendant of $\mathbf{t}$, then $f(\mathbf{s})$ precedes $\mathbf{t}$ in the total order $\mathscr{T}$, and (2) any exceptional descendant $\mathbf{s}$ of a given vertex $\mathbf{t}$ lies along an edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ whose edge product $\pi_{\mathbf{s},\mathbf{t}}$ can be expressed in terms of edge products prior to $\mathbf{s} \xrightarrow{i} \mathbf{t}$. While the hypotheses of the following theorem are somewhat involved, conditions of Theorem 5.1 put us in exactly this situation, where we have $\mathfrak{g} = G_2$, V a one-rowed representation of G_2, and L one of $L^{\mathrm{LM}}_{G_2}(k)$ or $L^{\mathrm{RS}}_{G_2}(k)$. Part (1) of Theorem 4.1 concludes that the product of coefficients on any edge of a certain kind of edge-labelled poset L is "uniquely determined" in a certain sense by the combinatorics of the poset. By Lemma 3.1.B of [1], this property is necessarily possessed by the representation diagram of any solitary basis for an irreducible $\mathfrak{g}$-module. Part (2) of Theorem 4.1 provides a partial converse to this result.

Theorem 4.1. *Let V be a* $\mathfrak{g}$*-module, and let L be the representation diagram for some given weight basis* $\{v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ *of V. Suppose that* $\pi_{\mathbf{s},\mathbf{t}} \neq 0$ *for each edge* $\mathbf{s} \xrightarrow{i} \mathbf{t}$ *in L, and suppose L is connected and has no open vees. Let* $\mathscr{T}$ *be a reverse linear extension of L and let f be an ancestor function. Suppose that L together with* $\mathscr{T}$ *and f is diamond-and-crossing friendly. Then*:

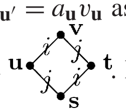
(1) *Let* $\{(a_{\mathbf{t},\mathbf{s}}, b_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathscr{E}(L)}$ *be another set of coefficients assigned to the edges of L such that the corresponding edge-labelled poset satisfies the diamond and crossing relations. Then on any edge* $\mathbf{s} \xrightarrow{i} \mathbf{t}$ *in L, it is the case that* $a_{\mathbf{t},\mathbf{s}} b_{\mathbf{s},\mathbf{t}} = \pi_{\mathbf{s},\mathbf{t}}$.
(2) *If V is an irreducible* $\mathfrak{g}$*-module, then the basis* $\{v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ *and the supporting graph L are solitary and edge-minimal.*

Our proof requires two preliminary results (Lemmas 4.2 and 4.3). Let $\{v_{\mathbf{s}}\}_{\mathbf{s}\in K}$ and $\{w_{\mathbf{t}}\}_{\mathbf{t}\in L}$ be two weight bases for a finite-dimensional $\mathfrak{g}$-module, where K and L are index sets. Let K and L also denote, respectively, the representation diagrams for these two bases. We say the representation diagram K and L are *edge product similar* if, under some correspondence between elements of the index sets, the edge-colored directed graphs K and L are isomorphic and the product of coefficients on any edge in K is the same as the product of coefficients on the corresponding edge in L.

Lemma 4.2. *Let L be the representation diagram for a weight basis* $\mathscr{B} = \{v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ *for a* $\mathfrak{g}$*-module V. Suppose that the product of the x-coefficient and y-coefficient on any edge in L is nonzero. Suppose L is connected and has no open vees. Let K be the representation diagram for another weight basis for V such that K and L are edge product similar. Let* $\mathbf{s}'$ *denote the vertex in K corresponding to the vertex* $\mathbf{s}$ *in L. Then there exist nonzero scalars* $\{a_{\mathbf{s}}\}_{\mathbf{s}\in L}$ *such that the representation diagram for the basis* $\{w_{\mathbf{s}'} := a_{\mathbf{s}} v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ *of V is K.*

Proof. Since L is connected and has no open vees, L has a unique maximal element $\mathbf{m}$. Set $w_{\mathbf{m}'} := v_{\mathbf{m}}$ (i.e. $a_{\mathbf{m}} := 1$). We can inductively produce scalars $a_{\mathbf{s}}$ such that $w_{\mathbf{s}'} = a_{\mathbf{s}} v_{\mathbf{s}}$ as follows: Let $\mathbf{t} \in L$ and suppose we have found an appropriate scalar $a_{\mathbf{t}}$ with $w_{\mathbf{t}'} := a_{\mathbf{t}} v_{\mathbf{t}}$ as required. If $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in L, then set

$$a_{\mathbf{s}} := a_{\mathbf{t}} \frac{c_{\mathbf{t}',\mathbf{s}'}}{c_{\mathbf{t},\mathbf{s}}} = a_{\mathbf{t}} \frac{d_{\mathbf{s},\mathbf{t}}}{d_{\mathbf{s}',\mathbf{t}'}}.$$

The quantities $a_{\mathbf{t}}(c_{\mathbf{t}',\mathbf{s}'}/c_{\mathbf{t},\mathbf{s}})$ and $a_{\mathbf{t}}(d_{\mathbf{s},\mathbf{t}}/d_{\mathbf{s}',\mathbf{t}'})$ are equal since $c_{\mathbf{t},\mathbf{s}} d_{\mathbf{s},\mathbf{t}}$ and $c_{\mathbf{t}',\mathbf{s}'} d_{\mathbf{s}',\mathbf{t}'}$ are the same nonzero product. The scalar $a_{\mathbf{s}}$ is well-defined in the following sense: Suppose $\mathbf{s} \xrightarrow{j} \mathbf{u}$, where $\mathbf{u} \in L$ is some other vertex for which we have an appropriate scalar $a_{\mathbf{u}}$ with $w_{\mathbf{u}'} = a_{\mathbf{u}} v_{\mathbf{u}}$ as required. Since L has no open vees, then we have a "diamond" of edges and vertices in L that looks like 

. By working within the representation diagrams L and K, respectively, we see that $x_i.y_j.(a_{\mathbf{t}}v_{\mathbf{t}}) = a_{\mathbf{t}}(x_i.y_j.v_{\mathbf{t}}) = a_{\mathbf{t}}(\cdots + c_{\mathbf{u},\mathbf{s}}d_{\mathbf{s},\mathbf{t}}v_{\mathbf{u}} + \cdots)$, while $x_i.y_j.(w_{\mathbf{t}'}) = \cdots + c_{\mathbf{u}',\mathbf{s}'}d_{\mathbf{s}',\mathbf{t}'}w_{\mathbf{u}'} + \cdots = \cdots + c_{\mathbf{u}',\mathbf{s}'}d_{\mathbf{s}',\mathbf{t}'}a_{\mathbf{u}}v_{\mathbf{u}} + \cdots$. Equate coefficients to get $a_{\mathbf{t}}c_{\mathbf{u},\mathbf{s}}d_{\mathbf{s},\mathbf{t}} = a_{\mathbf{u}}c_{\mathbf{u}',\mathbf{s}'}d_{\mathbf{s}',\mathbf{t}'}$. So

$$a_{\mathbf{s}} := a_{\mathbf{t}}\frac{c_{\mathbf{t}',\mathbf{s}'}}{c_{\mathbf{t},\mathbf{s}}} = a_{\mathbf{u}}\frac{c_{\mathbf{u}',\mathbf{s}'}d_{\mathbf{s}',\mathbf{t}'}}{c_{\mathbf{u},\mathbf{s}}d_{\mathbf{s},\mathbf{t}}}\frac{c_{\mathbf{t}',\mathbf{s}'}}{c_{\mathbf{t},\mathbf{s}}} = a_{\mathbf{u}}\frac{c_{\mathbf{u}',\mathbf{s}'}}{c_{\mathbf{u},\mathbf{s}}}.$$

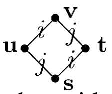
Thus we can produce the scalars $a_{\mathbf{s}}$ inductively by proceeding down through L one level at a time starting at the unique maximal vertex. Now for any $\mathbf{s}$ in L we have

$$\begin{aligned} x_i.w_{\mathbf{s}'} = x_i.(a_{\mathbf{s}}v_{\mathbf{s}}) = a_{\mathbf{s}}x_i.v_{\mathbf{s}} &= a_{\mathbf{s}}\sum_{\mathbf{t}:\mathbf{s}\xrightarrow{i}\mathbf{t}} c_{\mathbf{t},\mathbf{s}}v_{\mathbf{t}} = \sum_{\mathbf{t}:\mathbf{s}\xrightarrow{i}\mathbf{t}} a_{\mathbf{s}}c_{\mathbf{t},\mathbf{s}}v_{\mathbf{t}} \\ &= \sum_{\mathbf{t}:\mathbf{s}\xrightarrow{i}\mathbf{t}} a_{\mathbf{t}}\frac{c_{\mathbf{t}',\mathbf{s}'}}{c_{\mathbf{t},\mathbf{s}}}c_{\mathbf{t},\mathbf{s}}v_{\mathbf{t}} = \sum_{\mathbf{t}:\mathbf{s}\xrightarrow{i}\mathbf{t}} a_{\mathbf{t}}c_{\mathbf{t}',\mathbf{s}'}v_{\mathbf{t}} = \sum_{\mathbf{t}:\mathbf{s}\xrightarrow{i}\mathbf{t}} c_{\mathbf{t}',\mathbf{s}'}w_{\mathbf{t}'}, \end{aligned}$$

and similarly for any $\mathbf{t}$ in L we have $y_i.w_{\mathbf{t}'} = \sum_{\mathbf{s}:\mathbf{s}\xrightarrow{i}\mathbf{t}} d_{\mathbf{s}',\mathbf{t}'}w_{\mathbf{s}'}$. It follows that with the scalars $a_{\mathbf{s}}$ defined as above, the basis $\{w_{\mathbf{s}'} := a_{\mathbf{s}}v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ has K as its representation diagram. □

Lemma 4.3. *Let V*, $\mathscr{B}$, *and L be as in* Lemma 4.2. *In addition suppose V is irreducible. Suppose that for any representation diagram K for V which is isomorphic to L as an edge-colored directed graph, it is the case that K and L are edge product similar. Then the basis* $\mathscr{B}$ *is solitary.*

Proof. Let $\{w_{\mathbf{s}'}\}_{\mathbf{s}'\in K}$ be a weight basis for V with representation diagram K such that K and L are edge product similar. Let $\mathbf{s}$ and $\mathbf{s}'$ denote corresponding elements in L and K, respectively. By Lemma 4.2, there exists a basis $\{a_{\mathbf{s}}v_{\mathbf{s}}\}_{\mathbf{s}'\in K}$ which has representation diagram K. Then by Lemma 3.1.C of [1], it follows that $\{w_{\mathbf{s}'}\}_{\mathbf{s}\in L}$ and $\{a_{\mathbf{s}}v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ are scalar equivalent. Therefore, $\{w_{\mathbf{s}'}\}_{\mathbf{s}\in L}$ and $\{v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ are diagonally equivalent. Since this is true for any basis $\{w_{\mathbf{s}'}\}_{\mathbf{s}\in L}$ with supporting graph L, it follows that the basis $\mathscr{B} = \{v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ is solitary. □

Proof of Theorem 4.1. For part (1), let K denote the edge-labelled poset that coincides with L as an edge-colored directed graph and whose edge coefficients are $\{(a_{\mathbf{t},\mathbf{s}}, b_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s}\xrightarrow{i}\mathbf{t}\in\mathscr{E}(L)}$. We use induction on the totally ordered sequence $(\mathbf{t}_0, \ldots, \mathbf{t}_{|\mathscr{V}(L)|-1})$ of vertices of L. Here, $\mathbf{t}_p$ precedes $\mathbf{t}_q$ in the total order $\mathscr{T}$ if $p<q$. At a given vertex $\mathbf{t}$, we will argue that on each edge $\mathbf{s}\xrightarrow{i}\mathbf{t}$ below $\mathbf{t}$, the product $a_{\mathbf{t},\mathbf{s}}b_{\mathbf{s},\mathbf{t}}$ coincides with the product $\pi_{\mathbf{s},\mathbf{t}}$. So suppose that at each vertex $\mathbf{t}'$ which precedes a given vertex $\mathbf{t} := \mathbf{t}_j$ $(0 \leqslant j \leqslant |\mathscr{V}(L)| - 1)$ it is the case that $a_{\mathbf{t}',\mathbf{s}'}b_{\mathbf{s}',\mathbf{t}'} = \pi_{\mathbf{s}',\mathbf{t}'}$ on each edge $\mathbf{s}'\xrightarrow{i}\mathbf{t}'$ below $\mathbf{t}'$. Let $\mathbf{s}_{p_1}, \ldots, \mathbf{s}_{p_r}$ be the descendants of $\mathbf{t}$, where $\mathbf{s}_{p_1}$ precedes $\mathbf{s}_{p_2}$, etc. Let $1 \leqslant l \leqslant r$ and suppose that for each $\mathbf{s} \in \{\mathbf{s}_{p_1}, \ldots, \mathbf{s}_{p_{l-1}}\}$, it is the case that $a_{\mathbf{t},\mathbf{s}}b_{\mathbf{s},\mathbf{t}} = \pi_{\mathbf{s},\mathbf{t}}$. Now let $\mathbf{s} := \mathbf{s}_{p_l}$. Let $\mathbf{u} := f(\mathbf{s})$. If $\mathbf{u} \neq \mathbf{t}$, then since L is diamond-and-crossing friendly, $\mathbf{u}$ precedes $\mathbf{t}$ in the total order. Let $\mathbf{v}$ be the unique element in L which covers both $\mathbf{u}$ and $\mathbf{t}$. So we have a "diamond" in L which looks like 

and with $\mathbf{u}$ and $\mathbf{v}$ preceding $\mathbf{t}$ in the total order $\mathscr{T}$. The diamond conditions for K and L (cf. Lemma 3.1) together with the inductive hypothesis imply that

$$a_{\mathbf{t},\mathbf{s}}b_{\mathbf{s},\mathbf{t}} = \frac{a_{\mathbf{v},\mathbf{u}}b_{\mathbf{u},\mathbf{v}}\cdot a_{\mathbf{v},\mathbf{t}}b_{\mathbf{t},\mathbf{v}}}{a_{\mathbf{u},\mathbf{s}}b_{\mathbf{s},\mathbf{u}}} = \frac{\pi_{\mathbf{u},\mathbf{v}}\cdot\pi_{\mathbf{t},\mathbf{v}}}{\pi_{\mathbf{s},\mathbf{u}}} = \pi_{\mathbf{s},\mathbf{t}}.$$

Next suppose that $\mathbf{u} = \mathbf{t}$ and that $\mathbf{s}$ is an exceptional descendant of $\mathbf{t}$. Since it is the case that the edge product $\pi_{\mathbf{s},\mathbf{t}}$ can be expressed in terms of edge products prior to $\mathbf{s}\xrightarrow{i}\mathbf{t}$, one can use an inductive argument to conclude that $a_{\mathbf{t},\mathbf{s}}b_{\mathbf{s},\mathbf{t}} = \pi_{\mathbf{s},\mathbf{t}}$. Finally suppose $\mathbf{s}$ is a crossing descendant of $\mathbf{t}$. By the crossing conditions in K and L we see that

$$\begin{aligned} a_{\mathbf{t},\mathbf{s}}b_{\mathbf{s},\mathbf{t}} &= m_i(\mathbf{t}) + \sum_{\mathbf{v}:\mathbf{t}\xrightarrow{i}\mathbf{v}} a_{\mathbf{v},\mathbf{t}}b_{\mathbf{t},\mathbf{v}} - \sum_{\mathbf{r}:\mathbf{r}\xrightarrow{i}\mathbf{t}\text{ and }\mathbf{r}\neq\mathbf{s}} a_{\mathbf{t},\mathbf{r}}b_{\mathbf{r},\mathbf{t}} \\ &= m_i(\mathbf{t}) + \sum_{\mathbf{v}:\mathbf{t}\xrightarrow{i}\mathbf{v}} \pi_{\mathbf{t},\mathbf{v}} - \sum_{\mathbf{r}:\mathbf{r}\xrightarrow{i}\mathbf{t}\text{ and }\mathbf{r}\neq\mathbf{s}} \pi_{\mathbf{r},\mathbf{t}} = \pi_{\mathbf{s},\mathbf{t}}, \end{aligned}$$

since each of the edges $\mathbf{t} \xrightarrow{i} \mathbf{v}$ and $\mathbf{r} \xrightarrow{i} \mathbf{t}$ in the preceding sums has already been accounted for. This completes the proof of (1).

Part (1) implies that if K is any representation diagram for V such that K is isomorphic to L as an edge-colored directed graph, then K is edge product similar to L. Suppose now that V is irreducible. It follows from Lemma 4.3 that the basis $\{v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ and the supporting graph L are solitary. Now let L' be a representation diagram for a basis of V, and suppose that L' is an edge-colored subgraph of L. In particular, $\mathscr{V}(L') = \mathscr{V}(L)$ and $\mathscr{E}_i(L') \subseteq \mathscr{E}_i(L)$ for $1 \leqslant i \leqslant n$. Let $(c'_{\mathbf{t},\mathbf{s}}, d'_{\mathbf{s},\mathbf{t}})$ denote the pair of coefficients attached to the edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in $\mathscr{E}(L')$. Let $m'_i(\mathbf{s}) := 2\rho'_i(\mathbf{s}) - l'_i(\mathbf{s})$, where $l'_i(\mathbf{s})$ is the length of the i-component in L' containing $\mathbf{s}$ and $\rho'_i(\mathbf{s})$ is the rank of $\mathbf{s}$ within this i-component. Let $m_i(\mathbf{s})$ denote the corresponding quantity in L. By Lemma 3.2.C of [1], we have $m'_i(\mathbf{s}) = m_i(\mathbf{s})$ for $1 \leqslant i \leqslant n$ and any $\mathbf{s}$ in $\mathscr{V}(L)$. Let K be the edge-labelled poset with $\mathscr{V}(K) := \mathscr{V}(L)$, $\mathscr{E}_i(K) := \mathscr{E}_i(L)$ for $1 \leqslant i \leqslant n$, and$\{(p_{\mathbf{t},\mathbf{s}}, q_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathscr{E}(K)} := \{(c'_{\mathbf{t},\mathbf{s}}, d'_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathscr{E}(L')} \bigcup \{(0,0)\}_{\mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathscr{E}(K)\setminus\mathscr{E}(L')}$. Since $m'_i(\mathbf{s}) = m_i(\mathbf{s})$ for $1 \leqslant i \leqslant n$ and any $\mathbf{s}$ in $\mathscr{V}(L') = \mathscr{V}(K)$, and since the edge-labelled poset L' satisfies the crossing condition by Lemma 3.1, then K satisfies the crossing condition. Since L' satisfies the diamond condition by Lemma 3.1, it is easy to see that K also satisfies the diamond condition. Part (1) implies that for any edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in K, the product of coefficients $p_{\mathbf{t},\mathbf{s}} q_{\mathbf{t},\mathbf{s}}$ is the same as the nonzero product $\pi_{\mathbf{s},\mathbf{t}}$ of coefficients on the same edge in L. So both coefficients are nonzero on any edge in K. It follows that $\mathscr{E}(L') = \mathscr{E}(K) = \mathscr{E}(L)$, so L' and L coincide as edge-colored posets. Since L does not contain as a proper edge-colored subgraph the supporting graph for any other weight basis for V, it follows that L and the basis $\{v_{\mathbf{s}}\}_{\mathbf{s}\in L}$ are edge-minimal. □

Remark 4.4. Our proof of Theorem 4.1 gives an algorithmic or procedural approach for computing edge coefficients when an edge-labelled poset L is known to meet the hypotheses of the theorem statement. The reverse linear extension $\mathscr{T}$ of L induces the following ordering on the edges of L: edge $\mathbf{x} \to \mathbf{y}$ precedes edge $\mathbf{s} \to \mathbf{t}$ if (1) $\mathbf{y}$ precedes $\mathbf{t}$ in the total ordering $\mathscr{T}$ or (2) $\mathbf{y} = \mathbf{t}$ and $\mathbf{x}$ precedes $\mathbf{s}$ in the total ordering $\mathscr{T}$. A consequence of the proof is that one has enough information available to (uniquely) determine edge products in this order one edge at a time.

For the lattices we have studied, a crucial step in implementing this procedure has been to identify a set of boundary vertices which gives us a starting point for performing computations on edges below vertices of a given rank. For the G_2 lattices we consider in this paper, the boundary is defined in Section 2; in Section 6 we identify the boundary for certain odd orthogonal lattices. As in Figs. 15 and 16, we visualize boundary elements to be on the left side of the Hasse diagram. In these cases the algorithm of Remark 4.4 proceeds from the top of the lattice down, and from left to right (following the total order) across each rank of the lattice. Alverson [9] was able to implement this algorithm in a computer algebra system so that one can obtain edge products for certain lattices associated to the rank two simple Lie algebras.

5. Statement and proof of main G_2 result

In this section we refer to the edge-colored lattices $L_{G_2}^{\mathrm{LM}}(k)$ and $L_{G_2}^{\mathrm{RS}}(k)$ with the total orderings and ancestor function **rightmost_decrease** identified in Section 2. We apply Theorem 4.1 in these settings to obtain:

Theorem 5.1. *Let L be one of the edge-colored posets $L_{G_2}^{\mathrm{LM}}(k)$ or $L_{G_2}^{\mathrm{RS}}(k)$.*

(1) *Let $\{(X_{\mathbf{t},\mathbf{s}}, Y_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathscr{E}(L)}$ be the set of positive rational coefficients assigned to the edges of L in [2]. Suppose $\{(c_{\mathbf{t},\mathbf{s}}, d_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathscr{E}(L)}$ is another set of coefficients assigned to the edges of L such that the corresponding edge-labelled poset satisfies the diamond and crossing conditions. Then on any edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in L, it is the case that $c_{\mathbf{t},\mathbf{s}} d_{\mathbf{s},\mathbf{t}} = X_{\mathbf{t},\mathbf{s}} Y_{\mathbf{s},\mathbf{t}}$.*
(2) *The edge-colored poset L is a positive rational, solitary, and edge-minimal supporting graph for the irreducible representation of G_2 with highest weight $k\omega_1$.*

We need the following lemmas to prove Theorem 5.1.

Lemma 5.2. *Let L be one of* $L^{\text{LM}}_{G_2}(k)$ *or* $L^{\text{RS}}_{G_2}(k)$. *Let* $\mathbf{t}=(t_1,\ldots,t_7)$ *be an element of L, and let* $\mathbf{s}_1,\ldots,\mathbf{s}_6$ *be the possible descendants of* $\mathbf{t}$ *as depicted in Fig. 1, with* $\mathbf{s}_i=(t_1,\ldots,t_{i-1},t_i-1,t_{i+1}+1,t_{i+2},\ldots,t_7)$ *for* $1\leqslant i\leqslant 6$. *Then in the total order on L,* $\mathbf{s}_i$ *precedes* $\mathbf{s}_j$ *when* $i<j$.

Proof. Let $\mathbf{s}_i$ be a descendant of $\mathbf{t}$ in L. In particular, we must have $t_i\geqslant 1$. Let $\mathbf{b}=(b_1,\ldots,b_7)$ be the boundary vertex with the same rank as $\mathbf{s}_i$. Then we have $\mathbf{dist}(\mathbf{b},\mathbf{s}_i)=\left|1+\sum_{p=1}^{i}(b_p-t_p)\right|+\sum_{q=1,q\neq i}^{6}\left|\sum_{r=1}^{q}(b_r-t_r)\right|$. Let $\mathbf{s}_j$ be a descendant of $\mathbf{t}$ with $i<j$. To prove the lemma, it clearly suffices to show that $\mathbf{dist}(\mathbf{b},\mathbf{s}_i)\leqslant\mathbf{dist}(\mathbf{b},\mathbf{s}_j)$. We have $\mathbf{dist}(\mathbf{b},\mathbf{s}_j)-\mathbf{dist}(\mathbf{b},\mathbf{s}_i)=\left|\sum_{p=1}^{i}(b_p-t_p)\right|-\left|1+\sum_{p=1}^{i}(b_p-t_p)\right|+\left|1+\sum_{p=1}^{j}(b_p-t_p)\right|-\left|\sum_{p=1}^{j}(b_p-t_p)\right|$. Now suppose that $\sum_{p=1}^{i}(b_p-t_p)<0$. Then $\left|\sum_{p=1}^{i}(b_p-t_p)\right|-\left|1+\sum_{p=1}^{i}(b_p-t_p)\right|=1$, and so

$$\mathbf{dist}(\mathbf{b},\mathbf{s}_j)-\mathbf{dist}(\mathbf{b},\mathbf{s}_i)=\left|1+\sum_{p=1}^{j}(b_p-t_p)\right|-\left|\sum_{p=1}^{j}(b_p-t_p)\right|+1=\begin{cases}0 & \text{if } \sum_{p=1}^{j}(b_p-t_p)<0,\\ 2 & \text{if } \sum_{p=1}^{j}(b_p-t_p)\geqslant 0.\end{cases}$$

So in this case $\mathbf{dist}(\mathbf{b},\mathbf{s}_j)-\mathbf{dist}(\mathbf{b},\mathbf{s}_i)\geqslant 0$.

Next consider the case that $\sum_{p=1}^{i}(b_p-t_p)\geqslant 0$. Note that since $t_i\geqslant 1$, then $b_p\geqslant 1$ for at least one p, $p=1,\ldots,i$. Except possibly for the case $i=3$ and $j=4$ with $\mathbf{b}=(0,0,b_3,b_4,b_5,0,0)$ in $L^{\text{LM}}_{G_2}(k)$, we will then have $\sum_{p=1}^{j}b_p=k$, from which it follows that $\sum_{p=1}^{j}b_p\geqslant\sum_{p=1}^{j}t_p$. Since $\sum_{p=1}^{i}(b_p-t_p)\geqslant 0$ as well, then it follows that $\mathbf{dist}(\mathbf{b},\mathbf{s}_j)-\mathbf{dist}(\mathbf{b},\mathbf{s}_i)=0$. This argument will only fail in the case that $i=3$ and $j=4$ with $\mathbf{b}=(0,0,b_3,b_4,b_5,0,0)$ in $L^{\text{LM}}_{G_2}(k)$, $\sum_{p=1}^{3}(b_p-t_p)\geqslant 0$, and $\sum_{p=1}^{4}(b_p-t_p)<0$. The latter inequalities imply that $t_4>b_4$. In $L^{\text{LM}}_{G_2}(k)$, this means $t_4=1$ and $b_4=0$. But then $\mathbf{t}=(t_1,t_2,t_3,1,t_5,t_6,t_7)$, which has no descendant $\mathbf{s}_3$ in $L^{\text{LM}}_{G_2}(k)$. □

Lemma 5.3. *Let L be one of* $L^{\text{LM}}_{G_2}(k)$ *or* $L^{\text{RS}}_{G_2}(k)$. *Let* $\mathbf{s}\xrightarrow{i}\mathbf{t}$ *be an edge in L. Let* $\mathbf{t}':=$ **rightmost_decrease**$(\mathbf{s})$. *Suppose* $\mathbf{t}'\neq\mathbf{t}$. *Then* $\mathbf{t}'$ *precedes* $\mathbf{t}$ *in the total order on L.*

Proof. Write $\mathbf{s}=(s_1,\ldots,s_7)$. Then for some p, we have $\mathbf{t}=(s_1,\ldots,s_p+1,s_{p+1}-1,\ldots,s_7)$. Moreover, we have $\mathbf{t}'=(s_1,\ldots,s_q+1,s_{q+1}-1,\ldots,s_7)$ for some $q>p$. Let $\mathbf{u}$ be the least upper bound of $\mathbf{t}$ and $\mathbf{t}'$ in L. If $p<q-1$, then $\mathbf{u}=(s_1,\ldots,s_p+1,s_{p+1}-1,\ldots,s_q+1,s_{q+1}-1,\ldots,s_7)$.

If $p=q-1$, then $\mathbf{u}=(s_1,\ldots,s_p+1,s_{p+1},s_{p+2}-1,\ldots,s_7)$. In either case, when we view the descendants $\mathbf{t}'$ and $\mathbf{t}$ of $\mathbf{u}$ in the light of Lemma 5.2, then we see that $\mathbf{t}'$ precedes $\mathbf{t}$. □

Lemma 5.4. *There are no exceptional descendants in* $L^{\text{LM}}_{G_2}(k)$. *In* $L^{\text{RS}}_{G_2}(k)$, *a vertex* $\mathbf{s}$ *is an exceptional descendant of some vertex* $\mathbf{t}$ *only when* $\mathbf{t}=(a,b,c,d,0,0,0)$, $\mathbf{s}=(a,b,c-1,d+1,0,0,0)$, $c\geqslant 1$, *and* $d\geqslant 1$.

Proof. Let L be one of $L^{\text{LM}}_{G_2}(k)$ or $L^{\text{RS}}_{G_2}(k)$. Let $\mathbf{t}=(t_1,\ldots,t_7)$ be in L. If $\mathbf{s}$ is the only descendant of $\mathbf{t}$ along an edge of a given color i ($i=1,2$), then $\mathbf{s}$ cannot be an exceptional descendant. So suppose $\mathbf{t}$ has two descendants along edges of color 2; these are $\mathbf{s}_2$ and $\mathbf{s}_5$ in the notation of Lemma 5.2. In particular, $t_5>0$. Now observe that in this situation $\mathbf{s}_2$ is a diamond descendant of $\mathbf{t}$ since **rightmost_decrease**$(\mathbf{s}_2)$ is a vertex in L that differs from $\mathbf{t}$ in coordinate positions 5 and 6 or in coordinate positions 6 and 7. Now $\mathbf{s}_5$ will be a diamond descendant if $t_7>0$; otherwise, $\mathbf{s}_5$ is a crossing descendant of $\mathbf{t}$, since $\mathbf{s}_2$ precedes $\mathbf{s}_5$ in the total order by Lemma 5.2. Thus $\mathbf{t}$ has no exceptional descendants along edges of color 2.

Now consider the four possible descendants $\mathbf{s}_1$, $\mathbf{s}_3$, $\mathbf{s}_4$, $\mathbf{s}_6$ of $\mathbf{t}$ along edges of color 1. Observe that in any case, $\mathbf{s}_1$ is a diamond or a crossing descendant of $\mathbf{t}$. Suppose $\mathbf{s}_6$ is a descendant of $\mathbf{t}$ in L. In particular, $t_6>0$. Then $\mathbf{s}_6$ is a crossing descendant, since by Lemma 5.3 each of $\mathbf{s}_1$, $\mathbf{s}_3$, and $\mathbf{s}_4$ precedes $\mathbf{s}_6$ in the total order. Then, if $\mathbf{s}_i$ is a descendant of $\mathbf{t}$ for $i=1,3,4$, it is a diamond descendant since **rightmost_decrease**$(\mathbf{s}_i)$ will not coincide with $\mathbf{t}$. In this case, then, $\mathbf{t}$ has no exceptional descendants. So suppose $\mathbf{s}_6$ is not a descendant of $\mathbf{t}$ in L. If exactly one of $\mathbf{s}_3$ or $\mathbf{s}_4$ is a descendant of $\mathbf{t}$, then it is a diamond or a crossing descendant. Since $\mathbf{t}$ can have at most one of $\mathbf{s}_3$ or $\mathbf{s}_4$ as a descendant if $\mathbf{t}$ is in $L^{\text{LM}}_{G_2}(k)$, we conclude that an element $\mathbf{t}$ of $L^{\text{LM}}_{G_2}(k)$ has no exceptional descendants. So now assume $\mathbf{t}$ has both $\mathbf{s}_3$ and $\mathbf{s}_4$ as descendants; hence we are working with $L=L^{\text{RS}}_{G_2}(k)$. Note that $t_3>0$ and $t_4>0$. Observe that

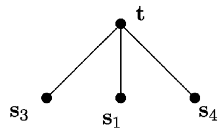
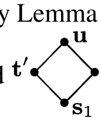


Fig. 2.

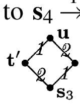
$\mathbf{s}_4$ is a diamond or a crossing descendant. For $\mathbf{s}_3$ to be an exceptional descendant of $\mathbf{t}$, then it must be the case that **rightmost_decrease**$(\mathbf{s}_3) = \mathbf{t}$. This will happen precisely when $t_5 = t_6 = t_7 = 0$. So, $\mathbf{t}$ has an exceptional descendant only when $\mathbf{t}$ is in $L^{\mathrm{RS}}_{G_2}(k)$ and has form $\mathbf{t} = (t_1, t_2, t_3, t_4, 0, 0, 0)$ with $t_3 > 0$ and $t_4 > 0$. The exceptional descendant here is $\mathbf{s}_3$. $\square$

Lemma 5.5. *Let* $\mathbf{t}$ *be an element of* $L^{\mathrm{RS}}_{G_2}(k)$ *of the form* $\mathbf{t} = (a, b, c, d, 0, 0, 0)$. *Suppose* $\mathbf{s} \stackrel{1}{\to} \mathbf{t}$. *Then the edge product* $\pi_{\mathbf{s},\mathbf{t}}$ *can be expressed in terms of edge products prior to* $\mathbf{s} \stackrel{1}{\to} \mathbf{t}$.

Proof. The vertex $\mathbf{t}$ has up to three descendants along edges of color 1. In the notation of Lemma 5.2, we will have $\mathbf{s}_1$ preceding $\mathbf{s}_3$ preceding $\mathbf{s}_4$ in the total order. For this proof it will be convenient to depict these edges as in Fig. 2.

Edge $\mathbf{s}_1 \stackrel{1}{\to} \mathbf{t}$ is easy to account for. Suppose $a \geqslant 1$. Then **rightmost_decrease**$(\mathbf{s}_1) = \mathbf{t}$ only if $c = d = 0$. In this case, $\mathbf{s}_1 \stackrel{1}{\to} \mathbf{t}$ is the only edge of color 1 below $\mathbf{t}$, and we can use the crossing relation for color 1 at vertex $\mathbf{t}$ to express $\pi_{\mathbf{s}_1,\mathbf{t}}$ in terms of edge products prior to $\mathbf{s}_1 \stackrel{1}{\to} \mathbf{t}$. If $c > 0$ or $d > 0$, then $\mathbf{t}' =$ **rightmost_decrease**$(\mathbf{s}_1) \neq \mathbf{t}$. Then by Lemma 5.3, $\mathbf{t}'$ precedes $\mathbf{t}$. Let $\mathbf{u}$ be the unique least upper bound of $\mathbf{t}'$ and $\mathbf{t}$. Use the diamond relation on the diamond 

to express $\pi_{\mathbf{s}_1,\mathbf{t}}$ in terms of edge products prior to $\mathbf{s}_1 \stackrel{1}{\to} \mathbf{t}$.

If $c = 0$ (respectively, $d = 0$), then $\pi_{\mathbf{s}_4,\mathbf{t}}$ (resp. $\pi_{\mathbf{s}_3,\mathbf{t}}$) can be expressed in terms of edge products prior to $\mathbf{s}_4 \stackrel{1}{\to} \mathbf{t}$ (resp. $\mathbf{s}_3 \stackrel{1}{\to} \mathbf{t}$) using the crossing relation for color 1 at vertex $\mathbf{t}$. So let us suppose $c \geqslant 2$ and $d \geqslant 1$. Consider 

, where $\mathbf{t}' = (a, b+1, c-2, d+1, 0, 0, 0)$ and $\mathbf{u}$ is the unique least upper bound of $\mathbf{t}'$ and $\mathbf{t}$. The edge product $\pi_{\mathbf{s}_3,\mathbf{t}'}$ can be expressed in terms of edge products prior to $\mathbf{s}_3 \stackrel{1}{\to} \mathbf{t}$ by the crossing relation for color 2 at vertex $\mathbf{t}'$. Then the edge product $\pi_{\mathbf{s}_3,\mathbf{t}}$ can be expressed in terms of edge products prior to $\mathbf{s}_3 \stackrel{1}{\to} \mathbf{t}$ by the diamond relation on this diamond. At this point $\pi_{\mathbf{s}_4,\mathbf{t}}$ can now be expressed in terms of edge products prior to $\mathbf{s}_4 \stackrel{1}{\to} \mathbf{t}$ using the crossing relation for color 1 at vertex $\mathbf{t}$.

The remainder of the proof handles the case $c = 1$ and $d \geqslant 1$. Set $m = \lfloor (d-1)/2 \rfloor$. Consider the following sequence of vertices in $L^{\mathrm{RS}}_{G_2}(k)$ that have the same rank as $\mathbf{t}$: $\mathbf{t}_0 := (a, b, 1, d, 0, 0, 0) = \mathbf{t}$; $\mathbf{t}_1 := (a, b, 2, d-2, 1, 0, 0)$; $\ldots$; $\mathbf{t}_i := (a, b, 1+i, d-2i, i, 0, 0)$; $\ldots$; $\mathbf{t}_m := (a, b, 1+m, 1 \text{ or } 2, m, 0, 0)$; $\mathbf{t}_{m+1} := (a, b+1, m, 0 \text{ or } 1, 1+m, 0, 0)$; $\mathbf{t}_{m+2} := (a, b+2, m-1, 0 \text{ or } 1, m, 1, 0)$; $\ldots$; $\mathbf{t}_{m+i} := (a, b+i, m-i+1, 0 \text{ or } 1, m-i+2, m, 0)$; $\ldots$; $\mathbf{t}_{2m+1} := (a, b+m+1, 0, 0 \text{ or } 1, 1, m, 0)$; $\mathbf{t}_{2m+2} := (a+1, b+m, 0, 0 \text{ or } 1, 0, m+1, 0)$. If d is even, then necessarily $m > 0$, and we set $\mathbf{t}_{2m+3} := (a+2, b+m-1, 0, 0, 1, m+1, 0)$. If d is odd we will not need $\mathbf{t}_{2m+3}$. We are only interested in the descendants of these vertices depicted in Figs. 3 and 4. The product on the rightmost edge in each of Figs. 3 and 4 can be expressed in terms of edge products prior to $\mathbf{s}_3 \stackrel{1}{\to} \mathbf{t}$ by a crossing relation. One can then use diamond and crossing relations in succession from the right to express the edge products on each of the depicted edges below vertices

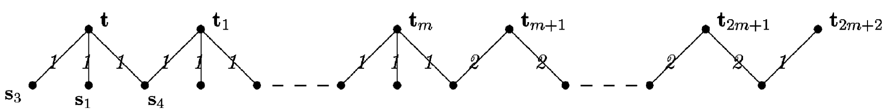


Fig. 3. d odd.

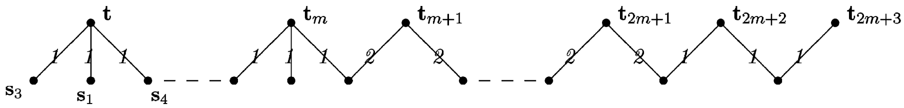


Fig. 4. d even.

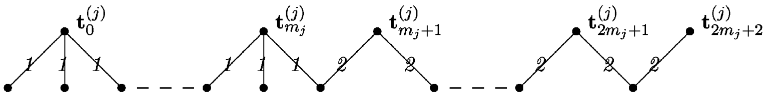


Fig. 5.

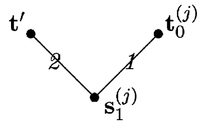


Fig. 6.

$\mathbf{t}_{2m+2}, \ldots, \mathbf{t}_{m+1}$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$. If $a=0$, then there are no "middle" edges below the vertices $\mathbf{t}_0, \ldots, \mathbf{t}_m$, in which case we can continue in succession from the right to express the remaining edge products below vertices $\mathbf{t}_m, \ldots, \mathbf{t}_0$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$. So now suppose $a>0$. We can complete the argument to express the edge products $\pi_{\mathbf{s}_3,\mathbf{t}}$ and $\pi_{\mathbf{s}_4,\mathbf{t}}$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$ if we can do so for the "middle" edge below each of $\mathbf{t}_m, \ldots, \mathbf{t}_1$. Before doing this, we need the following.

If $d>a$, then let $0 \leqslant j < a$. If $d \leqslant a$, then allow $a-d \leqslant j < a$. As long as $j > a-d$, we set $m_j = \lfloor (d-a+j-1)/2 \rfloor$ and consider the following sequence of vertices (all have the same rank in $L_{G_2}^{\mathrm{RS}}(k)$ as $\mathbf{t}$): $\mathbf{t}_0^{(j)} := (j, a+b-j, a+1-j, d-a+j, 0, 0, 0)$; $\mathbf{t}_1^{(j)} := (j, a+b-j, a+2-j, d-a+j-2, 1, 0, 0)$; $\ldots$; $\mathbf{t}_i^{(j)} := (j, a+b-j, a+1-j+i, d-a+j-2i, i, 0, 0)$; $\ldots$; $\mathbf{t}_{m_j}^{(j)} := (j, a+b-j, a+1-j+m_j, 1 \text{ or } 2, m_j, 0, 0)$; $\mathbf{t}_{m_j+1}^{(j)} := (j, a+b-j+1, a-j+m_j, 0 \text{ or } 1, 1+m_j, 0, 0)$; $\mathbf{t}_{m_j+2}^{(j)} := (j, a+b-j+2, a-j+m_j-1, 0 \text{ or } 1, m_j, 1, 0)$; $\ldots$; $\mathbf{t}_{m_j+i}^{(j)} := (j, a+b-j+i, a-j+m_j-i+1, 0 \text{ or } 1, m_j-i+2, i-1, 0)$; $\ldots$; $\mathbf{t}_{2m_j+1}^{(j)} := (j, a+b-j+m_j+1, a-j, 0 \text{ or } 1, 1, m_j, 0)$; $\mathbf{t}_{2m_j+2}^{(j)} := (j, a+b-j+m_j+2, a-j-1, 0 \text{ or } 1, 0, m_j+1, 0)$. If $j=a-d$, we consider the sequence $\mathbf{t}_0^{(a-d)} := (a-d, b+d, d+1, 0, 0, 0, 0)$; $\mathbf{t}_1^{(a-d)} = (a-d, b+d+1, d-1, 1, 0, 0, 0)$. Note that $t_{i_1}^{(j_1)} = t_{i_2}^{(j_2)}$ if and only if $i_1 = i_2$ and $j_1 = j_2$.We are only interested in the descendants of these vertices depicted in Fig. 5.

The product on the rightmost edge in Fig. 5 can be expressed in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$ by a crossing relation. Working from the right, one can then use diamond and crossing relations to express the products on each of the depicted edges below vertices $\mathbf{t}_{2m_j+2}^{(j)}, \ldots, \mathbf{t}_{m_j+1}^{(j)}$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$. Let $\mathbf{s}_1^{(j)} = (j-1, a+b+1-j, a+1-j, d-a+j, 0, 0, 0)$ and $\mathbf{t}' = (j-1, a+b+2-j, a-j, d-a+j, 0, 0, 0)$, so $\mathbf{s}_1^{(j)}$ is the common descendant of $\mathbf{t}'$ and $\mathbf{t}_0^{(j)}$, as depicted in Fig. 6. Then $\pi_{\mathbf{s}_1^{(j)},\mathbf{t}'}$ can be expressed in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$ by a crossing relation, and hence $\pi_{\mathbf{s}_1^{(j)},\mathbf{t}_0^{(j)}}$ can be expressed in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$ using a diamond relation. We can complete the argument to express products on each of the depicted edges below $\mathbf{t}_0^{(j)}$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$ if we can do so for the "middle" edge below each of $\mathbf{t}_m^{(j)}, \ldots, \mathbf{t}_1^{(j)}$. To do this we use induction on j. For $d>a$, the base case is $j=0$. Since the vertices $\mathbf{t}_0^{(0)}, \ldots, \mathbf{t}_m^{(0)}$have no descendants along "middle" edges, we get the picture in Fig. 7. Starting on the right and using diamond and crossing relations, we can now express the products on each of these edges in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$. When $d \leqslant a$, the base case

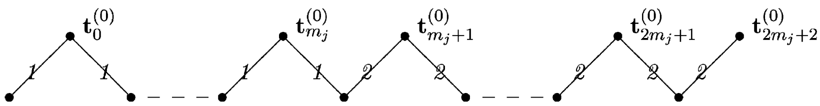


Fig. 7.

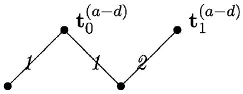


Fig. 8.

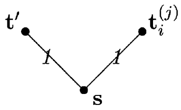


Fig. 9.

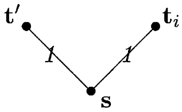


Fig. 10.

for the induction argument is $j = a - d$. In this case we have the following very simple picture of Fig. 8. Starting on the right and using diamond and crossing relations, we can now express the products on each of these edges in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$.

For the inductive hypothesis, assume that the products on the depicted edges below each $\mathbf{t}_i^{(r)}$ have been expressed in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$ for each $0 \leqslant r \leqslant j-1$ (respectively, $a-d \leqslant r \leqslant j-1$) when $0 < j < a$ and $d > a$ (respectively, $a-d < j < a$ and $d \leqslant a$). Consider now a vertex of the form $\mathbf{t}_i^{(j)}$ (where $1 \leqslant i \leqslant m_j$) along with the arrangement of vertices depicted in Fig. 9. In this figure, we have: $\mathbf{s} = (j-1, a+b-(j-1), a+1-(j-1)+(i-1), d-a+j-2i, i, 0, 0)$ and $\mathbf{t}' = (j-1, a+b-(j-1), a+1-(j-1)+(i-1), d-a+(j-1-2(i-1), i-1, 0, 0)$. Then $\mathbf{t}' = \mathbf{t}_{i-1}^{(j-1)}$, and the inductive hypothesis applies to the edge product $\pi_{\mathbf{s},\mathbf{t}'}$. Now use a diamond relation to express the edge product $\pi_{\mathbf{s},\mathbf{t}_i^{(j)}}$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$. We have now accounted for all the edges of interest below each of the vertices $\mathbf{t}_0^{(j)}, \ldots, \mathbf{t}_{2m_j+2}^{(j)}$. This completes the induction argument.

We can now return to the problem of expressing products on the "middle" edge below each of $\mathbf{t}_1, \ldots, \mathbf{t}_m$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$. Let $1 \leqslant i \leqslant m$ and consider the arrangement of vertices depicted in Fig. 10. In this figure, we have: $\mathbf{s} = (a-1, b+1, i+1, d-2i, i, 0, 0)$ and $\mathbf{t}' = (a-1, b+1, i+1, d-2i+1, i-1, 0, 0)$. Then $\mathbf{t}' = \mathbf{t}_{i-1}^{(a-1)}$, and the previous paragraph applies to the edge product $\pi_{\mathbf{s},\mathbf{t}'}$. Now use a diamond relation to express the edge product $\pi_{\mathbf{s},\mathbf{t}_i}$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$. Finally, we can now express $\pi_{\mathbf{s}_4,\mathbf{t}}$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$. Then we can use the crossing relation for color 1 at vertex $\mathbf{t}$ to express $\pi_{\mathbf{s}_3,\mathbf{t}}$ in terms of edge products prior to $\mathbf{s}_3 \xrightarrow{1} \mathbf{t}$. □

Proof of Theorem 5.1. From Corollary 3.3 (and its preceding paragraphs) in [2] we know that L is a positive rational supporting graph for the irreducible representation of G_2 with highest weight $k\omega_1$. Consider L together with its total ordering on vertices, its ancestor function **rightmost_decrease**, and the edge coefficients assigned in [2]. By Lemmas

5.3, 5.4, and 5.5, it follows that L is diamond-and-crossing friendly. Since L also meets the other hypotheses of Theorem 4.1, statements (1) and (2) of Theorem 5.1 follow. □

6. Re-deriving an odd orthogonal result

Theorem 6.1 re-derives parts of Corollary 3.1 of [2] by applying Theorem 4.1 to the edge-colored distributive lattices $L_B^{\mathrm{Mol}}(k, 2n)$ and $L_B^{\mathrm{RS}}(k, 2n)$ of [2]. In keeping with the notation of [2], we think of elements of $L_B^{\mathrm{Mol}}(k, 2n)$ and $L_B^{\mathrm{RS}}(k, 2n)$ as $(2n+1)$-tuples $\mathbf{s} = (s_1, \ldots, s_{2n+1})$. We define the *boundary for* $L_B^{\mathrm{Mol}}(k, 2n)$ to be the set

$$\left\{ (b_1, \ldots, b_{2n+1}) \;\middle|\; \begin{array}{l} b_j + b_{j+1} = k \text{ where } 1 \leqslant j \leqslant 2n \ (j \neq n,\ j \neq n+1); \\ \quad \text{or } \ b_n + b_{n+1} + b_{n+2} = k \end{array} \right\}.$$

For $L_B^{\mathrm{RS}}(k, 2n)$, the *boundary* is the set

$$\{(b_1, \ldots, b_{2n+1}) | b_j + b_{j+1} = k \text{ where } 1 \leqslant j \leqslant 2n\}.$$

Now let L be one of $L_B^{\mathrm{Mol}}(k, 2n)$ or $L_B^{\mathrm{RS}}(k, 2n)$, and let ρ be the rank function for L. One can see that there is a one-to-one correspondence between the elements of the boundary set and the ranks of L. The distance between $\mathbf{s}$ and another element $\mathbf{t} = (t_1, \ldots, t_{2n+1})$ in L is $\mathbf{dist}(\mathbf{s}, \mathbf{t}) = \sum_{i=1}^{2n} |\sum_{j=1}^{i} (s_i - t_i)|$. As in Section 3, we totally order the elements of L as follows: we say $\mathbf{s}$ *precedes* $\mathbf{t}$ if (1) $\rho(\mathbf{s}) > \rho(\mathbf{t})$; or (2) $\rho(\mathbf{s}) = \rho(\mathbf{t})$ and $\mathbf{dist}(\mathbf{s}, \mathbf{b}) < \mathbf{dist}(\mathbf{t}, \mathbf{b})$, where $\mathbf{b}$ is the unique boundary element of L for which $\rho(\mathbf{b}) = \rho(\mathbf{s}) = \rho(\mathbf{t})$; or (3) $\rho(\mathbf{s}) = \rho(\mathbf{t})$, $\mathbf{dist}(\mathbf{s}, \mathbf{b}) = \mathbf{dist}(\mathbf{t}, \mathbf{b})$, and there exists a j such that $s_i = t_i$ for $i > j$ and $s_j < t_j$. If $\mathbf{s}$ is a nonmaximal element of L, let i be the largest index $(1 \leqslant i \leqslant 2n)$ for which $\mathbf{t} = (s_1, \ldots, s_{i-1}, s_i + 1, s_{i+1} - 1, s_{i+2}, \ldots, s_{2n+1})$ is an element of the lattice L; in this case we write $\mathbf{t} = \mathbf{rightmost\_decrease}(\mathbf{s})$.

Theorem 6.1. *Let L be one of the edge-colored posets $L_B^{\mathrm{Mol}}(k, 2n)$ or $L_B^{\mathrm{RS}}(k, 2n)$.*

(1) *Let $\{(X_{\mathbf{t},\mathbf{s}}, Y_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathscr{E}(L)}$ be the set of positive rational coefficients assigned to the edges of L in* [2]. *Suppose $\{(c_{\mathbf{t},\mathbf{s}}, d_{\mathbf{s},\mathbf{t}})\}_{\mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathscr{E}(L)}$ is another set of coefficients assigned to the edges of L such that the corresponding edge-labelled poset satisfies the diamond and crossing conditions. Then on any edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in L, it is the case that $c_{\mathbf{t},\mathbf{s}} d_{\mathbf{s},\mathbf{t}} = X_{\mathbf{t},\mathbf{s}} Y_{\mathbf{s},\mathbf{t}}$.*

(2) *The edge-colored poset L is a positive rational, solitary, and edge-minimal supporting graph for the irreducible representation of B_n with highest weight $k\omega_1$.*

As with Theorem 5.1, the proof of Theorem 6.1 follows quickly from a sequence of lemmas. We suppress the details of the proofs of these lemmas since the arguments are analogous to the arguments given for the corresponding Section 5 lemmas.

Lemma 6.2. *Let L be one of $L_B^{\mathrm{Mol}}(k, 2n)$ or $L_B^{\mathrm{RS}}(k, 2n)$. Let $\mathbf{t} = (t_1, \ldots, t_{2n+1})$ be an element of L, and let $\mathbf{s}_1, \ldots, \mathbf{s}_{2n}$ be the possible descendants of $\mathbf{t}$, where $\mathbf{s}_i = (t_1, \ldots, t_{i-1}, t_i - 1, t_{i+1} + 1, t_{i+2}, \ldots, t_{2n+1})$ for $1 \leqslant i \leqslant 2n$. Then in the total order on L, $\mathbf{s}_i$ precedes $\mathbf{s}_j$ when $1 \leqslant i < j \leqslant 2n$.*

Remarks on proof. This proof is analogous to the proof of Lemma 5.2; one only needs to change indices in the appropriate places.

Lemma 6.3. *Let L be one of $L_B^{\mathrm{Mol}}(k, 2n)$ or $L_B^{\mathrm{RS}}(k, 2n)$. Let $\mathbf{s} \xrightarrow{i} \mathbf{t}$ be an edge in L. Let $\mathbf{t}' := \mathbf{rightmost\_decrease}(\mathbf{s})$. Suppose $\mathbf{t}' \neq \mathbf{t}$. Then $\mathbf{t}'$ precedes $\mathbf{t}$ in the total order on L.*

Remarks on proof. Follows from Lemma 6.2 in the exactly the same way that Lemma 5.3 follows from Lemma 5.2.

Lemma 6.4. *There are no exceptional descendants in $L_B^{\mathrm{Mol}}(k, 2n)$. In $L_B^{\mathrm{RS}}(k, 2n)$, a vertex $\mathbf{s}$ is an exceptional descendant of some vertex $\mathbf{t}$ only when $\mathbf{t} = (t_1, \ldots, t_{n+1}, 0, \ldots, 0)$, $\mathbf{s} = (t_1, \ldots, t_n - 1, t_{n+1} + 1, 0, \ldots, 0)$, $t_n \geqslant 1$, and $t_{n+1} \geqslant 1$.*

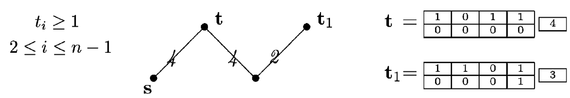


Fig. 11.

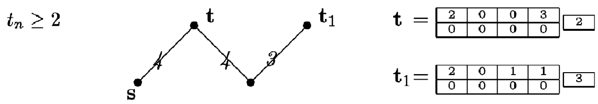


Fig. 12.

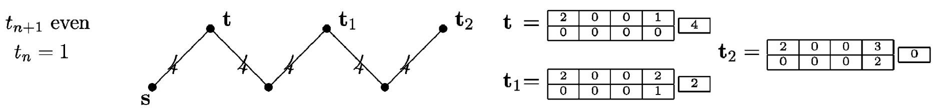


Fig. 13.

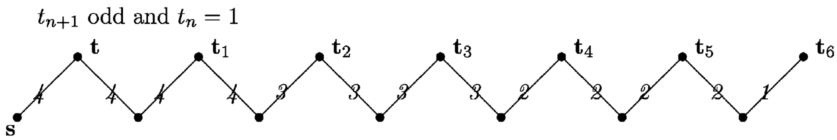


Fig. 14.

Remarks on proof. Analogous to our proof of Lemma 5.4. Analysis of descendants along edges of a given color i (for $1 \leqslant i < n$) in the B_n lattices is entirely similar to our analysis of descendants along edges of color 2 in the G_2 lattices. Descendants along edges of color n in the B_n lattices can be handled in a fashion similar to our treatment of descendants along edges of color 1 in the G_2 lattices.

Lemma 6.5. *Let* $\mathbf{t}$ *be an element of* $L_B^{\mathrm{RS}}(k, 2n)$ *of the form* $\mathbf{t} = (t_1, \ldots, t_{n+1}, 0, \ldots, 0)$. *Suppose* $\mathbf{s} \xrightarrow{n} \mathbf{t}$. *Then the edge product* $\pi_{\mathbf{s},\mathbf{t}}$ *can be expressed in terms of edge products prior to* $\mathbf{s} \xrightarrow{n} \mathbf{t}$.

Remarks on proof. The proof here is easier than the proof of Lemma 5.5 since it is not possible for a vertex $\mathbf{t}$ in $L_B^{\mathrm{RS}}(k, 2n)$ to have more than two descendants along edges of a given color. Figs.11–14 illustrate the various cases our argument considers in the setting of $L_B^{\mathrm{RS}}(k, 2n)$ with $n = 4$ and $k = 7$. In Fig. 14, we have $\mathbf{t} = \begin{array}{|c|c|c|c|}\hline 3&0&0&1\\\hline 0&0&0&0\\\hline\end{array}\,\boxed{3}$, $\mathbf{t}_1 = \begin{array}{|c|c|c|c|}\hline 3&0&0&2\\\hline 0&0&0&1\\\hline\end{array}\,\boxed{1}$, $\mathbf{t}_2 = \begin{array}{|c|c|c|c|}\hline 3&0&1&1\\\hline 0&0&0&2\\\hline\end{array}\,\boxed{0}$, $\mathbf{t}_3 = \begin{array}{|c|c|c|c|}\hline 3&0&2&0\\\hline 0&0&1&1\\\hline\end{array}\,\boxed{0}$, $\mathbf{t}_4 = \begin{array}{|c|c|c|c|}\hline 3&1&1&0\\\hline 0&0&2&0\\\hline\end{array}\,\boxed{0}$, $\mathbf{t}_5 = \begin{array}{|c|c|c|c|}\hline 3&2&0&0\\\hline 0&1&1&0\\\hline\end{array}\,\boxed{0}$, and $\mathbf{t}_6 = \begin{array}{|c|c|c|c|}\hline 4&1&0&0\\\hline 0&2&0&0\\\hline\end{array}\,\boxed{0}$. In each figure we can express the product on the "rightmost" edge in terms of products prior to $\mathbf{s} \to \mathbf{t}$ by a crossing relation. We can then use diamond and crossing relations in succession from the right to express the edge products on each of the depicted edges in terms of edge products prior to $\mathbf{s} \to \mathbf{t}$.

Proof of Theorem 6.1. See the first paragraph in the proof of Corollary 3.1 of [2] for an argument that L is a positive rational supporting graph for the irreducible representation of B_n with highest weight $k\omega_1$. That argument only depends on Theorem 2.1 of [2] and is independent of the other conclusions of Corollary 3.1. Now consider L together with its total ordering on vertices and the ancestor function **rightmost_decrease**. By Lemmas 6.3, 6.4 and 6.5, it follows that L is diamond-and-crossing friendly. Since L also meets the other hypotheses of Theorem 4.1, statements (1) and (2) of Theorem 6.1 follow. □

7. Examples of Littelmann–Molev and Reiner–Stanton G_2 lattices

The figures (Figs. 15 and 16) that appear in this section follow the notation of Section 2 for the G_2 lattices $L_{G_2}^{\mathrm{LM}}(k)$ and $L_{G_2}^{\mathrm{RS}}(k)$. Here we take $k=2$. Each lattice has 27 vertices identified as $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{26}$ so that $\mathbf{t}_i$ precedes $\mathbf{t}_j$ in the total order if and only if $i < j$. The larger dots for the vertices along the left side of each figure indicate boundary vertices. In the edge-colored graphs depicted in these two figures, each edge has one coefficient attached. This coefficient is the product of the x- and y-coefficients that were obtained for G_2 lattice edges in [2]. These products can also be computed following the procedure described in Remark 4.4. A consequence of Theorem 5.1 is that for any representation diagram for G_2 whose underlying edge-colored poset is one of these G_2 lattices, the product of the coefficients on any edge must agree with the product given in these figures.

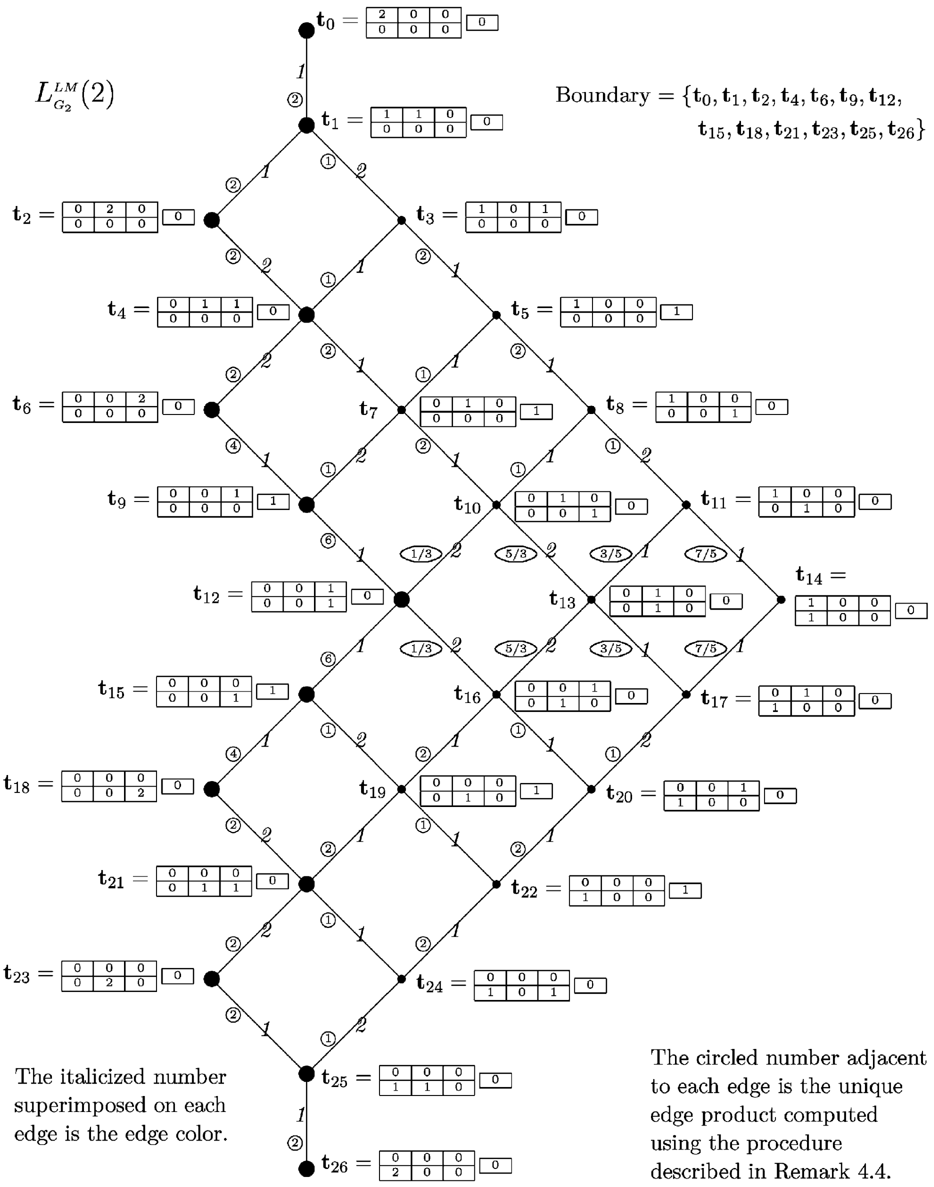


Fig. 15.

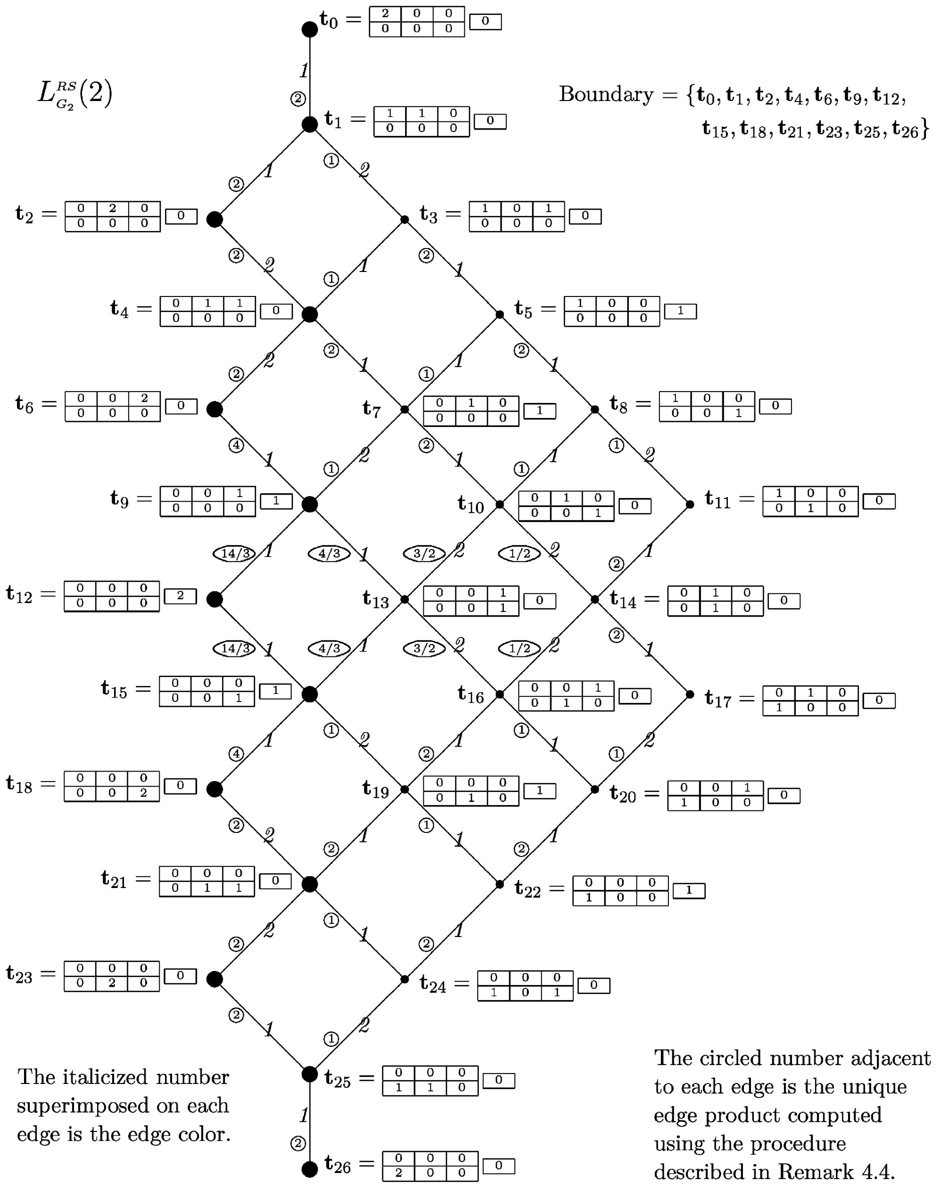


Fig. 16.

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