Note

On the Class of Entire Functions Defined by Dirichlet Series of Several Complex Variables

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In this note, we consider functions of only two variables, although the results can easily be extended to any finite number of variables. Let

\[ f(s_1, s_2) = \sum_{m,n} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2) \]  

be an entire function defined by Dirichlet series \([1]\) of two complex variables \(s_1, s_2\), where the coefficients \(a_{m,n}\) are complex numbers and

\[ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m \to \infty \quad \text{with } m \]

\[ 0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty \quad \text{with } n \]

and further \([2, 3]\)

\[ \limsup_{m+n \to \infty} \frac{\log(m+n)}{\lambda_m + \mu_n} = D < +\infty \]

\[ \limsup_{m+n \to \infty} \frac{\log |a_{m,n}|}{\lambda_m + \mu_n} = -\infty. \]

Let \(X\) be a class of entire functions defined by Dirichlet series \((1)\) which satisfy the condition

\[ \sum_{m,n=1}^{\infty} |a_{m,n}| \exp(m\lambda_m + n\mu_n) < \infty. \]

Definition. (I) We use the notation \(f \sim a_{m,n}\) to denote \((1)\). The algebraic operations in \(X\) are defined by:

(i) \(f + g \sim a_{m,n} + b_{m,n}\)

(ii) \(af \sim aa_{m,n}\)

(iii) \(f \ast g \sim a_{m,n}b_{m,n} \exp(m\lambda_m + n\mu_n),\)

where \(f \sim a_{m,n}\) and \(g \sim b_{m,n}\) are in \(X.\)
(II) We can define also for each \( f \in X \), the following function:

\[
\| f \| = \sum_{m, n = 1}^{\infty} |a_{m, n}| \exp(m\lambda_m + n\mu_n).
\]  \hspace{1cm} (6)

Observe that \( \| f \| \) is defined and it is easy to see that \( \| f \| \) defines a norm on \( X \).

**Theorem 1.** \( X \) is a Banach-algebra.

**Proof.** It sufficient to show that \( X \) is complete; for this, assume that \( \{f_p\} \) is a Cauchy sequence in \( X \), where for \((s_1, s_2) \in \mathbb{C}^2\)

\[
f_p(s_1, s_2) = \sum_{m, n = 1}^{\infty} a_{m, n}^{(p)} \exp(\lambda_m s_1 + \mu_n s_2).
\]

Given \( \varepsilon > 0 \), then there exists some constant \( p_0 \geq 1 \) such that

\[
\| f_p - f_q \| < \varepsilon \quad \text{for } p, q \geq p_0,
\]

i.e.,

\[
\sum_{m, n = 1}^{\infty} |a_{m, n}^{(p)} - a_{m, n}^{(q)}| \exp(m\lambda_m + n\mu_n) < \varepsilon, \quad p, q \geq p_0.
\]  \hspace{1cm} (7)

This implies \( \{a_{m, n}^{(p)}\} \) forms a Cauchy sequence in \( \mathbb{C} \) for every \( m, n \) and hence, owing to the completeness of \( \mathbb{C} \), converges to a complex number, say \( a_{m, n} \). In (7) let \( q \to \infty \); we get

\[
\sum_{m, n = 1}^{\infty} |a_{m, n}^{(p)} - a_{m, n}| \exp(m\lambda_m + n\mu_n) < \varepsilon \quad p \geq p_0,
\]

therefore

\[
f_p \to f \sim a_{m, n}.
\]

Moreover \( f \sim a_{m, n} \in X \), since

\[
\sum_{m, n = 1}^{\infty} |a_{m, n}| \exp(m\lambda_m + n\mu_n)
\leq \sum_{m, n = 1}^{\infty} |a_{m, n}^{(p)} - a_{m, n}| \exp(m\lambda_m + n\mu_n)
+ \sum_{m, n = 1}^{\infty} |a_{m, n}^{(p)}| \exp(m\lambda_m + n\mu_n).
\]

It can now be verified that \( X \) is a Banach-algebra.
DEFINITION. A complex-valued function $\Phi(f)$ defined for $f \in X$ is called a functional; this functional is said to be linear if

$$\Phi(af + bg) = a\Phi(f) + b\Phi(g).$$

**Theorem 2.** Every continuous linear functional $\Phi$ on $X$ is of the form

$$\Phi(f) = \sum_{m,n=1}^{\infty} a_{m,n} d_{m,n} \exp(m\lambda_m + n\mu_n)$$

with

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2),$$

where $\{d_{m,n}\}$ is a bounded sequence.

**Proof.** We denote the dual space of $X$ by $X^*$; let $\Phi \in X^*$. Define

$$f_{m,n} = \exp\{ (s_1 - m) \lambda_m + (s_2 - n) \mu_n \}$$

$$f^{(N)} = \sum_{m,n=1}^{N} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2).$$

Obviously $f^{(N)} \to f$ as $N \to \infty$.

Let

$$\Phi(f_{m,n}) = d_{m,n};$$

then

$$\Phi(f) = \Phi\left( \lim_{N \to \infty} f^{(N)} \right)$$

$$= \Phi \left( \lim_{N \to \infty} \sum_{m,n=1}^{N} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2) \right)$$

$$= \Phi \left( \lim_{N \to \infty} \sum_{m,n=1}^{N} a_{m,n} f_{m,n} \exp(m\lambda_m + n\mu_n) \right)$$

$$= \sum_{m,n=1}^{\infty} a_{m,n} d_{m,n} \exp(m\lambda_m + n\mu_n).$$

Moreover

$$|d_{m,n}| = |\Phi(f_{m,n})| \leq M \| f_{m,n} \| = M;$$

hence, $\{d_{m,n}\}$ is a bounded sequence.
Conversely, let \( \{d_{m,n}\} \) be a bounded sequence. The functional defined by (8) is well defined and linear. Further, we note

\[
|\Phi(f)| \leq \sum_{m,n=1}^{\infty} |a_{m,n}d_{m,n}| \exp(m\lambda_m + n\mu_n) \\
\leq M \|f\|; \quad (9)
\]

hence the theorem.

Remark. This characterization helps us in formulating an alternative expression for the norm in \( X^* \). We know [4] that \( X^* \) is a Banach space with the same operations as in \( X \) and norm define as

\[
\|\Phi\| = \sup_{\|f\| \leq 1} |\Phi(f)|/\|f\|.
\]

**Lemma.** We have

\[
\|\Phi\| = \sup_{m,n \geq 1} |d_{m,n}|.
\]

**Proof.** By (9), we infer that

\[
\|\Phi\| = \sup_{\|f\| \leq 1} |\Phi(f)|/\|f\| \leq \sup_{m,n \geq 1} |d_{m,n}|;
\]
on the other hand

\[
|d_{m,n}| = |\Phi(f_{m,n})| \leq \|\Phi\| \|f_{m,n}\| = \|\Phi\|.
\]

Hence the Lemma.

**Theorem 3.** Let

\[
f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2),
\]

where \( a_{m,n} \neq 0, \forall m, n \geq 1 \). Let \( D \subset \mathbb{C}^2 \) having at least one finite point. Define

\[
f_{\alpha, \beta}(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp\left\{(s_1 + \alpha - m) \lambda_m + (s_2 + \beta - n) \mu_n\right\};
\]

then the set

\[A_f = \{f_{\alpha, \beta} : \alpha, \beta \in D\}\]

is a total set in \( X \).
Proof. Note that \( f_{\alpha, \beta} \in X, \forall \alpha, \beta \in D \), since

\[
f_{\alpha, \beta}(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp \{ (s_1 + \alpha - m) \lambda_m + (s_2 + \beta - n) \mu_n \}
\]

and

\[
\sum_{m,n=1}^{\infty} \exp(m \lambda_m + n \mu_n) |a_{m,n}\exp(\alpha - m) \lambda_m + (\beta - n) \mu_n| = \sum_{m,n=1}^{\infty} |a_{m,n}| \exp(\gamma_1 \lambda_m + \gamma_2 \mu_n),
\]

where \( \gamma_1 = \text{Re} \, \alpha, \gamma_2 = \text{Re} \, \beta \), which must converge for every \( \alpha, \beta \in D \), because \( f(s_1, s_2) \) is an entire Dirichlet series. Let \( \Phi^* \in X^* \) be such that \( \Phi^*(A_f) = 0 \), i.e.,

\[
\Phi^*(f_{\alpha, \beta}) = 0, \quad \forall \alpha, \beta \in D.
\]

This implies that

\[
\sum_{m,n=1}^{\infty} a_{m,n} d_{m,n} \exp(m \lambda_m + n \mu_n) \exp \{ (\alpha - m) \lambda_m + (\beta - n) \mu_n \} = 0, \quad \forall \alpha, \beta \in D,
\]

i.e.,

\[
\sum_{m,n=1}^{\infty} a_{m,n} d_{m,n} \exp(\alpha \lambda_m + \beta \mu_n) = 0, \quad \forall \alpha, \beta \in D. \tag{10}
\]

Now define \( h \sim a_{m,n} d_{m,n} \), since \( \{d_{m,n}\} \) is bounded sequence and \( f \sim a_{m,n} \in X \). \( h \sim a_{m,n} d_{m,n} \in X \). But, owing to (10),

\[
h(\alpha, \beta) = 0, \quad \forall \alpha, \beta \in D.
\]

Since \( D \) has a finite limit point, this means that \( h = 0 \); this, however, implies that \( a_{m,n} d_{m,n} = 0, \forall m, n \geq 1 \) and as \( a_{m,n} \) is not zero for every \( m, n \), we get the result.

**Theorem 4.** Every element in \( X \) is a topological zero divisor in \( X \).

**Proof.** For the definition of the topological zero divisor, we refer to [5]. Consider the sequence \( \{g_{m,n}\} \) where

\[
g_{m,n} = \exp - (m \lambda_m + n \mu_n) \exp(\lambda_m s_1 + \mu_n s_2), \quad m, n \geq 1.
\]
Obviously $g_{m,n} \in X$ and $\| g_{m,n} \| = 1, \forall m, n \geq 1$. Also
\[ f \ast g_{m,n} = g_{m,n} \ast f = a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2) \]
and
\[ \| f \ast g_{m,n} \| = \| g_{m,n} \ast f \| = |a_{m,n}| \exp(m\lambda_m + n\mu_n) \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty; \]
hence the theorem.

REFERENCES